A model-theoretic proof of an incompleteness theorem

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The fragment of Peano Arithmetic with bounded induction is denoted by $\mathrm{I}\Delta_0$. The axiom Ω_1 expresses the totality of the function $\omega_1(x) = x^{\log x}$, and Ω_2 states the totality of $\omega_2(x) = x^{(\log x)^{\log\log x}}$ (see [2]).

A model-theoretic proof of the fact that $\mathrm{I}\Delta_0 + \Omega_2$ does not prove its Herbrand Consistency, is was given by Adamowicz [1].

Let $\log^m$ be the cut consisting of all x such that the m-th exponential of x, $\exp^m(x)$ exists. Theorem 1.1 of [1] implies the existence of a bounded formula $\theta(x)$ such that $\mathrm{I}\Delta_0 + \Omega_i + \exists x \in \log^{i+1} \theta(x)$ is consistent but $\mathrm{I}\Delta_0 + \Omega_i + \exists x \in \log^{i+2} \theta(x)$ is not $(i = 1, 2)$.

For a suitable predicate $\mathrm{HCon}(T)$ expressing the Herbrand Consistency of a theory T (relativized to a cut), it is shown in [1] that for any bounded formula $\theta(x)$, if $\mathrm{I}\Delta_0 + \Omega_2 + \exists x \in \log^3 \theta(x) + \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_2)$ is consistent, then so is $\mathrm{I}\Delta_0 + \Omega_2 + \exists x \in \log^4 \theta(x)$.

By these two theorems, the main theorem of [1], that $\mathrm{I}\Delta_0 + \Omega_2 \nvdash \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_2)$, follows.

In this paper, we modify the predicate $\mathrm{HCon}(T)$, so that it can be shown that for any bounded formula $\theta(x)$, if $\mathrm{I}\Delta_0 + \Omega_1 + \exists x \in \log^2 \theta(x) + \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$ is consistent, then so is $\mathrm{I}\Delta_0 + \Omega_1 + \exists x \in \log^3 \theta(x)$. Hence, the unprovability of Herbrand Consistency of $\mathrm{I}\Delta_0 + \Omega_1$ in itself can be proved by model-theoretical tools.

References

[1] Adamowicz, Z.; "Herbrand consistency and bounded arithmetic", *Fundamenta Mathematica* **171**, N. 3 (2002) pp. 279–292.

[2] Hájek, P. & Pudlák, P.; *Metamathematics of first-order arithmetic*, Second printing, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.