Generalized Hilbert Boundary Value Problem

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Abstract

Let Γ be a simple closed Lyapunov curve dividing the complex plane $\mathbb{C}$ in two parts D^+ and D^-. The generalized Hilbert boundary value problem consists in finding a function

$$\Phi^+(z) = v(x, y) + iw(x, y)\ ,\ z = x + iy\ ,$$

analytic in the domain D^+, whose limit values belong to $H_\mu(\Gamma)$ and satisfy on Γ the condition

$$Re\{\mathcal{A}(t)\Phi^+(t) + \mathcal{B}(t)\Phi^+(\alpha(t))\} = h(t) \tag{1}$$

where $\mathcal{A}(t) = a(t) - ic(t)$ and $\mathcal{B}(t) = b(t) - id(t)$, with real functions a, b, c, d, h of the class $H_\mu(\Gamma)$, and α is a direct or inverse shift on Γ, such that $\alpha'(t) \neq 0,\ t \in \Gamma$ and $\alpha' \in H_\mu(\Gamma)$.

This problem was proposed by E. G. Khasabov and G. S. Litvinchuk [1]. In their papers [1] and [2], the Noetherity conditions and the index formula of problem (1) with a direct or inverse Carleman shift ($\alpha(\alpha(t)) \equiv t$) on Γ were obtained.

Our main goal is to obtain the defect numbers of problem (1) on the unit circle $\mathbb{T}$, with a direct or inverse linear fractional Carleman shift ($\alpha(\alpha(t)) \equiv t$). To this end we start by reducing this problem to the study of the singular integral operator with shift

$$(\mathcal{A}I + \mathcal{B}W)\, P_+ - \left(t\overline{\mathcal{A}}I + \alpha(t)\,\overline{\mathcal{B}}W\right) P_-\ , \tag{2}$$

where $(W\varphi)(t) = \varphi(\alpha(t))$, $P_\pm$ are projections of Riesz.

It is well known that the study of operator (2) can be reduced to the study of a singular integral operator (without shift) of the form

$$P_+ + CP_-$$

where C is a matrix function.

The matrix C obtained can be represented as a product of an Hermitean matrix function with negative determinant by diagonal rational matrix functions.

We estimate the partial indexes of matrix C and use these results to obtain estimates of the defect numbers of problem (1). Afterwords we consider the direct Carleman shift $\alpha(t) = -t$ and the inverse Carleman shift $\alpha(t) = \frac{1}{t}$ to show that the mentioned estimates are exact.

References

[1] Khasabov, E. G., Litvinchuk, G. S., *On Hilbert boundary value problem with a shift.* Dokl. Akad. Nauk SSSR, 142(6), 274-277, 1962 (in Russian).

[2] Khasabov, E. G., Litvinchuk, G. S., *On the index of generalized Hilbert boundary value problem.* Uspekhi matem. nauk, 20(2), 124-130, 1965 (in Russian).