# Generalized Carleman Boundary Value Problem

Jorge N. Ferreira and Gueorgui S. Litvinchuk

**Abstract**

We consider the following boundary value problem for a simply connected domain in the complex plane $\mathbb{C}$:

*Find a function $\Phi^+$ analytic in a domain $D^+$, bounded by a simple closed Lyapunov curve $\Gamma$, and satisfying a Hölder condition in $D^+ \cup \Gamma$, by the boundary condition*

$$\Phi^+(\alpha(t)) = a(t)\,\Phi^+(t) + b(t)\,\overline{\Phi^+(t)} + h(t)\,, \tag{1}$$

*where $\alpha$ is a direct or an inverse Carleman shift, $\alpha'(t) \neq 0$, $\alpha'(t) \in H_\mu(\Gamma)$; $a(t)$, $b(t)$, $h(t) \in H_\mu(\Gamma)$.* This problem was considered for the first time by N. P. Vekua [2] and is called the *generalized Carleman boundary value problem.* Note that if $\alpha$ is a inverse Carleman shift and $b(t) = 0$, we obtain the *Carleman boundary value problem* that was considered by T. Carleman [1].

In this work we study the solvability theory of problem (1) on the unit circle with a direct or an inverse linear fractional Carleman shift. To this end, we reduce problem (1) to the singular integral operator with shift

$$K \equiv WP_+ - a(t)\,P_+ + b(t)\,tP_-,$$

where $(W\varphi)(t) = \varphi(\alpha(t))$, $P_\pm$ are projections of Riesz. It is well know that, in this case, this operator can be reduced to a singular operator without shift

$$P_+ + AP_-,$$

where $A$ is a matrix function. The matrix $A$ can be represented as the product of an hermitian matrix function with negative determinant by diagonal rational matrix functions. We estimate the partial indices of the matrix function $A$, and then we obtain estimates of the defects numbers of problem (1). Afterwards, we consider the direct shift $\alpha(t) = -t$ and the inverse shift $\alpha(t) = \frac{1}{t}$ to show that the mentioned estimates are exacts.

## References

[1] Carleman, T., *Sur la theorie des equations integrales et ses applictions*, Verhandl. des Internat. Math. Kong., I, Zürich, 138-151, 1932.

[2] Vekua, N. P., *Systems of singular integral equations*, Nauka: Moscow, 1970.