TWO-STEP METHODS FOR NUMERICAL SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

Mahmoud Mohseni Moghadam
Mohammad Shafie Dahaghin
Mathematics Department
Kerman University, Kerman, IRAN
e-mail: mohseni@mail.uk.ac.ir

Abstract

Recently fuzzy initial value problems or fuzzy differential equations has received considerable amount of attentions ([3], [4], [7] and [8]). In the first three sections we present the necessary and introductory materials to deal with the fuzzy initial value differential equations. In section four a modified two-step Simpson method and the corresponding convergence theorem of our method is presented. In the last section we will present two examples of fuzzy differential equations and compare our numerical results with the results of the existing methods.

Keywords: Fuzzy differential equations, two-step methods, ordinary differential equations.

1 Preliminaries

A general definition of fuzzy numbers may be found in [1], [5]. However our fuzzy numbers will be almost always triangular or triangular shaped fuzzy numbers. A triangular fuzzy number N is defined by three real numbers $a < b < c$ where the base of the triangle is the interval $[a, c]$ and its vertex is at $x = b$. Triangular fuzzy numbers will be written as $N = (a/b/c)$. The membership function for the triangular fuzzy number $N = (a/b/c)$ is defined as the following:

$$N(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{x-c}{b-c}, & b \le x \le c \end{cases} \tag{1}$$

For triangular shaped fuzzy number P we write $P \approx (a/b/c)$ which is only partially specified by the three numbers a, b, c since the graph on $[a, b]$ and $[b, c]$ is not a straight line segment.

To be a triangular shaped fuzzy number, we require the graph of the corresponding membership function to be continuous and

(1) monotonically increasing on $[a, b]$

(2) monotonically decreasing on $[b, c]$.

The core of a fuzzy number is the set of values where the membership value equals one.

If $N = (a/b/c)$ or $N \approx (a/b/c)$ then the core of N is the single point b. Let T be the set of all triangular or triangular shaped fuzzy numbers and $u \in T$. We define the r-level set

$$[u]_r = \{x | u(x) \ge r\}, \qquad 0 \le r \le 1 \tag{2}$$

which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. It is clear that the following statements are true.

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded right continuous non increasing function over $[0, 1]$,
3. $\underline{u}(r) \le \overline{u}(r)$ for all $r \in [0, 1]$.

For more details see [1], [2].

Let S be the set of all closed bounded intervals in R and $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be two members of S. The interval metric d_1 on S is defined as:

$$d_I(I_1, I_2) = |a_1 - a_2| + |b_1 - b_2|.$$

2 Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation given by

$$\begin{cases} y'(t) = f(t, y(t)) \qquad , t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. Therefore we have a fuzzy Cauchy problem [6].

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of $y(t)$ for $t \in [t_0, T]$ is $[y(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)]$. Also

$$[y'(t)]_r = [\underline{y}'(t; r), \overline{y}'(t; r)], \qquad [f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)].$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and $\underline{f}(t, y) = F[t, \underline{y}, \overline{y}]$, $\overline{f}(t, y) = G[t, \underline{y}, \overline{y}]$. Because of $y' = f(t, y)$ we have

$$\underline{y}'(t; r) = \underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \overline{y}(t; r)] \tag{4}$$

$$\overline{y}'(t; r) = \overline{f}(t, y(t); r) = G[t, \underline{y}(t; r), \overline{y}(t; r)]. \tag{5}$$

Also we write

$$[y(t_0)]_r = [\underline{y}(t_0; r), \overline{y}(t_0; r)] \quad , \quad [y_0]_r = [\underline{y}_0(r), \overline{y}_0(r)]$$

$$\underline{y}(t_0; r) = \underline{y}_0(r) \quad , \quad \overline{y}(t_0; r) = \overline{y}_0(r)$$

By using the extension principle we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\}, \quad s \in R \tag{6}$$

so fuzzy number $f(t, y(t))$. From this is follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \quad r \in [0, 1] \tag{7}$$

where

$$\underline{f}(t, y(t); r) = \min\{f(t, u) | u \in [y(t)]_r\} \tag{8}$$

$$\overline{f}(t, y(t); r) = \max\{f(t, u) | u \in [y(t)]_r\} \tag{9}$$

3 A Modified Simpson Method

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)) & , t \in I = [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{10}$$

It is known that, the sufficient conditions for the existence of a unique solution to Eq(10) are that f to be continuous function satisfying the Lipschitz condition of the following form:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \qquad L > 0.$$

We replace the interval $[t_0, T]$ by a set of discrete equally spaced grid points

$$t_0 < t_1 < t_2 < \ldots < t_N = T, \qquad h = \frac{T - t_0}{N}, \qquad t_i = t_0 + ih \quad i = 0, 1, \ldots, N.$$

To obtain the Simpson method for numerical solution of system (10), we integrate the system from t_{i-1} to t_{i+1} and use the Simpson method for right hand side of

$$\int_{t_{i-1}}^{t_{i+1}} y'(s)ds = \int_{t_{i-1}}^{t_{i+1}} f(s, y(s))ds.$$

Therefore

$$\begin{aligned} y(t_{i+1}) - y(t_{i-1}) &= \frac{h}{3}[f(t_{i-1}, y(t_{i-1})) + 4f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))] \\ &\quad -\frac{h^5}{90} f^{(4)}(\xi_1, y(\xi_1)) \quad t_{i-1} \leq \xi_1 \leq t_{i+1}. \end{aligned} \tag{11}$$

Equation (11) is an implicit equation in term of $y(t_{i+1})$. To avoid of solving such implicit equation we will substitute $y(t_{i+1})$ by $y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))$ in right hand side of (11) where $\xi_2 \in [t_i, t_{i+1}]$. Therefore,

$$\begin{aligned} y(t_{i+1}) &= y(t_{i-1}) + \frac{h}{3} f(t_{i-1}, y(t_{i-1})) + \frac{4h}{3} f(t_i, y(t_i)) \\ &\quad + \frac{h}{3} f\left(t_{i+1}, y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))\right) \\ &\quad - \frac{h^5}{90} f^{(4)}(\xi_1, y(\xi_1)), \quad t_{i-1} \leq \xi_1 \leq t_{i+1}, \quad t_i \leq \xi_2 \leq t_{i+1}. \end{aligned} \tag{12}$$

But we have

$$\begin{aligned} & f\left(t_{i+1}, y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))\right) \\ & = f(t_{i+1}, y(t_i) + hf(t_i, y(t_i))) + \frac{h^2}{2} f'(\xi_2, y(\xi_2)) f_y(t_{i+1}, \xi_3) \end{aligned} \tag{13}$$

where ξ_3 is in between $y(t_i) + hf(t_i, y(t_i))$ and $y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))$. As the result of above we will have

$$\begin{aligned} y(t_{i+1}) \;\; = \;\; & y(t_{i-1}) + \frac{h}{3} f(t_{i-1}, y(t_{i-1})) + \frac{4h}{3} f(t_i, y(t_i)) \\ & + \frac{h}{3} f\left(t_{i+1}, y(t_i) + hf(t_i, y(t_i))\right) \\ & + \frac{h^3}{6} f'(\xi_2, y(\xi_2)) f_y(t_{i+1}, \xi_3) - \frac{h^5}{90} f^{(4)}(\xi_1, y(\xi_1)). \end{aligned} \tag{14}$$

where $t_{i-1} \le \xi \le t_{i+1}$, $t_i \le \xi_2 \le t_{i+1}$ and ξ_3 is in between $y(t_i) + hf(t_i, y(t_i))$ and $y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))$.

Thus we have the following two-step explicit equation for calculation y_{i+1} using y_{i-1} and y_i:

$$y_{i+1} = y_{i-1} + \frac{h}{3} f(t_{i-1}, y_{i-1}) + \frac{4h}{3} f(t_i, y_i) + \frac{h}{3} f(t_{i+1}, y_i + hf(t_i, y_i)) \tag{15}$$

with initial value $y_0 = y(t_0)$ and $y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} f'(t_0, y_0)$.

4 A Modified two-step Simpson Method for Numerical Solution of Fuzzy Differential Equations

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximation solution of the initial value equation (3) by using the two-step modified Simpson method. Let,

$$[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)] \quad , \quad [y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)].$$

Also we note that throughout each integration step, the value of r is unchanged. The exact and approximation solution at t_n are denoted by

$$[Y_n]_r = [\underline{Y}_n(r), \overline{Y}_n(r)] \quad , \quad [y_n]_r = [\underline{y}_n(r), \overline{y}_n(r)] \qquad (0 \le n \le N),$$

respectively. The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N} \quad , \quad t_i = t_0 + ih \quad 0 \le i \le N.$$

By using the modified Simpson method we obtain:

$$\begin{aligned} &\underline{Y}_{n+1}(r) = \underline{Y}_{n-1}(r) + \frac{h}{3}F[t_{n-1}, \underline{Y}_{n-1}(r), \overline{Y}_{n-1}(r)] + \frac{4h}{3}F[t_n, \underline{Y}_n(r), \overline{Y}_n(r)] \\ &+\frac{h}{3}F[t_{n+1}, \underline{Y}_n(r) + hF[t_n, \underline{Y}_n(r), \overline{Y}_n(r)], \overline{Y}_n(r) + hG[t_n, \underline{Y}_n(r), \overline{Y}_n(r)]] \\ &+h^3\underline{A}(r) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\overline{Y}_{n+1}(r) = \overline{Y}_{n-1}(r) + \frac{h}{3}G[t_{n-1}, \underline{Y}_{n-1}(r), \overline{Y}_{n-1}(r)] + \frac{4h}{3}G[t_n, \underline{Y}_n(r), \overline{Y}_n(r)] \\ &+\frac{h}{3}G[t_{n+1}, \underline{Y}_n(r) + hF[t_n, \underline{Y}_n(r), \overline{Y}_n(r)], \overline{Y}_n(r) + hG[t_n, \underline{Y}_n(r), \overline{Y}_n(r)]] \\ &+h^3\overline{A}(r) \end{aligned} \tag{17}$$

where $A = [\underline{A}, \overline{A}]$, $[A]_r = [\underline{A}(r), \overline{A}(r)]$ and

$$[A]_r = \left[\frac{1}{6}f'(\xi_2, Y(\xi_2)) \cdot f_y(t_{i+1}, \xi_3) - \frac{h^2}{90}f^{(4)}(\xi_1, Y(\xi_1))\right]_r . \tag{18}$$

Also we have

$$\begin{aligned} &\underline{y}_{n+1}(r) = \underline{y}_{n-1}(r) + \frac{h}{3}F[t_{n-1}, \underline{y}_{n-1}(r), \overline{y}_{n-1}(r)] + \frac{4h}{3}F[t_n, \underline{y}_n(r), \overline{y}_n(r)] \\ &+\frac{h}{3}F[t_{n+1}, \underline{y}_n(r) + hF[t_n, \underline{y}_n(r), \overline{y}_n(r)], \overline{y}_n(r) + hG[t_n, \underline{y}_n(r), \overline{y}_n(r)]] \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\overline{y}_{n+1}(r) = \overline{y}_{n-1}(r) + \frac{h}{3}G[t_{n-1}, \underline{y}_{n-1}(r), \overline{y}_{n-1}(r)] + \frac{4h}{3}G[t_n, \underline{y}_n(r), \overline{y}_n(r)] \\ &+\frac{h}{3}G[t_{n+1}, \underline{y}_n(r) + hF[t_n, \underline{y}_n(r), \overline{y}_n(r)], \overline{y}_n(r) + hG[t_n, \underline{y}_n(r), \overline{y}_n(r)]] \end{aligned} \tag{20}$$

The following lemma will be applied to show the convergence of our method. For more details see [3], [6].

Lemma 1: Let a sequence of non negative numbers $\{W_n\}_{n=0}^N$ satisfy

$$W_{n+1} \le AW_n + B, \qquad 0 \le n \le N - 1,$$

for some given positive constants A and B. Then

$$W_{n+1} \leq A^n W_1 + B\frac{A^n - 1}{A - 1}, \qquad 0 \leq n \leq N - 1.$$

Now, we prove the following lemma which will be used to prove the convergence of our method.

Lemma 2: Suppose that a sequence of non negative $\{P_n\}_{n=0}^{N}$ satisfy

$$\{P_{n+1} \leq AP_n + BP_{n-1} + C, \qquad 1 \leq n \leq N - 1,$$

for some given positive constants A, B and C. Then

$$P_{n+1} + (\alpha - A)P_n \leq \alpha^n[P_1 + (\alpha - A)P_0] + C\frac{\alpha^n - 1}{\alpha - 1},$$

where $\alpha = \dfrac{\sqrt{A^2 + 4B} + A}{2}$.

Proof. It is obvious that $A = \dfrac{\sqrt{A^2 + 4B} + A}{2} - \dfrac{\sqrt{A^2 + 4B} - A}{2}$. Therefore we have:

$$P_{n+1} + \frac{\sqrt{A^2 + 4B} - A}{2}P_n \leq \frac{\sqrt{A^2 + 4B} + A}{2}P_n + BP_{n-1} + C,$$

and consequently,

$$P_{n+1} + \frac{\sqrt{A^2 + 4B} - A}{2}P_n \leq \frac{\sqrt{A^2 + 4B} + A}{2}\left(P_n + \frac{\sqrt{A^2 + 4B} - A}{2}P_{n-1}\right) + C.$$

If we set $T_{n+1} = P_{n+1} + \dfrac{\sqrt{A^2 + 4B} - A}{2}P_n$, then $T_{n+1} \leq \alpha T_n + C$, $1 \leq n \leq N - 1$, where $\alpha = \dfrac{\sqrt{A^2 + 4B} + A}{2}$. By using lemma 1 we have $T_{n+1} \leq \alpha^n T_1 + C\dfrac{\alpha^n - 1}{\alpha - 1}$ and consequently,

$$P_{n+1} + (\alpha - A)P_n \leq \alpha^n[P_1 + (\alpha - A)P_0] + C\frac{\alpha^n - 1}{\alpha - 1}$$

where $\alpha = \dfrac{\sqrt{A^2 + 4B} + A}{2}$.$\blacksquare$

Our next result determines the point wise convergence of the modified Simpson approximations to the exact solution. Let $F[t, u, v]$ and $G[t, u, v]$ be the functions which are given by the equations (4), (5) where u and v are constants and $u \leq v$. Thus the domain of F and G are defined as the following:

$$K = \{(t, u, v)|t_0 \leq t \leq T, -\infty < u \leq v, -\infty < v < \infty\}.$$

With the above notations in the following we will present the convergence theorem.

Theorem 1: Let $F[t,u,v]$ and $G[t,u,v]$ belong to $C^1(K)$ and suppose that the partial derivatives of F, G are bounded on K. Then for arbitrary fixed r, $0 \le r \le 1$, the two-step modified Simpson approximations y_N converge to the exact solution $Y(T)$ uniformly in t. In other word,

$$\lim_{h\to 0} d_I\left(\left[\underline{y}_N(r), \overline{y}_N\right], \left[\underline{Y}_N(r), \overline{Y}_N(r)\right]\right) = 0.$$

Proof: Let $W_n = \underline{Y}_n(r) - \underline{y}_n(r)$, $V_n = \overline{Y}_n(r) - \overline{y}_n(r)$. By using the equations (16), (17), (19) and (20), we conclude that:

$$\begin{aligned} |W_{n+1}| \le |W_{n-1}| &+ \frac{2Lh}{3}\max\{|W_{n-1}|, |V_{n-1}|\} + \frac{8Lh}{3}\max\{|W_n|, |V_n|\} \\ &+ \frac{2Lh}{3}[2Lh\max\{|W_n|, |V_n|\} + \max\{|W_n|, |V_n|\}] + h^3\underline{M}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} |V_{n+1}| \le |V_{n-1}| &+ \frac{2Lh}{3}\max\{|W_{n-1}|, |V_{n-1}|\} + \frac{8Lh}{3}\max\{|W_n|, |V_n|\} \\ &+ \frac{2Lh}{3}[2Lh\max\{|W_n|, |V_n|\} + \max\{|W_n|, |V_n|\}] + h^3\overline{M}, \end{aligned} \tag{22}$$

where $\underline{M}$, $\overline{M}$ are upper bound for $\underline{A}(r)$, $\overline{A}(r)$ respectively which

$$[A]_r = [\underline{A}(r), \overline{A}(r)] = \left[\frac{1}{6}f(\xi_2, y(\xi_2)) \cdot f_y(t_{i+1}, \xi_3) - \frac{h^2}{90}f^{(4)}(\xi_1, Y(\xi_1))\right]_r.$$

Consequently

$$\begin{aligned} |W_{n+1}| \le |W_{n-1}| &+ \frac{2Lh}{3}(|W_{n-1}| + |V_{n-1}|) + \frac{8Lh}{3}(|W_n| + |V_n|) \\ &+ \frac{2Lh}{3}(1+2Lh)(|W_n| + |V_n|) + h^3\underline{M}, \end{aligned} \tag{23}$$

and

$$\begin{aligned} |V_{n+1}| \le |V_{n-1}| &+ \frac{2Lh}{3}(|W_{n-1}| + |V_{n-1}|) + \frac{8Lh}{3}(|W_n| + |V_n|) \\ &+ \frac{2Lh}{3}(1+2Lh)(|W_n| + |V_n|) + h^3\overline{M}. \end{aligned} \tag{24}$$

By adding above two equations and setting $U_n = |W_n| + |V_n|$ we obtain,

$$U_{n+1} \le \frac{4Lh}{3}(5+2Lh)U_n + \left(1 + \frac{4Lh}{3}\right)U_{n-1} + 2h^3M,$$

where $M = \max\{\underline{M}, \overline{M}\}$. By using lemma 2 we have:

$$U_{n+1} + (\alpha - A)U_n \le \alpha^n[U_1 + (\alpha - A)U_0] + C\frac{\alpha^n - 1}{\alpha - 1}$$

where $\alpha = \dfrac{\sqrt{A^2 + 4B} + A}{2}$, $\quad A = \dfrac{4Lh}{3}(5 + 2Lh)$, $\quad B = 1 + \dfrac{4Lh}{3}$ and $C = 2h^3M$.

Because of $U_0 = 0$, for $n = N - 1$ we have

$$\lim_{h\to 0}\left[\alpha^{N-1}[U_1 + (\alpha - A)U_0] + C\frac{\alpha^{N-1} - 1}{\alpha - 1}\right] = 0.$$

Therefore we have $\lim_{h\to 0}[U_N + (\alpha - A)U_{N-1}] = 0$ and consequently, $\lim_{h\to 0} U_N = 0$ or $\lim_{h\to 0}(|W_N| + |V_N|) = 0$. In other word,

$$\lim_{h\to 0} d_I\left(\left[\underline{y}_N(r), \overline{y}_N(r)\right], \left[\underline{Y}_N(r), \overline{Y}_N(r)\right]\right) = 0.\blacksquare$$

5 Numerical Results

In this section we will present two numerical examples. For each of them the theoretical exact solution and the numerical solution via our method are shown in the tables at the end of this section. In order to compare the accuracy we devoted tables (3) and (6) for the corresponding errors of examples one and two respectively. It should be mentioned that the difference of two r-level sets $[a_1, b_1]$ and $[a_2, b_2]$ is denoted by $E = |a_1 - a_2| + |b_1 - b_2|$.

As well as the convergence theorem shows, the numerical results also show that for smaller stepsize h we get smaller errors and hence better results.

Example 1: consider the initial value problem

$$\begin{cases} y'(t) = y(t) & t \in [0, 1] \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r). \end{cases}$$

The exact solution at $t = 1$ is given by

$$Y(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], \qquad 0 \le r \le 1.$$

Using the two-step modified Simpson method approximation and denote:

$$\underline{y}_0 = 0.75 + 0.25r, \quad \overline{y}_0 = 1.125 - 0.125r, \quad \underline{y}_1 = \underline{y}_0 + h\underline{y}_0 + \frac{h^2}{2}\underline{y}_0$$

and $\overline{y}_1 = \overline{y}_0 + h\overline{y}_0 + \frac{h^2}{2}\overline{y}_0$ as initial values, we have:

$$\underline{y}_{i+1} = \underline{y}_{i-1} + \frac{h}{3}\underline{y}_{i-1} + \frac{4h}{3}\underline{y}_i + \frac{h}{3}(\underline{y}_i + h\underline{y}_i)$$

$$\overline{y}_{i+1} = \overline{y}_{i-1} + \frac{h}{3}\overline{y}_{i-1} + \frac{4h}{3}\overline{y}_i + \frac{h}{3}(\overline{y}_i + h\overline{y}_i).$$

Example 2: The solution of

$$\begin{cases} y'(t) = ty(t) & t \in [-1, 1] \\ y(-1) = [\sqrt{e} - 0.5(1-r), \sqrt{e} + 0.5(1-r)] \end{cases}$$

is separated between two steps $-1 \leq t \leq 0$ and $0 \leq t \leq 1$. The exact solution at $t = 1$ is given in [6]. To get the Simpson approximation we divide $[-1, 1]$ into even number N equally spaced subintervals, define $\underline{y}_0 = \underline{x}_0$, $\overline{y}_0 = \overline{x}_0$ and calculate

$$\underline{y}_1 = \underline{y}_0 + ht_0\overline{y}_0 + \frac{h^2}{2}(1 + t_0^2)\underline{y}_0$$

$$\overline{y}_1 = \overline{y}_0 + ht_0\underline{y}_0 + \frac{h^2}{2}(1 + t_0^2)\overline{y}_0$$

as initial value of two step method. While $t_i < 0$ we have

$$\underline{y}_{i+1} = \underline{y}_{i-1} + \frac{h}{3}t_{i-1}\overline{y}_{i-1} + \frac{4h}{3}t_i\overline{y}_i + \frac{h}{3}t_{i+1}(\overline{y}_i + ht_i\underline{y}_i)$$

$$\overline{y}_{i+1} = \overline{y}_{i-1} + \frac{h}{3}t_{i-1}\underline{y}_{i-1} + \frac{4h}{3}t_i\underline{y}_i + \frac{h}{3}t_{i+1}(\underline{y}_i + ht_i\overline{y}_i)$$

if $t_i = 0$ then

$$\underline{y}_{i+1} = \underline{y}_{i-1} + \frac{h}{3}t_{i-1}\overline{y}_{i-1} + \frac{h}{3}t_{i+1}\underline{y}_i$$

$$\overline{y}_{i+1} = \overline{y}_{i-1} + \frac{h}{3}t_{i-1}\underline{y}_{i-1} + \frac{h}{3}t_{i+1}\overline{y}_i$$

and for $t_i > 0$ we have

$$\underline{y}_{i+1} = \underline{y}_{i-1} + \frac{h}{3}t_{i-1}\underline{y}_{i-1} + \frac{4h}{3}t_i\underline{y}_i + \frac{h}{3}t_{i+1}(\underline{y}_i + ht_i\underline{y}_i)$$

$$\overline{y}_{i+1} = \overline{y}_{i-1} + \frac{h}{3}t_{i-1}\overline{y}_{i-1} + \frac{4h}{3}t_i\overline{y}_i + \frac{h}{3}t_{i+1}(\overline{y}_i + ht_i\overline{y}_i).$$

h / r	0.1	0.01	0.001	0.0001
0	$2.0369278, 3.0553918$	$2.0386942, 3.0580414$	$2.0387111, 3.0580668$	$2.0387113, 3.0580670$
0.2	$2.1727230, 2.9874942$	$2.1746072, 2.9900849$	$2.1746252, 2.9901097$	$2.1746254, 2.9901099$
0.4	$2.3085182, 2.9195966$	$2.3105201, 2.9221284$	$2.3105393, 2.9221527$	$2.3105395, 2.9221529$
0.6	$2.4443134, 2.8516990$	$2.4464331, 2.8541720$	$2.4464534, 2.8541956$	$2.4464536, 2.8541958$
0.8	$2.5801086, 2.7838014$	$2.5823460, 2.7862155$	$2.5823675, 2.7862386$	$2.5823676, 2.7862388$
1	$2.7159038, 2.7159038$	$2.7182590, 2.7182590$	$2.7182815, 2.7182815$	$2.7182817, 2.7182817$

Table (1)

r	Exact solution
0	$2.0387113, 3.0580670$
0.2	$2.1746254, 2.9901100$
0.4	$2.3105395, 2.9221529$
0.6	$2.4464536, 2.8541959$
0.8	$2.5823677, 2.7862388$
1	$2.7182818, 2.7182818$

Table (2)

h / r	0.1	0.01	0.001	0.0001
0	0.0044587	0.0000427	0.0000004	0.0000000
0.2	0.0045181	0.0000432	0.0000004	0.0000001
0.4	0.0045776	0.0000438	0.0000004	0.0000001
0.6	0.0046370	0.0000444	0.0000004	0.0000000
0.8	0.0046965	0.0000450	0.0000004	0.0000001
1	0.0047559	0.0000455	0.0000004	0.0000001

Table (3)

h / r	0.2	0.02	0.01	0.004
0	$0.2812197, 3.0102137$	$0.2894315, 0.0080038$	$0.2895434, 3.0078982$	$0.2895744, 3.0078680$
0.2	$0.5541191, 2.7373143$	$0.5612888, 2.7361465$	$0.5613789, 2.7360627$	$0.5614038, 2.7360386$
0.4	$0.8270185, 2.4644149$	$0.8331460, 2.4642893$	$0.8332143, 2.4642272$	$0.8332331, 2.4642093$
0.6	$1.0999179, 2.1915155$	$1.1050032, 2.1924321$	$1.1050498, 2.1923917$	$1.1050625, 2.1923799$
0.8	$2.5801086, 2.7838014$	$2.5823460, 2.7862155$	$2.5823675, 2.7862386$	$1.3768918, 1.9205505$
1	$1.6457167, 1.6457167$	$1.6487177, 1.6487177$	$1.6487208, 1.6487208$	$1.6487212, 1.6487212$

Table (4)

r	Exact solution
0	$0.2895803, 3.0078621$
0.2	$0.5614085, 2.7360340$
0.4	$0.8332367, 2.4642058$
0.6	$1.1050649, 2.1923776$
0.8	$1.3768930, 1.9205494$
1	$1.6487212, 1.6487212$

Table (5)

h / r	0.2	0.02	0.01	0.004
0	0.0107121	0.0002904	0.0000729	0.0000116
0.2	0.0085697	0.0002323	0.0000583	0.0000093
0.4	0.0064272	0.0001742	0.0000437	0.0000070
0.6	0.0060090	0.0001161	0.0000291	0.0000046
0.8	0.0060090	0.0000580	0.0000145	0.0000023
1	0.0060090	0.0000071	0.0000009	0.0000000

Table (6)

References

[1] J.J. Buckley and E.Eslami, Introduction to Fuzzy Logic and Fuzzy Sets, Physica-Verlag, Heidelberg, Germany, 2001.

[2] J.J. Buckley, E.Eslami and T. Feuring, Fuzzy Mathematics in Economics and Engineering, Physica-Verlag, Heidelberg, Germany, 2002.

[3] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, Vol. 24, pp. 301-317, 1987.

[4] O. Kaleva, The Cauchy problem for fuzzy differential equations. Fuzzy Sets and Systems, Vol. 35, pp. 389-396, 1990.

[5] A. Kaufmann and M.M. Gupta, Introduction to Fuzzy Arithmetic, Theory and Application, Van Nostrand Reinhold, New York, 1985.

[6] M. Ma, M. Friedman and A. Kandel, Numerical solutions of fuzzy differential equation. Fuzzy Sets and Systems, Vol. 105, pp. 133-138, 1999.

[7] Ouyang he, Wu Yi, On fuzzy differential equations, Fuzzy Sets and Systems, Vol. 32, pp. 321-325, 1989.

[8] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, Vol. 24, pp. 319-330, 1987.