ON ASYMPTOTICAL STRUCTURE OF FREE ABELIAN GROUPS

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Let $\zeta(z)$ denote the Riemann zeta function. Consider the free abelian group $\mathbb{Z}^d$. Let $H \subset \mathbb{Z}^d$ be a subgroup of finite index. We have a decomposition into cyclic factors $\mathbb{Z}^d/H \cong C_{\alpha_1} \oplus C_{\alpha_2} \oplus \cdots \oplus C_{\alpha_d}$, where $\alpha_{i+1}|\alpha_i$ for $i = 1, \ldots, d-1$. Denote these numbers as $\alpha_i(H)$, $i = 1, \ldots, d$. We study the following *multiple zeta function*

$$\zeta_{\mathbb{Z}^d}(s_1, \ldots, s_d) = \sum_{H \subset \mathbb{Z}^d} \alpha_1(H)^{-s_1} \cdots \alpha_d(H)^{-s_d}.$$

Theorem 1. *Denote the smallest d-th factor by* $\omega(H) = \alpha_d(H)$ *and* $\alpha(H) = \alpha_1(H) \cdots \alpha_{d-1}(H)$. *Denote* $\zeta_{\mathbb{Z}^d}(s, z) = \sum_{H \subset \mathbb{Z}^d} \alpha(H)^{-s} \omega(H)^{-z}$. *Then*

$$\zeta_{\mathbb{Z}^d}(s, z) = \zeta(s)\zeta(s-1) \cdots \zeta(s-(d-1)) \frac{\zeta((d-1)s+z)}{\zeta(ds)}.$$

By $s = z$ we get the formula for ordinary zeta function of $\mathbb{Z}^d$ [1], [2]. This result also yields the multiple zeta function for $d = 2$; but starting with $d = 3$ the multiple zeta function looks rather complicated.

Theorem 2. *Denote* $z_j = s_1 + \cdots + s_j - j(d-j)$, *for* $j = 1, \ldots, d$. *Then*

1. $$\begin{aligned} \zeta_{\mathbb{Z}^d}(s_1, \ldots, s_d) &= \zeta(z_1)\zeta(z_2) \cdots \zeta(z_d) \cdot f(s_1, \ldots, s_{d-1}), \quad \textit{where} \\ f(s_1, \ldots, s_{d-1}) &= \prod_{p\ prime} \Big(1 + \sum_\lambda w_\lambda(p^{-1}) \prod_{j \in \lambda} p^{-z_j}\Big), \end{aligned}$$

   the sum is taken over nonempty subsets $\lambda \subset \{1, 2, \ldots, d-1\}$;
2. $w_\lambda(q)$ *are polynomials in* q *with nonnegative integer coefficients that enumerate some permutations of multisets.*
3. *The multiple zeta function converges absolutely for* $\Re(z_j) > 1$, $j = 1, \ldots, d$.
4. *The product over primes converges absolutely for* $\Re(z_j) > 0$, $j = 1, \ldots, d-1$.

Theorem 3. *Consider subgroups of finite index* $H \subset \mathbb{Z}^d$ *with a cyclic factor-group. The zeta function that enumerates such subgroups equals*

$$\zeta^c_{\mathbb{Z}^d}(x) = \zeta(x-(d-1)) \prod_{p\ prime} (1 + p^{-x}(1 + p + \cdots + p^{d-2})).$$

As a corollary, we compute the probability M_d that a factorgroup $\mathbb{Z}^d/H$ is cyclic. The number M_d tends to $0.84693\ldots$ as $d \to \infty$. Also, it follows from Theorem 2 that a random factorgroup $\mathbb{Z}^d/H$ has a huge first cyclic factor, while the overage value of the product of other factors tends to some small constant.

References

[1] Grunewald F.J., Segal D., and Smith G.C., Subgroups of finite index in nilpotent groups, *Invent. Math.* **93** (1988), 185–223.

[2] Lubotzky A. and Segal D., Subgroup growth, New York etc.: Springer-Verlag, 2003.