Linear functions and scalar products

Kramer Alpar-Vajk
Institut for Informatics and Practical Mathematics,Kiel,Germany

February 13, 2004

In what follows let $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$.
Theorem:Let X a linear space (vector space) over $\mathbb{K}$, let $< \cdot, \cdot >: X \times X \to \mathbb{K}$ a scalar product and let $f : X \to X$ a linear function such that

$$\forall x \in X \setminus \{0_X\}, f(x) \neq 0_X. \tag{1}$$

Then,the function $\| \cdot \| : X \to \mathbb{R}, \forall x \in X, \|x\| := \sqrt{< f(x), f(x) >}$ is a norm.
Proof:Let $\alpha_1, \alpha_2 \in \mathbb{K}, x_1, x_2 \in X$.
$\|\alpha_1 x_1 + \alpha_2 x_2\|^2 = < f(\alpha_1 x_1 + \alpha_2 x_2), f(\alpha_1 x_1 + \alpha_2 x_2) > \overset{f\ linear}{=} < \alpha_1 f(x_1) + \alpha_2 f(x_2), \alpha_1 f(x_1) + \alpha_2 f(x_2) > = \alpha_1 \overline{\alpha_1} < f(x_1), f(x_1) > + \alpha_1 \overline{\alpha_2} < f(x_1), f(x_2) > + \overline{\alpha_1} \alpha_2 < f(x_2), f(x_1) > + \alpha_2 \overline{\alpha_2} < f(x_2), f(x_2) > = |\alpha_1|^2 \|x_1\|^2 + 2Re(\alpha_1 \overline{\alpha_2} < f(x_1), f(x_2) > + |\alpha_2|^2 \|x_2\|^2 \leq |\alpha_1|^2 \|x_1\|^2 + 2|\alpha_1||\alpha_2| \cdot | < f(x_1), f(x_2) > | + |\alpha_2|^2 \|x_2\|^2 \overset{Cauchy-B.-Schwarz}{\leq} |\alpha_1|^2 \|x_1\|^2 + 2|\alpha_1||\alpha_2| \cdot | < f(x_1), f(x_1) > | \cdot | < f(x_2), f(x_2) > | + |\alpha_2|^2 \|x_2\|^2 = |\alpha_1|^2 \|x_1\|^2 + 2|\alpha_1||\alpha_2| \|x_1\|^2 \|x_2\|^2 + |\alpha_2|^2 \|x_2\|^2 = (|\alpha_1| \|x_1\| + |\alpha_2| \|x_2\|)^2 \Rightarrow$
$\Rightarrow \|\alpha_1 x_1 + \alpha_2 x_2\| \leq |\alpha_1| \|x_1\| + |\alpha_2| \|x_2\| \Rightarrow \| \cdot \|$ is a seminorm.
In light of (1) and from the positive definitness of the scalar product we have that $\forall x \neq 0_X \Rightarrow < f(x), f(x) >> 0 \Rightarrow \|x\| > 0$.Therefore we can finally say that $\| \cdot \|$ is a norm.
Remark:If we consider the identic function $f : X \to X, \forall x \in X, f(x) := x$ then we obtain the well known scalar product-norm connection,(which is a basic idea in the definition of the Hilbert space)namely that $\forall x \in X, \|x\| := \sqrt{< x, x >}$ is a norm.
Basic calculus show that if in a vector space X over $\mathbb{K}$ a norm $\| \cdot \| : X \to \mathbb{R}$ is generated by a scalar product$< \cdot, \cdot >: X \times X \to \mathbb{K}$ besides a linear function $f : X \to X$ with property (1),then the parallelogram-equation holds,namely $\forall x \in X, \forall y \in X : \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.Therefore the parallelogram-equation is invariant over the set of scalar products and over the set of linear functions with property (1).
Let's take a look on a nice application.Let $n \in \mathbb{N}^*$. On the vector space $\mathbb{R}^n$ we consider the usual norm,$\forall x \in \mathbb{R}^n, \|x\| := (\sum_{i=1}^{n} \|x_i\|^2)^{\frac{1}{2}}$,and the usual scalar product $\forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^n, < x, y >:= \sum_{i=1}^{n} x_i \cdot y_i$. Let further $\sigma \in S_n$, apermutation.

Consider the function $f : \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$\forall x \in \mathbb{R}^n, x := (x_1, x_2, ..., x_n), f(x) := (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}). \tag{2}$$

Obviously function f is linear and condition (1) is also fulfilled ,so in the light of the Theorem,$\forall x \in \mathbb{R}^n$, $\sqrt{< f(x), f(x) >}$ is a norm.Moreover,if we complete the calculus we will obtain that $\forall x \in \mathbb{R}^n$, $\sqrt{< f(x), f(x) >} = \|x\|$. Since $\sigma \in S_n$ was arbitrary chosen we conclude that for every permutation $\sigma \in S_n$ the function defined in (2) generate the same norm,namely the usual norm on $\mathbb{R}^n$. If $\sigma \in S_n$ is the identic permutation then f is the identitic function on $\mathbb{R}^n$ and we obtain once again the well-known norm-scalar product connection.As conclusion,we can say that $\forall n \in \mathbb{N}^*$ there are at least $n!$ different linear functions with property (1) which generate the usual norm on $\mathbb{R}^n$. In fact there are more than $n!$ but this aspect will be discussed elsewhere.