Proceedings of the 41st IEEE
Conference on Decision and Control
Las Vegas, Nevada USA, December 2002

Identifiability of shaping filters from covariance lags, cepstral windows and Markov parameters[1]

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Abstract

In this paper, which is an abbreviated version of [8], we study the well-posedness of the problems of determining shaping filters from combinations of finite windows of cepstral coefficients, covariance lags, or Markov parameters. For example, we determine whether there exists a shaping filter with prescribed window of Markov parameters and a prescribed window of covariance lags. We show that several such problems are well-posed in the sense of Hadamard; that is, one can prove existence, uniqueness (identifiability) and continuous dependence of the model on the measurements. Our starting point is the global analysis of linear systems, where one studies an entire class of systems or models as a whole, and where one views measurements, such as covariance lags and cepstral coefficients or Markov parameters, from data as functions on the entire class. This enables one to pose such problems in a way that tools from calculus, optimization, geometry and modern nonlinear analysis can be used to give a rigorous answer to such problems in an algorithm-independent fashion.

1 Introduction

In this paper we review some basic results in [8] while omitting most proofs. We refer the reader to [8] for these as well as to additional references.

It is common to model a (real, zero-mean) stationary process $\{y(t) \mid t \in \mathbb{Z}\}$ as a convolution

$$y(t) = \sum_{k=-\infty}^{t} w_{t-k} u_k$$

of an excitation signal $\{u(t) \mid t \in \mathbb{Z}\}$ which is a white noise, i.e., $\mathrm{E}\{u(t)u(s)\} = \delta_{ts}$, where δ_{ts} is one if $t = s$ and zero otherwise. In the language of systems and control, under suitable finiteness conditions this amounts to passing the white noise u through a linear filter with the transfer function $w(z)$ having the Laurent expansion

$$w(z) = \sum_{k=0}^{\infty} w_k z^{-k} \tag{1}$$

for all $z \geq 1$, thus obtaining the process y as the output. In addition, we assume that $w_0 \neq 0$ and that $w(z)$ is a rational function, the latter assumption being the finiteness condition required in systems and control theory. Such a filter will be called a *shaping filter* and the coefficients $w_0, w_1, w_2, \cdots$ the *Markov parameters.* Clearly, any shaping filter must be *stable* in the sense that $w(z)$ has all its poles in the open unit disc. To begin, we also assume that all zeros are located in the open unit disc. Such a shaping filter will be called a *minimum-phase shaping filter.*

Then the stationary stochastic process y has a rational spectral density

$$\Phi(e^{i\theta}) = |w(e^{i\theta})|^2, \tag{2}$$

which is positive for all θ. It is well-known that the spectral density has a Fourier expansion

$$\Phi(e^{i\theta}) = r_0 + 2\sum_{k=1}^{\infty} r_k \cos k\theta,$$

where the Fourier coefficients

$$r_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta \tag{3}$$

are the covariance lags $r_k = \mathrm{E}\{y(t+k)y(t)\}$.

The spectral density $\Phi(z)$ is analytic in an annulus containing the unit circle and has there the representation

$$\Phi(z) = f(z) + f(z^{-1}),$$

where f is a rational function with all its poles and zeros in the open unit disc. Moreover, $\Phi(e^{i\theta}) = 2\mathrm{Re}\{f(e^{i\theta})\} > 0$ for all θ, and therefore f is a real function which maps $\{|z| \geq 0\}$ into the right half-plane $\{\mathrm{Re}\, z > 0\}$; such a function is called *positive real.* For this to hold, the Toeplitz matrices

$$T_n = \begin{bmatrix} r_0 & r_1 & \cdots & r_n \\ r_1 & r_0 & \cdots & r_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_{n-1} & \cdots & r_0 \end{bmatrix} \tag{4}$$

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[1] This research was supported in part by grants from AFOSR, VR, the Göran Gustafsson Foundation, and Southwestern Bell.

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must be positive definite for $n = 0, 1, 2, \ldots$.

Another way of representing the distribution of the stationary process is via the so-called *cepstrum*

$$\log \Phi(e^{i\theta}) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos k\theta. \tag{5}$$

The Fourier coefficients

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\theta \tag{6}$$

are known as the *cepstral coefficients.*

Finite windows of covariance lags and cepstral coefficient can be estimated from an observed data record

$$y_0, y_1, y_2, \ldots, y_N$$

of the process $\{y(t) \mid t \in \mathbb{Z}\}$. In fact, a limited number of covariance lags can be estimated via some ergodic estimate

$$r_k = \frac{1}{N+1-n} \sum_{t=0}^{N-n} y_{t+k} y_k. \tag{7}$$

However, we can only estimate

$$r_0, r_1, \ldots, r_n, \tag{8}$$

where $n << N$, with some precision. A complementary set of observables are given by the window

$$c_0, c_1, \ldots, c_n \tag{9}$$

of cepstral coefficients. One topic considered in this paper is to investigate the conditions under which these estimated coefficients can be used to determine minimum-phase shaping filters, i.e., to determine the identifiability of such shaping filters from covariance and cepstral windows.

For minimum-phase shaping filters, the cepstral coefficients used in signal processing are closely related to the Markov parameters $w_0, w_1, w_2, \cdots$ defined by (1). In more general systems problems, the minimum phase requirement is relaxed to allow the numerator polynomial of w to be an arbitrary (monic) polynomial. In this case, a record

$$w_0, w_1, \ldots, w_n \tag{10}$$

of Markov parameters are typically determined from the impulse response of an underlying system.

In this paper we are interested in the mathematical nature of the transformation of measurements, such as covariance lags and cepstral coefficients or Markov parameters, from data into the parameters of systems which produce such data. Our starting point will be the global analysis of linear systems, where one studies an entire class of systems or models as a whole, and where one views measurements from data or model parameters as functions on the entire class. To this end, we shall need some basic spaces of systems. Suppose the transfer function w is given by

$$w(z) = \frac{\sigma(z)}{a(z)}, \tag{11}$$

where

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_n \tag{12}$$
$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n \tag{13}$$

are (real) polynomials of degree n with all roots in the open unit disc. We shall denote the class of such monic (Schur) polynomials by $\mathcal{S}_n$.

The set $\mathcal{P}_n$ of all $(a, \sigma) \in \mathcal{S}_n \times \mathcal{S}_n$ is a smooth, connected, real manifold of dimension $2n$ that is diffeomorphic to $\mathbb{R}^{2n}$. Moreover, we denote by $\mathcal{P}_n^*$ the (dense) open subspace of $\mathcal{P}_n$ consisting of those pairs (a, σ) of polynomials which are coprime. It can be seen that $\mathcal{P}_n$ is diffeomorphic to the space $\text{Rat}(n)$, first studied in [2], which is a $2n$-dimensional manifold with $n+1$ path-connected components, some of which have a rather complicated topology. For nonminimum-phase systems, we need to allow (a, σ) to vary over the larger space

$$\mathcal{Q}_n := \mathcal{S}_n \times \Pi_n,$$

where Π_n is the space of real, monic, degree n-polynomials. We shall also need to consider the space $\mathcal{Q}_n^*$, the (dense) open subspace of $\mathcal{Q}_n$ consisting of those pairs (a, σ) of polynomials which are coprime.

2 Main Results

Our first results show that it is possible to parameterize minimum-phase shaping filters in terms of a window of cepstral coefficients and a window of covariance lags, both of which can be estimated from data. It is tempting, of course, to argue the plausibility of this result by counting parameters. This method typically works only when there is a rigorous way to compute the dimension of some geometric object – in this case the smooth $2n$-dimensional manifold $\mathcal{P}_n$. In this setting, the implicit function theorem enables one to compute dimensions by computing the rank of certain Jacobian matrices. The following theorem is proved in Section 3.

Theorem 2.1 *The normalized covariance lags $r_1, r_2, \ldots, r_n$ and the cepstral coefficients $c_1, c_2, \ldots, c_n$ form a bona fide smooth coordinate system on the open subset $\mathcal{P}_n^*$ of $\mathcal{P}_n$, i.e., the map from $\mathcal{P}_n^*$ to $\mathbb{R}^{2n}$ with components $(r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n)$ has an everywhere invertible Jacobian matrix.*

Accordingly, when viewed as functions on $\mathcal{P}_n^*$, $(r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n)$ form local coordinates for the space $\mathcal{P}_n^*$ of pole-zero filters of degree n. At this point, one might hope to be able to use a global inverse function theorem, such as Hadamard's Theorem, to show that these data define a global coordinate system. In part because of the complicated topology of $\mathcal{P}_n^*$, this is not possible, and instead we use a convex optimization scheme to conclude one of the important features of a global inverse function theorem. Indeed, the very nontrivial consequence of our next observation, to be proved in Section 4, is that there is a one-to-one correspondence between the $2n$ coefficients $r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n$ of the minimum-phase shaping filter (11) and the $2n$ coefficients $a_1, a_2, \ldots, a_n, \sigma_1, \sigma_2, \ldots, \sigma_n$ of the denominator and numerator polynomials of (11), provided the degree of w is *exactly* n.

Theorem 2.2 *Each shaping filter in $\mathcal{P}_n^*$ determines and is uniquely determined by its window $r_1, r_2, \ldots, r_n$ of normalized covariance lags and its window $c_1, c_2, \ldots, c_n$ of cepstral coefficients.*

As we have indicated, uniqueness follows from the remarkable fact that such a modeling filter arises as the minimum of a (strictly) convex optimization problem (see Section 4). This optimization problem has, of course, antecedents in the literature, beginning with maximum entropy methods. Recall that linear predictive coding (LPC) is the most common method for determining shaping filters in signal processing. Given the window of (unnormalized) covariance data (8) with a positive definite Toeplitz matrix T_n, find the (unnormalized) shaping filter $w(z)$, and the corresponding spectral density (2), which maximizes the entropy gain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \Phi(e^{i\theta})d\theta, \tag{14}$$

subject to the covariance-matching condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta = r_k, \quad k = 0, 1, \ldots, n. \tag{15}$$

For this reason, the LPC filter is often called the *maximum-entropy filter*.

Now, observe that the entropy gain (14) is precisely the zeroth cepstral coefficient c_0. However, in cepstral analysis, one is interested not only in c_0 but in a finite window (9) of cepstral coefficients. It is therefore natural to maximize instead some (positive) linear combination

$$p_0c_0 + p_1c_1 + \cdots + p_nc_n \tag{16}$$

of the cepstral coefficients in the window (9). In view of (6), this may be written as a generalized entropy gain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta})d\theta, \tag{17}$$

where P is the symmetric pseudopolynomial

$$P(z) = p_0 + \tfrac{1}{2}p_1(z + z^{-1}) + \cdots + \tfrac{1}{2}p_n(z^n + z^{-n}), \tag{18}$$

We shall say that $P \in \mathcal{D}$ if P is nonnegative on the unit circle and $P \in \mathcal{D}_+$ if it is positive there.

Now, it can be shown that the problem of maximizing (16) subject to (15) has a finite solution only if the pseudo-polynomial (18) belongs to $\mathcal{D}$. Moreover, if $P \in \mathcal{D}_+$, there is a unique solution Φ, and this solution has the form

$$\Phi(z) = \frac{P(z)}{Q(z)}, \tag{19}$$

where $Q \in \mathcal{D}_+$. To determine Q we need to solve the dual problem

$$\min_{Q \in \mathcal{D}} \mathbb{J}_P(Q), \tag{20}$$

where $\mathbb{J}_P$ is the strictly convex functional

$$\mathbb{J}_P(q) = r_0q_0 + \cdots + r_nq_n - \frac{1}{2\pi}\int_{-\pi}^{\pi} P \log Q \, d\theta. \tag{21}$$

This is precisely the optimization problem considered in [6], where it was shown that the dual problem has a unique solution that belongs to $\mathcal{D}_+$. In view of (2), we have therefore determined the unique shaping filter (11) that matches the covariance data (8). Hence, we have the following result that first appeared in [5].

Theorem 2.3 *Let $r_0, r_1, \ldots, r_n$ be a partial covariance sequence. Then, to any stable polynomial* (13) *of degree n, there corresponds a unique real stable polynomial*

$$a(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$$

of degree n such that $\Phi(e^{i\theta}) := \left|\frac{\sigma(e^{i\theta})}{a(e^{i\theta})}\right|^2$ satisfies (15).

Theorem 2.3 was conjectured by Georgiou [12] as a solution to the partial covariance extension problem posed by Kalman [13]. Georgiou had already established the existence part, but a complete proof of the conjecture was given much later in [5]. Similarly, in [5] we also showed

Theorem 2.4 *The normalized covariance lags $r_1, r_2, \ldots, r_n$ and the zero coefficients $\sigma_1, \sigma_2, \ldots, \sigma_n$ form a bona fide smooth coordinate system on the open manifold $\mathcal{P}_n$, i.e., the map from $\mathcal{P}_n$ to $\mathbb{R}^{2n}$ with components $(r_1, r_2, \ldots, r_n, \sigma_1, \sigma_2, \ldots, \sigma_n)$ has an everywhere invertible Jacobian matrix.*

While the *stochastic realization problem* amounts to determining shaping filters w having a fixed window of covariance lags $r_0, r_1, \cdots, r_n$, the object of the *deterministic realization problem* is to find shaping filters w

with a fixed window $w_0, w_1, \cdots, w_n$ of Markov parameters (10). An important question is whether the two problems can be solved simultaneously so that both interpolation conditions are satisfied at the same time. This problem has been studied in the literature as the Q-Markov Cover problem (see [15, 14, 1], where it has been used as a tool for performing model reduction). In Section 3, we derive the following results for coordinization by covariance data and Markov parameters.

Theorem 2.5 *The normalized covariance lags $r_1, r_2, \ldots, r_n$ and the normalized Markov parameters $w_1, w_2, \cdots, w_n$ form a bona fide smooth coordinate system on $\mathcal{Q}_n^*$, i.e., the map from $\mathcal{Q}_n^*$ to $\mathbb{R}^{2n}$ with components $(r_1, r_2, \ldots, r_n, w_1, w_2, \cdots, w_n)$ has an everywhere invertible Jacobian matrix. For each choice of a covariance window and a Markov window, there exists exactly one shaping filter matching these windows.*

3 Global analysis on $\mathcal{P}_n$ and $\mathcal{Q}_n$

For each $a \in \mathcal{S}_n$, define $\mathcal{P}_n(a)$ to be the space of all points in $\mathcal{P}_n$ with the polynomial a fixed. If we define $\mathcal{P}_n(\sigma)$ analogously, then $\mathcal{P}_n(a)$ and $\mathcal{P}_n(\sigma)$ are real, smooth, connected n-manifolds. In fact, both are clearly diffeomorphic to $\mathcal{S}_n$, and hence to $\mathbb{R}^n$. Now, the n-manifolds $\{\mathcal{P}_n(a) \mid a \in \mathcal{S}_n\}$ form the leaves of a foliation of $\mathcal{P}_n$, as do the n-manifolds $\{\mathcal{P}_n(\sigma) \mid \sigma \in \mathcal{S}_n\}$. Moreover, these two foliations are complementary, in the sense that if a leaf of one intersects a leaf of the other, the tangent spaces intersect in just $(0,0)$. This transversality property is equivalent to the fact that the functions (a, σ) form a local system of coordinates.

We now turn to the cepstral functions and the covariance functions. Let $g : \mathcal{P}_n \to \mathbb{R}^n$ be the map which sends (a, σ) to the vector $c \in \mathbb{R}^n$ with components

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \log |w(e^{i\theta})|^2 d\theta, \quad k = 1, 2, \cdots, n, \tag{22}$$

and let $\mathcal{C}_n := g(\mathcal{P}_n)$. Then, for each $c \in \mathcal{C}_n$, the subset

$$\mathcal{P}_n(c) = g^{-1}(c)$$

can be seen shown [8] to be a smooth, connected manifold of dimension n.

Next, let $h : \mathcal{P}_n \to \mathbb{R}^n$ be the map which sends (a, σ) to the vector $r \in \mathbb{R}^n$ of normalized covariance lags with components

$$r_k = \frac{\int_{-\pi}^{\pi} e^{ik\theta} \left|w(e^{i\theta})\right|^2 d\theta}{\int_{-\pi}^{\pi} \left|w(e^{i\theta})\right|^2 d\theta}, \quad k = 1, 2, \cdots, n \tag{23}$$

and let $\mathcal{R}_n := h(\mathcal{P}_n)$. Of course, any $r \in \mathcal{R}_n$ satisfies the positivity condition (4) with $r_0 = 1$. In [8] we prove that

$$\mathcal{P}_n(r) = h^{-1}(r)$$

is a smooth, connected manifold of dimension n for each $r \in \mathcal{R}_n$. Let $T_{(a,\sigma)}\mathcal{P}_n(r)$ and $T_{(a,\sigma)}\mathcal{P}_n(c)$ be the tangent spaces at (a, σ) of $\mathcal{P}_n(r)$ and $\mathcal{P}_n(c)$, respectively.

Theorem 3.1 *The two families of n-manifolds, $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$ and $\{\mathcal{P}_n(r) \mid r \in \mathcal{R}_n\}$, each form the leaves of a foliation of $\mathcal{P}_n$. For each $(a, \sigma) \in \mathcal{P}_n(r) \cap \mathcal{P}_n(c)$, the dimension of*

$$\Theta := T_{(a,\sigma)}\mathcal{P}_n(r) \cap T_{(a,\sigma)}\mathcal{P}_n(c)$$

equals the degree of the greatest common divisor of the polynomials $a(z)$ and $\sigma(z)$.

Consequently, the foliations $\{\mathcal{P}_n(r) \mid r \in \mathcal{R}_n\}$ and $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$ are complementary at any point $(a, \sigma) \in \mathcal{P}_n$ where a and σ are coprime. From this it follows that the kernels of $\text{Jac}(g)|_{(a,\sigma)}$ and $\text{Jac}(f)|_{(a,\sigma)}$ are complementary at any point (a, σ) in $\mathcal{P}_n^*$. In particular, the Jacobian of the joint map $(a, \sigma) \to (r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n)$ has full rank, and, by the Inverse Function Theorem, the joint map forms a smooth local coordinate system on $\mathcal{P}_n^*$. This proves Theorem 2.1.

As an illustration of this result, the shaded region of Figure 1 depicts the cepstral (dotted line) and the covariance (solid line) matching foliations of $\mathcal{P}_1$. The rest of Figure 1 will be explained next.

Next, we determine whether the windows $r_0, r_1, \cdots, r_n$ $w_0, w_1, \cdots, w_n$ of covariance lags and and Markov parameters, respectively, provide a bona fide set of smooth coordinates of $\mathcal{Q}_n$. Thus let $\psi : \mathcal{Q}_n \to \mathbb{R}^n$ be the map which sends (a, σ) to $w = (w_1, w_2, \ldots, w_n)$, and let $\mathcal{W}_n := \psi(\mathcal{Q}_n)$. Given any $w \in \mathcal{W}_n$, it is not hard to see that

$$\mathcal{Q}_n(w) := \psi^{-1}(w)$$

is a smooth, connected n-manifold. Moreover, the n-manifolds $\{\mathcal{P}_n(w) \mid w \in \psi(\mathcal{P}_n)\}$, where

$$\mathcal{P}_n(w) := \mathcal{Q}_n(w) \cap \mathcal{P}_n,$$

form the leaves of a foliation of $\mathcal{P}_n$, which is identical to $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$; see [8]. The dotted lines in Figure 1 also represent $\{\mathcal{P}_1(w)\}$, which hence coincides with $\{\mathcal{P}_1(c)\}$ in the shaded region.

The covariance matching foliation can also be extended to non-minimum phase shaping filters, as illustrated in Figure 1 (solid lines) for the case $n = 1$. In fact, let $\phi : \mathcal{Q}_n \to \mathbb{R}^n$ be the map that sends (a, σ) to the vector $r \in \mathbb{R}^n$ of normalized covariance lags (23). Clearly, $\phi(\mathcal{Q}_n) = \mathcal{R}_n := h(\mathcal{P}_n)$. Given any $r \in \mathcal{R}_n$, define

$$\mathcal{Q}_n(r) := \phi^{-1}(r).$$

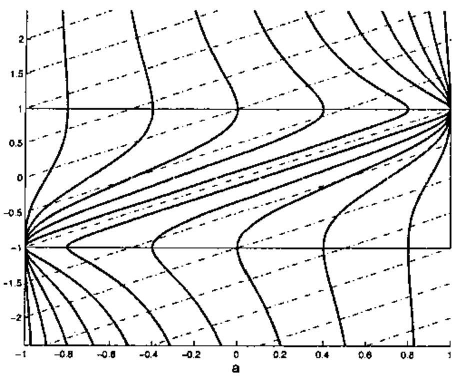

Figure 1.

Theorem 3.1 *The two families of n-manifolds, $\{\mathcal{Q}_n(w) \mid w \in \mathcal{W}_n\}$ and $\{\mathcal{Q}_n(r) \mid r \in \mathcal{R}_n\}$, each form the leaves of a foliation of $\mathcal{Q}_n$. Moreover,*

$$T_{(a,\sigma)}\mathcal{Q}_n(w) \cap T_{(a,\sigma)}\mathcal{Q}_n(r) = 0 \tag{24}$$

for any coprime $(a,\sigma) \in \mathcal{Q}_n(w) \cap \mathcal{Q}_n(r)$.

This establishes that the Jacobian of the joint map $(a,\sigma) \to (r_1, r_2, \ldots, r_n, w_1, w_2, \ldots, w_n)$ has full rank, and, by the Inverse Function Theorem, the joint map forms a smooth local coordinate system on $\mathcal{Q}_n^*$. This proves the first statement of Theorem 2.5.

Figure 1 illustrates the fact that the covariance foliation (solid line) and the Markov foliation (dotted line) are everywhere transverse. This figure also suggests that each leaf of the Markov foliation meets each leaf of the covariance matching foliation, a fact we shall now establish in a slightly generalized form. As above, $\overline{\mathcal{Q}_n(r)}$ and $\overline{\mathcal{Q}_n(w)}$ denote the closures of the submanifolds $\mathcal{Q}_n(r)$ and $\mathcal{Q}_n(w)$, respectively.

Theorem 3.2 *The closure of every leaf of the Markov foliation intersects the closure of any leaf of the covariance matching foliation. Moreover, either the leaves themselves intersect, or every point of intersection is of the form (a,σ) where a has some roots on the unit circle and σ vanishes at each of these roots while the ratio has the prescribed covariance and Markov windows.*

Since, according to Theorem 3.2, any intersection between $\overline{\mathcal{Q}_n(r)}$ and $\overline{\mathcal{Q}_n(w)}$ on the boundary of $\mathcal{Q}_n$ defines a pair (a,σ) of polynomials whose roots on the unit circle are common, after cancellation, $w(z) = \sigma(z)/a(z)$ has all its poles in open unit disc and is thus a bona fide shaping filter. Consequently, Theorem 3.2 establishes the existence part of the last statement of Theorem 2.5. The uniqueness part needs a separate proof that can be found in [8].

4 Identifiability of shaping filters from cepstral and covariance windows

The analysis preceding Theorem 2.3 motivates the construction of a functional that will be the key in establishing uniqueness of minimum-phase shaping filters having prescribed windows $r_0, r_1, \ldots, r_n$ and $c_1, c_2, \ldots, c_n$ of covariance lags and cepstral coefficients, respectively. More precisely, following Enqvist [11], consider the (primal) problem of finding a spectral density

$$\Phi(e^{i\theta}) = f_0 + 2\sum_{k=1}^{\infty} f_k \cos k\theta \tag{25}$$

which maximizes the entropy gain

$$\mathbb{I}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \Phi(e^{i\theta}) d\theta \tag{26}$$

over the class $\mathcal{F}_+$ of f such that (25) is positive for all θ, subject to the covariance-lag matching condition (15) and the cepstral matching condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\theta = c_k, \quad k = 1, \ldots, n. \tag{27}$$

This is a simpler primal optimization problem than that considered in [8], but it has the same dual.

Taking $q_0, q_1, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$ to be the Lagrange multipliers for the constraints (15) and (27), respectively, we obtain the Lagrangian

$$\begin{aligned} L(f,p,q) &= \mathbb{I}(f) + \sum_{k=0}^{n} q_k \left[r_k - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta}) d\theta \right] \\ &+ \sum_{k=1}^{n} p_k \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\theta - c_k \right], \end{aligned}$$

which, setting $p_0 = 1$, can be written in the more compact form

$$\begin{aligned} L(f,p,q) &= r_0 q_0 + \cdots + r_n q_n - c_1 p_1 - \cdots - c_n p_n \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[P \log \Phi - Q\Phi \right] d\theta, \end{aligned}$$

which clearly can have a finite minimum only for those values of the Lagrange multipliers for which both P and Q belong to $\mathcal{D}$. For such Lagrange multipliers, if the function $f \mapsto L(f,p,q)$ has a minimum, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta} \left[P\Phi^{-1} - Q \right] d\theta = 0, \quad k = 0, 1, 2, \ldots,$$

or, equivalently, (19) in the minimizing point, which inserted into the Lagrangian yields the dual functional

$$\inf_{f \in \mathcal{F}_+} L(f,p,q) = \mathbb{J}(P,Q) + 1,$$

where the functional

$$\begin{aligned}\mathbb{J}(P,Q) &= r_0q_0 + \cdots + r_nq_n - c_1p_1 - \cdots - c_np_n \\ &+\frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\log\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta \qquad (28)\end{aligned}$$

is convex, but not necessarily strictly convex.

Now, a straightforward calculation shows that

$$\frac{\partial \mathbb{J}}{\partial q_k} = r_k - \int_{-\pi}^{\pi} e^{ik\theta}\frac{P}{Q}\frac{d\theta}{2\pi}, \quad k = 0,1,\dots,n, \quad (29)$$

$$\frac{\partial \mathbb{J}}{\partial p_k} = \int_{-\pi}^{\pi} e^{ik\theta}\log\frac{P}{Q}\frac{d\theta}{2\pi} - c_k, \; k = 1,\dots,n. \quad (30)$$

Consequently, if $\mathbb{J}$ has a stationary point at $(\hat{P},\hat{Q})$, then this point will define a Φ that satisfies both the covariance matching and cepstral matching conditions. Unfortunately, while there is always a point $(\hat{P},\hat{Q})$ such that (29) is zero, it may happen that (30) is never zero.

Theorem 4.1 *The dual problem to minimize $\mathbb{J}(P,Q)$ over all $(P,Q) \in \mathcal{D}\times\mathcal{D}$ such that $p_0 = 1$ has a solution $(\hat{P},\hat{Q})$, and, for any such solution, $\hat{Q} \in \mathcal{D}_+$, and*

$$\Phi(z) = \frac{\hat{P}(z)}{\hat{Q}(z)} \qquad (31)$$

satisfies the covariance matching condition (15). If, in addition, $\hat{P} \in \mathcal{D}_+$, then (31) is a solution of the primal problem, i.e., there is both covariance matching and cepstral matching. A minimizing point $(\hat{P},\hat{Q}) \in \mathcal{D}_+\times\mathcal{D}_+$ is unique if and only if $\hat{P}$ and $\hat{Q}$ are coprime.

Therefore, in particular, we have proved Theorem 2.2. In fact, given any $(a,\sigma) \in \mathcal{P}_n^*$, a window $(r_1,\cdots,r_n,c_1,\cdots,c_n)$ is uniquely determined from (23) and (22). Conversely, given $(r_1,\cdots,r_n,c_1,\cdots,c_n)$, the optimization problem of Theorem 4.1 yields a $(a,\sigma) \in \mathcal{P}_n$ which matches this window and is unique if and only if $(a,\sigma) \in \mathcal{P}_n^*$.

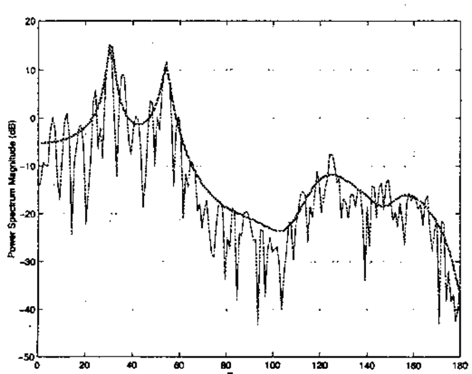

Figure 2.

Figure 2 shows the periodogram of a frame of speech for the phoneme [s] together with a 10th order spectral envelope produced by this method. In this case, $\hat{P} \in \mathcal{D}_+$, so there is both covariance and cepstral matching. In general, however, this is not the case, but then good approximate results can still be achieved by stopping the iterations before P reaches the boundary of $\mathcal{D}$ [10].

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