Veech groups and Teichmüller curves of origamis

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Translation surfaces and their Veech groups appear already in the work of Thurston and Veech and have been studied by a number of other authors (e.g. [GJ] and references therein). Interesting examples are given by square-tiled surfaces also called *origamis* ([Lo]). They can be defined as follows: Take finitely many euclidean unit squares and glue them such that each left edge is identified with a right one and each top edge with a bottom one.

Mapping each of the squares to the standard torus E yields a finite covering $p : X \to E$, unramified over $E^* = E - \{P\}$. Variation of the translation structure on E leads to a *Teichmüller curve* in the moduli space $M_{g,n}$ (g the genus of X, $n = |p^{-1}(P)|$).

The image of the group of affine diffeomorphisms of $X^* := p^{-1}(E^*)$ in $\mathrm{SL}_2(\mathbb{R})$, the *Veech group* $\Gamma(X)$, is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index ([GJ]). The quotient $\mathbb{H}/\Gamma(X)$ is an affine algebraic curve that turns out to be the normalization of the corresponding Teichmüller curve (e.g. [McM] and references therein).

In [Sm] we prove the following description of the Veech group: If H denotes the fundamental group $\pi_1(X^*)$ considered as subgroup of $\pi_1(E^*) \cong F_2$, then

$$\Gamma(X) = \{A \in \mathrm{SL}_2(\mathbb{Z}) : \gamma_A(H) = H \text{ for some lift } \gamma_A \in \mathrm{Aut}(F_2) \text{ of } A\}.$$

From this we deduce an algorithm determining generators and coset representatives for $\Gamma(X)$ as well as the genus and the number of cusps of $\mathbb{H}/\Gamma(X)$.

We present among others the following results:

- A series of origamis with the congruence group $\bar{\Gamma}^1(4k + 2)$ as projective Veech group; the genus of the corresponding Teichmüller curves is unbounded.
- Origamis whose Veech group is not a congruence group.
- Infinitely many origamis with Veech group equal to $\mathrm{SL}_2(\mathbb{Z})$; the smallest nontrivial example has 108 squares and shows a high symmetry.

References:

[GJ] Gutkin/Judge, Duke Math. J., Vol. 103, 2000.

[Lo] Lochak, Preprint 2003, http://www.math.jussieu.fr/∼lochak.

[McM] McMullen, J. Amer. Math. Soc. 16, 2003.

[Sm] Schmithüsen, arXiv:math.AG/0401185.