Double subordination-preserving integral operators

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If $H(\mathrm{U})$ denotes the space of analytic functions in the unit disk U, for the given function $h \in \mathcal{A}$ and the number $\beta \in \mathbb{C}$, we define the integral operator $\mathrm{I}_{h;\beta} : \mathcal{K} \subset H(\mathrm{U}) \to H(\mathrm{U})$ by

$$\mathrm{I}_{\beta,\gamma}(f)(z) = \left[\beta \int_0^z f^\beta(t) h^{-1}(t) h'(t)\,\mathrm{d}\,t\right]^{1/\beta}.$$

We determined simple sufficient conditions on g_1, g_2 and β such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$
$$\Rightarrow \qquad \mathrm{I}_{h;\beta}[g_1](z) \prec \mathrm{I}_{h;\beta}[f](z) \prec \mathrm{I}_{h;\beta}[g_2](z), \tag{1}$$

where the symbol "$\prec$" stands for subordination.

In this case we say that $\mathrm{I}_{h;\beta}$ is a *double subordination-preserving integral operator* and we call such a result a *sandwich-type theorem.*

Moreover, the above implication is *sharp*, in the sense that $\mathrm{I}_{h;\beta}[g_1]$ is the *largest* function and $\mathrm{I}_{h;\beta}[g_2]$ the *smallest* function so that the left-hand side, respectively the right-hand side of the above implication hold.

As applications, we give some particular cases obtained for appropriate choices of the h, that also generalize classic results of the theory of differential subordination and superordination.

The concept of differential superordination was introduced by S. S. Miller and P. T. Mocanu in [4] like a dual problem of differential subordination, and many results were obtained in [1], [2] and [3].

The main result is given by the following *sandwich-type theorem.*

Theorem 1 *Let $\alpha, \beta, \theta \in \mathbb{R}$, with $\beta > 0$, $\alpha\beta \geq 1$ and $0 \leq \theta < 1$. Let $g_1, g_2 \in \mathcal{K}_{h;\beta}$, where $h \in \mathcal{A}$, and suppose that the next two conditions are satisfied:*

$$g_1, g_2 \in \mathcal{M}_\alpha(\theta)$$
$$\operatorname{Re} J(0,h)(z) > -\frac{\theta}{\alpha},\ z \in \mathrm{U}.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{I}_{h;\beta}[f](z)$ are univalent functions in U.

Then, the implication (1) holds. Moreover, the functions $\mathrm{I}_{h;\beta}[g_1]$ and $\mathrm{I}_{h;\beta}[g_2]$ are respectively the best subordinant and the best dominant.

References

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