# ON THE CONTROLLABILITY FOR NONLINEAR ABSTRACT EQUATIONS OF EVOLUTION

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## 1 The abstract equation.

The space $E$ is a Banach lattice with a cone $E_+$ .

$$u'(t) = [Au](t) + F(t). \tag{1lin}$$

We consider the nonlinear systeme

$$u'(t) = [Au](t) + Gu(t) + F(t) \quad (0 \leq t \leq T), \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$\int_0^T u(t)d\mu(t) = \psi \qquad (\psi \in D(A)), \tag{3}$$

where $A : D(A) \to E$ is a linear closed operator with a dense domain of definition $D(A) \subset E$ ;
$u \in C^1([0,T]; E) \cap C([0,T]; D(A)) \stackrel{note}{=} X$;
$F(t) = \Phi(t)f \in C^1([0,T], E)$;
$f \in E$ is the control desired;
$\Phi(t) \in C^1([0,T]; \mathcal{L}(E))$ , $\mathcal{L}(E)$ is a Banach algebra of linear closed operators mapping $E$ into $E$ , equipped with the ordinary operator norm;
$\Phi \geq 0$ for $0 \leq t \leq T$ ;
$\mu(t)$ is a nondecreasing on $[0,T]$ scalar function continuous from the right for $t = 0$ ;
(3) is a vector-valued Riemann-Stieltjes integral;
$G : X \to C([0,T], E)$ is strong Fréchet differentiable nonlinear operator such that
$G(0) = 0$ , $G'(0) = 0$ .

The linear analogue of this problem has been considered by **Прилепко А.И., Тихонов И.В. "Восстановление неоднородного слагаемого в абстрактном эволюционном уравнении"Известия Академии Наук. Серия математическая. 1994. Т. 58, №2. С. 167-188.**

***Def.*** The element $f \in E$ is called a solution of the problem (1)- (3) with a fixed $\psi \in D(A)$ if the solution of Cauchy problem (1), (2) with this $f$ satisfies (3).

The condition of redetermination (3) includes the following subcases:

**Final redetermination**

$$u(T) = \psi$$

If $\mu(t)$ from (3) is a step function, we obtaine

$$\alpha_1 u(t_1) + \alpha_2 u(t_2) + ... = \psi,$$

where $t_i \in [0,T]$ , $\alpha_i :\ \Sigma|\alpha_i| < \infty$ .

**Integral redetermination**

If $\mu(t)$ from (3) is an absolutely continuous function we obtaine

$(B) \int\limits_0^T \omega(t)u(t)dt = \psi$ , where $\omega(t) = \mu'(t)$ is a summable function.

***Theorem.*** Let $J = \int\limits_0^T \Phi(t)d\mu(t)$ be an operator such that $J^{-1} \in \mathcal{L}(E)$ , $J^{-1} \geq 0$ i. e., $J^{-1}(E_+) \subset E_+$ ; the operator $A$ generate a semigroup $S(t)$ of $C_0$ -class for the linear problem (1 lin)- (2); the semigroup be positive $S(t) \geq 0 \quad \forall t \geq 0$ and compact for $t > 0$ ; the spectrum of $A$ be $\{\lambda \in C : Re\lambda < 0\}$ and suppose one of the following conditions holds:

(a) $\Phi'(t) \geq 0$ for $0 \leq t \leq T$ ;

(b) the function $\mu(t)$ is convex upwards on $[0,T]$ .

Then the nonlinear problem (1)- (3) has an unique solution $f$ in a neighborhood of zero in $E$ for a sufficiently small by norm $\psi$ from $D(A)$ .

The proof of the theorem is based on known properties of linear problems (1 lin), (2) and (1 lin)- (3) and on double application of inverse function theorem in corresponding function spaces, one of which is the space of the linear Cauchy problem's solution

$$H \stackrel{def}{=} \{v \in X | \exists F \in C([0,T],E) : v \quad \text{is solution of} \ \ (1 \text{ lin})\text{- } (2)\ \}.$$

As it is known $\forall F \in C([0,T],E)$ there exists and is unique solution of Cauchy problem (1 lin), (2):

$$\|u\|_{C^1([0,T],E)} \leq K\|F(t)\|_{C([0,T],E)},$$

where $K > 0$ is a constant,

then the operator $L = d/dt - A$ is an isometric isomorphism $H$ onto $LH = C([0,T],E)$ with the norm

$$\|u\|_H = \|Lu\|_{C([0,T],E)}.$$

Therefore $(H, \|\cdot\|_H)$ is a Banach space and the inclusion $H \subset C^1([0,T],E)$ is continuous.

Having assumed in addition the validity of estimate

$$\|G'(u)\|_{\mathcal{L}(H,LH)} \leq ?\vartheta(r)$$

$$\text{for} \quad \|u\|_H \leq r,$$

where the function $?\vartheta : [0,\infty[ \to [0,\infty[$ is monotonically nondecreasing, we also obtained with the help of stated below theorem the sufficient conditions for a dimension of the environment from which it is possible to take a function $\psi$ from the condition of redetermination (3) for one-valued solvability of (1)- (3).

**Theorem. (Sukhinin M.F.**

***S*olvability of nonlinear stationary transfer equation.//Theoretical and Mathematical Physics. Vol. 103, No. 1, 1995)**

Let $X$ be a Banach space, $Y$ be a separable topological vector space, $A : X \to Y$ be a linear continuous operator, $U$ be the open unit ball in $X$, $\mathrm{P}_{AU} : AX \to [0, +\infty[$ be a functional of the Minkowski set $AU$, and the mapping $\Psi : X \to AX$ satisfy the condition

$$\mathrm{P}_{AU}\big(\Psi(x) - \Psi(?x)\big) \le \Theta(r)\|x - ?x\|$$

$$\text{for} \quad \|x - x_0\| \leqslant r, \quad \|?x - x_0\| \le r$$

for some $x_0 \in X$, where the function

$\Theta : [0, \infty[ \to [0, \infty[$ is monotonically nondecreasing.

We set $b(r) = \max\big(1 - \Theta(r), 0\big)$ for $r \ge 0$.

Let $w = \int\limits_0^\infty b(r)\, dr \in ]0, \infty]$,

$r_* = \sup\{r \ge 0 |\ b(r) > 0\}$, $w(r) = \int\limits_0^r b(t)dt \quad (r \ge 0)$ and $f(x) = Ax + \Psi(x)$ for $x \in X$.

Then $\forall r \in [0, r_*[ \quad \forall y \in f(x_0) + w(r)AU$

$\exists x \in x_0 + rU : \quad f(x) = y$.

This abstract result can be applied for solving, for example, similarly posed inverse problem for Navier-Stoks equation.

## 2 The Heat equation.

For a bounded domain of one connectivity $\Omega \subset R^n$ with a boundary $\partial\Omega \in C^2 \quad (n \ge 2)$, $S_T = \partial\Omega \times [0, T]$ (where $T > 0$),

we consider the following inverse problem:

$$u_t - \Delta u - \Phi(u) = F(x, t),$$
$$u|_{t=0} = 0,$$
$$u|_{S_T} = 0,$$
$$\int\limits_\Omega u(x, t)\omega(x)dx = \chi(t),$$

where $F(x, t) = h(x, t)f(t)$, $f(t)$ is a control; the function $\Phi \in C^1(R)$ is such that

$$|\Phi(u)| \le C_1 |u|^{\alpha_1} + C_2 |u|^{\alpha_2},$$

$$|\Phi'(u)| \le \alpha_1 C_1 |u|^{\alpha_1 - 1} + \alpha_2 C_2 |u|^{\alpha_2 - 1},$$

with $1 < \alpha_1 \le \alpha_2 \le (n + 1)/(n - 1)$; $h$, $g$, $\omega$, $\chi$ - are given functions, but $u$ and $f$ - are unknown functions.

This inverse problem consists in selecting the control $f(t)$, guaranteeing the value of integral known at any moment of time, of product of the solution by the function who depends only from space variables.

The linear analogue of this problem has been considered by **Костин А.Б. Обратные задачи для математических моделей физических процессов. — М.: МИФИ, 1991. — С. 45-49.**

We proved that if

$$\omega \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega),$$

$$\chi \in W_2^1(0,T), \qquad \chi(0) = 0,$$

$h \in L_2(Q_T)$, $\|h(\cdot,t)\|_{2,\Omega}$ is bounded in $[0,T]$ and $\left|\int\limits_\Omega h(x,t)\omega(x)dx\right| \geq \delta > 0$ for almost every $t \in [0,T]$, $u \in W_2^{2,1}(Q_T)$, then there exists unique solution $f \in L_2(0,T)$ of the nonlinear problem in a neighborhood of zero for a sufficiently small by norm $\chi$.

The sufficient conditions for a dimension of the environment from which it is possible to take a function $\chi(t)$ from the condition of redetermination so that the inverse problem was solvable are also obtained.