Range-support uniform approximations for continuous vector-valued functions

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Let T be a topological space, X a locally convex space over the scalar field $\Gamma \in \{\mathbf{R}, \mathbf{C}\}$, and $C_X(T)$ the space of all X-valued continuous functions on T.

Definition. *Let us define the* <u>*local tensor product*</u>

$$(C_\Gamma(T) \otimes X)_{\mathrm{loc}} := \left\{ u \in C_X(T) \,\middle|\, \begin{array}{l} \textit{for every } t \in T, \textit{ we have } u|_V = v|_V \\ \textit{for some } V \in \mathcal{V}_T(t) \textit{ and } v \in C_\Gamma(T) \otimes X \end{array} \right\}$$

(functions which locally coincide with elements from $C_\Gamma(T) \otimes X$).

1. The main result

Theorem 1 ([3], Th.1). *Let $u \in C_X(T)$. Assume that one of the following conditions holds:*

(i) *one of the sets $T, u(T)$ is paracompact,*

(ii) *$u(T)$ is totally bounded in X.*

Then for every $W \in \mathcal{V}_X(0)$, there exists $u_W \in (C_\Gamma(T) \otimes X)_{\mathrm{loc}}$ satisfying

$$(u - u_W)(T) \subset W, \quad u_W(T) \subset \mathrm{co}(u(T)), \quad \mathrm{supp}\, u_W \subset u^{-1}(X \setminus \{0\}).$$

If $u(T)$ is totally bounded, we can find $u_W \in C_\Gamma(T) \otimes X$.

2. Applications

Three distinct applications of Theorem 1 are described in [3]:

1. A new proof of Schauder-Tihonov's fixed point theorem.
2. If T, X are both Hausdorff, then the inclusion[1] $C_0(T, \Gamma) \otimes X \subset C_0(T, X)$ is dense for the inductive limit topology (stronger than the uniform).
3. Several Tietze-Dugundji type extension theorems. We reproduce below one of them.

Theorem 2 ([3], Th.4). *Assume that T is normal and that X is a locally convex Fréchet space. Consider a closed subset $F \subset T$ and $u \in C_X(F)$. If T is paracompact or if $u(T)$ is totally bounded, then there exists $\widetilde{u} \in C_X(T)$, such that*

$$\widetilde{u}|_F = u, \quad \widetilde{u}(T) \subset \mathrm{co}(\overline{u(F)}).$$

[1] For functions with compact support.

References

[1] J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353–367.

[2] J. B. Prolla, "Approximation of vector valued functions", North-Holland Publishing Co., Amsterdam/New York/Oxford, 1977.

[3] V. Timofte, Special uniform approximations of continuous vector-valued functions. Part I: Approximations of functions from $C_X(T)$, *J. Approx. Theory* **119** (2002), 291–299.

[4] V. Timofte, Special uniform approximations of continuous vector-valued functions. Part II: Approximations of functions from $C_X(T) \otimes C_Y(S)$, *J. Approx. Theory* **123** (2003), 270–275.