A Homotopy Continuation Solution of the Covariance Extension Equation

Christopher I. Byrnes[1], Giovanna Fanizza[2], and Anders Lindquist[2]

[1] Washington University, St. Louis, MO, USA, chrisbyrnes@seas.wustl.edu
[2] Royal institute of Technology, Stockholm, Sweden, alq@math.kth.se

Dedicated to Clyde Martin on the occation of his 60th birthday

Algebraic geometry plays an important role in the theory of linear systems for (at least) three reasons. First, the Laplace transform turns expressions about linear differential systems into expressions involving rational functions. In addition, many of the transformations studied in linear systems theory, like changes of coordinates or feedback, turn out to be the action of algebraic groups on algebraic varieties. Finally, when we study linear quadratic problems in optimization and estimation, all roads eventually lead either to the Riccati equation or to spectral factorization.

Clyde Martin was a pioneer in applying algebraic geometry to linear systems in all three of these theaters. Perhaps the work which is closest to the results we discuss in this paper was his joint study, with Bob Hermann, of the matrix Riccati equation as a flow on a Grassmannian.

In this paper[3] we study the steady state form of a discrete-time matrix Riccati-type equation, connected to the rational covariance extension problem and to the partial stochastic realization problem. This equation, however, is nonstandard in that it lacks the usual kind of definiteness properties which underlie the solvability of the standard Riccati equation. Nonetheless, we prove the existence and uniqueness of a positive semidefinite solution. We also show that this equation has the proper geometric attributes to be solvable by homotopy continuation methods, which we illustrate in several examples.

1 The covariance extension equation

We will consider real sequences

$$c = (c_0, c_1, \dots, c_n) \tag{1}$$

[3] This research was supported in part by grants from AFOSR, NSF, VR, the Göran Gustafsson Foundation, and Southwestern Bell.

that are positive in the sense that

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} > 0,$$

and we consider Schur polynomials

$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n; \tag{2}$$

i.e, polynomials with all their roots in the open unit disc. For simplicity, we normalize by taking $c_0 = 1$. Motivated by the partial stochastic realization problem and the rational covariance extension problem, which we will briefly review in Section 3, we form the following n vectors and $n \times n$ matrix:

$$\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}, \quad h = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad \Gamma = \begin{bmatrix} -\sigma_1 & 1 & 0 & \cdots & 0 \\ -\sigma_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{n-1} & 0 & 0 & \cdots & 1 \\ -\sigma_n & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3}$$

Defining $u_1, u_2, \dots, u_n$ via

$$\frac{z^n}{z^n + c_1 z^{n-1} + \cdots + c_n} = 1 - u_1 z^{-1} - u_2 z^{-2} - u_3 z^{-3} - \dots \tag{4}$$

we also form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad U = \begin{bmatrix} 0 & & & & \\ u_1 & 0 & & & \\ u_2 & u_1 & & & \\ \vdots & \vdots & \ddots & & \\ u_{n-1} & u_{n-2} & \cdots & u_1 & 0 \end{bmatrix}. \tag{5}$$

We shall also need the function $g : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ defined by

$$g(P) = u + U\sigma + U\Gamma P h. \tag{6}$$

From these quantities, in [4], we formed the Riccati-like matrix equation

$$P = \Gamma(P - Phh'P)\Gamma' + g(P)g(P)', \tag{7}$$

which we sought to solve in the space of positive semidefinite matrices satisfying the additional constraint

$$h'Ph < 1, \tag{8}$$

where $'$ denotes transposition. We refer to this equation as the covariance extension equation (CEE).

To this end, define the semialgebraic sets

$$X = \{(c, \sigma) \mid T_n > 0,\ \sigma(z) \text{ is a Schur polynomial}\}$$

and

$$Y = \{P \in \mathbb{R}^{n \times n} \mid P \geq 0,\ h'Ph < 1\}.$$

On $X \times Y$ we define the rational map

$$F(c, \sigma, P) = P - \Gamma(P - Phh'P)\Gamma' - g(P)g(P)'$$

Of course its zero locus

$$Z = F^{-1}(0) \subset X \times Y$$

is the solution set to the covariance extension equations. We are interested in the projection map restricted to Z

$$\pi_X(c, \sigma, P) = (c, \sigma).$$

For example, to say that π_X is surjective is to say that there is always a solution to CEE, and to say that π_X is injective is to say that solutions are unique. One of the main results of this paper is the following, which, in particular, implies that CEE has a unique solution $P \in Y$ for each $(c, \sigma) \in X$ [4, Theorem 2.1].

Theorem 1. *The solution set Z is a smooth semialgebraic manifold of dimention $2n$. Moreover, π_X is a diffeomorphism between Z and X.*

In particular the map π_X is smooth with no branch points and every smooth curve in X lifts to a curve in Z. These observations imply that the homotopy continuation method will apply to solving the covariance extension equation [1].

2 Proof of Theorem 1

The rational covariance extension problem is to find polynomials

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_n \tag{9a}$$

$$b(z) = z^n + b_1 z^{n-1} + \cdots + b_n \tag{9b}$$

satisfying the interpolation condition

$$\frac{b(z)}{a(z)} = \frac{1}{2} + c_1 z^{-1} + \cdots + c_n z^{-n} + O(z^{-n-1}) \tag{10}$$

and the positivity condition

$$\frac{1}{2}\left[a(z)b(z^{-1})+b(z)a(z^{-1})\right]=\rho^2\sigma(z)\sigma(z^{-1}) \tag{11}$$

for some positive real number ρ. Given (9), let a and b be the n-vectors $a:=(a_1,a_2,\dots,a_n)$ and $b:=(b_1,b_2,\dots,b_n)$, respectively.

In [4] we proved:

Theorem 2. *There is a one-to-one correspondence between symmetric solutions P of the covariance extension equation* (7) *such that $h'Ph<1$ and pairs of monic polynomials* (9a)-(9b) *satisfying the interpolation condition* (10) *and the positivity condition* (11). *Under this correspondence*

$$a=(I-U)(\Gamma Ph+\sigma)-u, \tag{12a}$$

$$b=(I+U)(\Gamma Ph+\sigma)+u, \tag{12b}$$

$$\rho=(1-h'Ph)^{1/2}, \tag{12c}$$

and P is the unique solution of the Lyapunov equation

$$P=JPJ'-\frac{1}{2}(ab'+ba')+\rho^2\sigma\sigma', \tag{13}$$

where

$$J=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix} \tag{14}$$

is the upward shift matrix. Moreover the following conditions are equivalent

1. *$P\ge 0$*
2. *$a(z)$ is a Schur polynomial*
3. *$b(z)$ is a Schur polynomial*

and, if they are fulfilled,

$$\deg v(z)=\text{rank } P. \tag{15}$$

We can now prove Theorem 1. Let $\mathcal{P}_n$ be the space of pairs (a,b) whose quotient is positive real. Clearly, the mapping

$$f:\,\mathcal{P}_n\to X,$$

sending (a,b) to the corresponding (c,σ), is smooth. Our main result in [5] asserts that f is actually a diffeomorphism. In particular, for each positive sequence (1) and each monic Schur polynomial (2), there is a unique pair of polynomials, (a,b), satisfying (10) and (11), and consequently (a,b) solves the rational covariance extension problem corresponding to (c, σ). Moreover, by Theorem 2, there is a unique corresponding solution to the covariance extension equation, which is positive semi-definite.

Since J is nilpotent, the Lyapunov equation (13) has a unique solution, P, for each right hand side of equation (13). Moreover, the right hand side is a smooth function on X and, using elementary methods from Lyapunov theory, we conclude that P is also smooth as a function on X. As the graph in $X \times Y$ of a smooth mapping defined on X, Z is a smooth manifold of dimension $2n = \dim X$. Moreover, this mapping has the smooth mapping π_X as its inverse. Therefore, π_X is a diffeomorphism.

Remark 1. Our proof, together with the results in [5], shows more. Namely, that Z is an analytic manifold and that π_X is an analytic diffeomorphism with an analytic inverse.

3 Rational covariance extension and the CEE

As described above, given a positive sequence (1), the rational covariance extension problem – or the covariance extension problem with degree constraint – amounts to finding a pair (a, b) of Schur polynomials (9a)-(9b) satisfying the interpolation condition

$$\frac{b(z)}{a(z)} = \frac{1}{2} + c_1 z^{-1} + \cdots + c_n z^{-n} + O(z^{-n-1}) \tag{16}$$

and the positivity condition

$$\frac{1}{2}\left[a(z)b(z^{-1}) + b(z)a(z^{-1})\right] > 0 \quad \text{on } \mathbb{T}. \tag{17}$$

Then there is a Schur polynomial (2) such that

$$\frac{1}{2}\left[a(z)b(z^{-1}) + b(z)a(z^{-1})\right] = \rho^2 \sigma(z)\sigma(z^{-1}) \tag{18}$$

for some positive normalizing coefficient ρ, and

$$\mathrm{Re}\left\{\frac{b(e^{i\theta})}{a(e^{i\theta})}\right\} = \left|\rho \frac{\sigma(e^{i\theta})}{a(e^{i\theta})}\right|^2. \tag{19}$$

Georgiou [9, 10] raised the question whether there exists a solution for each choice of σ and answered this question in the affirmative. He also conjectured that this assignment is unique. This conjecture was proven in [5] in a more general context of well-posedness.

In [4] we showed that, for any $(c, \sigma) \in X$, CEE has a unique solution $P \in Y$ and that the unique solution corresponding to σ to the rational covariance extension problem is given by

$$a = (I - U)(\Gamma P h + \sigma) - u, \tag{20a}$$

$$b = (I + U)(\Gamma P h + \sigma) + u. \tag{20b}$$

Clearly the interpolation condition (16) can be written

$$b = 2c + (2C_n - I)a, \tag{21}$$

where

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad C_n = \begin{bmatrix} 1 & & & & \\ c_1 & 1 & & & \\ c_2 & c_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ c_{n-1} & c_{n-2} & c_{n-3} & \dots & 1 \end{bmatrix}.$$

Using the fact that $C_n u = c$ and $C_n(I - U) = I$, it was shown in [4] that (21) can be written

$$a = \frac{1}{2}(I - U)(a + b) - u. \tag{22}$$

For a fixed $(c, \sigma) \in X$, let $H : Y \to \mathbb{R}^{n \times n}$ be the map sending P to $F(c, \sigma, P)$, and let

$$dH(P; Q) := \lim_{t \to 0} \frac{H(P + tQ) - H(P)}{t}$$

be the derivative in the direction $Q = Q'$. A key property needed in the homotopy continuation solution of the CEE is the fact that this derivative is full rank.

Proposition 1. *Given $(c, \sigma) \in X$, let $P \in Y$ be the corresponding solution of CEE. Then, if $dH(P; Q) = 0$, $Q = 0$.*

Proof. Suppose that $dH(P; Q) = 0$ for some Q. Then

$$H(P) + \lambda dH(P; Q) = 0$$

for any $\lambda \in \mathbb{R}$. Since

$$dH(P; Q) = Q - \Gamma Q \Gamma' + \Gamma P h h' Q \Gamma' + \Gamma Q h h' P \Gamma' - g(P) h' Q \Gamma' U' - U \Gamma Q h g(P)',$$

this can be written

$$H(P_\lambda) = \lambda^2 R(Q), \tag{23}$$

where $P_\lambda := P + \lambda Q$ and

$$R(Q) := 2\Gamma Q h h' Q \Gamma' - 2U \Gamma Q h h' Q \Gamma' U'.$$

Proceeding as in the proof of Lemma 4.6 in [4], (23) can be written

$$P_\lambda = J P_\lambda J' - \frac{1}{2}(a_\lambda b_\lambda' + b_\lambda a_\lambda') + \rho_\lambda^2 \sigma \sigma' - \lambda^2 R(Q), \tag{24}$$

where

$$a_\lambda = (I-U)(\mathit{\Gamma} P_\lambda h + \sigma) - u, \tag{25a}$$

$$b_\lambda = (I+U)(\mathit{\Gamma} P_\lambda h + \sigma) + u, \tag{25b}$$

$$\rho_\lambda = (1 - h' P_\lambda h)^{1/2}. \tag{25c}$$

Observe that

$$a_\lambda = \frac{1}{2}(I-U)(a_\lambda + b_\lambda) - u, \tag{26}$$

and hence (a_λ, b_λ) satisfies the interpolation condition (22), or, equivalently, (16), for all $\lambda \in \mathbb{R}$.

Multipying (24) by $z^{j-i} = z^{n-i} z^{-(n-j)}$ and summing over all $i,j = 1,2,\dots,n$, we obtain

$$\frac{1}{2}\left[a_\lambda(z)b_\lambda(z^{-1}) + b_\lambda(z)a_\lambda(z^{-1})\right] = \rho_\lambda^2 \sigma(z)\sigma(z^{-1}) - \lambda^2 \sum_{i=1}\sum_{j=1} R_{ij}(Q) z^{j-i} \tag{27}$$

again along the calculations of the proof of Lemma 4.6 in [4]. Since $\sigma(z)\sigma(z^{-1}) > 0$ on $\mathbb{T}$,

$$\rho_\lambda^2 \sigma(z)\sigma(z^{-1}) - \lambda^2 \sum_{i=1}\sum_{j=1} R_{ij}(Q) z^{j-i} > 0 \quad \text{on } \mathbb{T}$$

for $|\lambda|$ sufficiently small. Then there is a Schur polynomial σ_λ and a positive constant $\hat\rho_\lambda$ such that

$$\hat\rho_\lambda^2 \sigma_\lambda(z)\sigma_\lambda(z^{-1}) = \rho_\lambda^2 \sigma(z)\sigma(z^{-1}) - \lambda^2 \sum_{i=1}\sum_{j=1} R_{ij}(Q) z^{j-i}.$$

Therefore,

$$\frac{1}{2}\left[a_\lambda(z)b_\lambda(z^{-1}) + b_\lambda(z)a_\lambda(z^{-1})\right] = \hat\rho_\lambda^2 \sigma_\lambda(z)\sigma_\lambda(z^{-1}) \tag{28}$$

for $|\lambda|$ sufficiently small.

Now recall that $a_0 = a$ and $b_0 = b$ are Schur polynomials and that the Schur region is open in $\mathbb{R}^n$. Hence there is an $\varepsilon > 0$ such that $a_\varepsilon(z), a_{-\varepsilon}(z), b_\varepsilon(z)$ and $b_{-\varepsilon}(z)$ are also Schur polynomials and (28) holds for $\lambda = \pm\varepsilon$.

Consequently, $(a_\varepsilon, b_\varepsilon)$ and $(a_{-\varepsilon}, b_{-\varepsilon})$ both satisfy the interpolation condition (16) and the positivity condition (18) corresponding to the same $\sigma := \sigma_\varepsilon = \sigma_{-\varepsilon}$. Therefore, since the solution to the rational covariance extension problem corresponding to σ is unique, we must have $a_\varepsilon = a_{-\varepsilon}$ and $b_\varepsilon = b_{-\varepsilon}$, and hence in view of (24), $P_\varepsilon = P_{-\varepsilon}$; i.e, $Q = 0$, as claimed.

4 Reformulation of the Covariance Extension Equation

Solving the covariance extension equation (7) amounts to solving $\frac{1}{2}n(n-1)$ nonlinear scalar equations, which number grows rapidly with increasing n. As in the theory of fast filtering algorithms [11, 12], we may replace these equations by a system of only n equations. In fact, setting

$$p = Ph \tag{29}$$

the covariance extension equation can be written

$$P - \Gamma P \Gamma' = -\Gamma p p' \Gamma' + (u + U\sigma + U\Gamma p)(u + U\sigma + U\Gamma p)' \tag{30}$$

If we could first determine p, P could be obtained from (30), regarded as a Lyapunov equation. We proceed to doing precisely this.

It follows from Theorem 2 that (30) may also be written

$$P = JPJ' - \frac{1}{2}(ab' + ba') + \rho^2 \sigma\sigma', \tag{31}$$

with a, b and ρ given by (12). Multiplying (31) by $z^{j-i} = z^{n-i}z^{-(n-j)}$ and summing over all $i, j = 1, 2, \dots, n$, we obtain precisely (11), which in matrix form becomes

$$S(a) \begin{bmatrix} 1 \\ b \end{bmatrix} = 2\rho^2 \begin{bmatrix} d \\ \sigma_n \end{bmatrix} \tag{32}$$

or, symmetrically,

$$S(b) \begin{bmatrix} 1 \\ a \end{bmatrix} = 2\rho^2 \begin{bmatrix} d \\ \sigma_n \end{bmatrix}, \tag{33}$$

where

$$S(a) = \begin{bmatrix} 1 & \dots & a_{n-2} & a_{n-1} & a_n \\ a_1 & \dots & a_{n-1} & a_n & \\ a_2 & \dots & a_n & & \\ \vdots & \iddots & & & \\ a_n & & & & \end{bmatrix} + \begin{bmatrix} 1 & a_1 & a_2 & \dots & a_n \\ & 1 & a_1 & \dots & a_{n-1} \\ & & 1 & \dots & a_{n-2} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \tag{34}$$

and

$$d = \begin{bmatrix} 1 + \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2 \\ \sigma_1 + \sigma_1\sigma_2 + \sigma_{n-1}\sigma_n \\ \sigma_2 + \sigma_1\sigma_3 + \sigma_{n-2}\sigma_n \\ \vdots \\ \sigma_{n-1} + \sigma_1\sigma_n \end{bmatrix}. \tag{35}$$

Inserting (12) and (29) in (32) yields

$$S(a(p)) \begin{bmatrix} 1 \\ b(p) \end{bmatrix} = 2(1 - h'p) \begin{bmatrix} d \\ \sigma_n \end{bmatrix}, \tag{36}$$

where

$$a(p) = (I - U)(\Gamma p + \sigma) - u, \tag{37a}$$

$$b(p) = (I + U)(\Gamma p + \sigma) + u \tag{37b}$$

are functions of p. More precisely, (36) are $n+1$ equations in the n unknown p. However, from (12) we have

$$\frac{1}{2}(a_n + b_n) = \rho^2 \sigma_n,$$

which is precisely the last equation in (32). Hence (36) is redundant and can be deleted to yield

$$ES(a(p)) \begin{bmatrix} 1 \\ b(p) \end{bmatrix} = 2(1 - h'p)d, \tag{38}$$

where E is the $n \times (n+1)$ matrix

$$E = \begin{bmatrix} I_n \; 0 \end{bmatrix}. \tag{39}$$

These n equations in n unkowns $p_1, p_2, \ldots, p_n$ clearly has a unique solution $\hat{p}$, for CEE has one.

5 Homotopy continuation

Suppose that $(c, \sigma) \in X$. To solve the corresponding covariance extension equation

$$P = \Gamma(P - Phh'P)\Gamma' + g(P)g(P)' \tag{40}$$

for its unique solution $\hat{P}$, we first observe that the solution is particularly simple if $c = c_0 = 0$. Then $u = 0$, $U = 0$ and (40) reduces to

$$P = \Gamma(P - Phh'P)\Gamma' \tag{41}$$

having the unique solution $P = 0$ in Y. Consider the deformation

$$c(\nu) = \nu c, \quad \nu \in [0, 1].$$

Clearly, $(c(\nu), \sigma) \in X$, and consequently the equation

$$H(P, \nu) := P - \Gamma(P - Phh'P)\Gamma' - g(P, \nu)g(P, \nu)' = 0, \tag{42}$$

where

$$g(P, \nu) = u(\nu) + U(\nu)\sigma + U(\nu)\Gamma P h$$

with

$$u(\nu)=\begin{bmatrix} 1 & & & & \\ \nu c_1 & 1 & & & \\ \nu c_2 & \nu c_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \nu c_{n-1} & \nu c_{n-2} & \nu c_{n-3} & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} \nu c_1 \\ \nu c_2 \\ \vdots \\ \nu c_n \end{bmatrix}$$

and

$$U(\nu)=\begin{bmatrix} 0 & & & & \\ u_1(\nu) & 0 & & & \\ u_2(\nu) & u_1(\nu) & & & \\ \vdots & \vdots & \ddots & & \\ u_{n-1}(\nu) & u_{n-2}(\nu) & \cdots & u_1(\nu) & 0 \end{bmatrix},$$

has a unique solution $\hat{P}(\nu)$ in Y.

The function $H : Y \times [0,1] \to \mathbb{R}^{n\times n}$ is a homotopy between (40) and (41). In view of Theorem 1, the trajectory $\{\hat{P}(\nu)\}_{\nu=0}^{1}$ is continuously differentiable and has no turning points or bifurcations. Consequently, homotopy continuation can be used to obtain a computational procedure. However, the corresponding ODE will be of dimension $O(n^2)$. Therefore, it is better to work with the reduced equation (38), which yields an ODE of order n.

To this end, setting

$$V := \{p \in \mathbb{R}^n \mid p = Ph, P \in Y\},$$

consider instead the homotopy $G : V \times [0,1] \to \mathbb{R}^n$ defined by

$$G(p,\nu) := ES(a(p)) \begin{bmatrix} 1 \\ b(p) \end{bmatrix} - 2(1-h'p)d,$$

where $a(p)$ and $b(p)$ are given by (37). *A fortiori* the corresponding trajectory $\{\hat{p}(\nu)\}_{\nu=0}^{1}$ is continuously differentiable and has no turning points or bifurcations. Differentiating

$$G(p,\nu)=0$$

with respect to ν yields

$$ES(a)\begin{bmatrix} 0 \\ \dot{b} \end{bmatrix} + ES(b)\begin{bmatrix} 0 \\ \dot{a} \end{bmatrix} + 2h'\dot{p}d = 0,$$

where dot denotes derivative and

$$\dot{a} = (I-U)\mathit{\Gamma}\dot{p} - \dot{U}(\mathit{\Gamma}p+\sigma) - \dot{u}, \tag{43a}$$

$$\dot{b} = (I+U)\mathit{\Gamma}\dot{p} + \dot{U}(\mathit{\Gamma}p+\sigma) + \dot{u}, \tag{43b}$$

or, which is the same,

$$ES\left(\frac{a+b}{2}\right)\begin{bmatrix}0\\ \Gamma\dot{p}\end{bmatrix} - ES\left(\frac{b-a}{2}\right)\begin{bmatrix}0\\ U\Gamma\dot{p}\end{bmatrix} + dh'\dot{p} =$$
$$= ES\left(\frac{b-a}{2}\right)\begin{bmatrix}0\\ \dot{U}\Gamma p + \dot{U}\sigma + \dot{u}\end{bmatrix} .$$

In view of (37), this may be written

$$\left[\hat{S}(\Gamma p+\sigma) - \hat{S}(U\Gamma p + U\sigma + u) + dh'\right]\dot{p} = \hat{S}(U\Gamma p + U\sigma + u)(\dot{U}\Gamma p + \dot{U}\sigma + \dot{u}),$$

where $\hat{S}(a)$ is the $n\times n$ matrix obtained by deleting the first column and the last row in (34). Hence we have proven the following theorem.

Theorem 3. *The differential equation*

$$\dot{p} = \left[\hat{S}(\Gamma p+\sigma) - \hat{S}(U(\nu)\Gamma p + U(\nu)\sigma + u(\nu)) + dh'\right]^{-1} \times$$
$$\hat{S}(U(\nu)\Gamma p + U(\nu)\sigma + u(\nu))(\dot{U}(\nu)\Gamma p + \dot{U}(\nu)\sigma + \dot{u}(\nu)),$$
$$p(0) = 0$$

has a unique solution $\{\hat{p}(\nu);\ 0 \le \nu \le 1\}$. *Moreover, the unique solution of the Lyapunov equation*

$$P - \Gamma P \Gamma' = -\Gamma\hat{p}(1)\hat{p}(1)'\Gamma' + (u + U\sigma + U\Gamma\hat{p}(1))(u + U\sigma + U\Gamma\hat{p}(1))',$$

where $U = U(1)$ *and* $u = u(1)$, *is also the unique solution of the covariance extension equation* (7).

The differential equation can be solved by methods akin to those in [3].

6 Simulations

We illustrate the method described above by two examples, in which we use covariance data generated in the following way. Pass white noise through a given stable filter

$$\text{white noise} \xrightarrow{w} \boxed{w(z)} \xrightarrow{y}$$

with a rational transfer function

$$w(z) = \frac{\hat{\sigma}(z)}{\hat{a}(z)}$$

of degree $\hat{n}$, where $\hat{\sigma}(z)$ is a (monic) Schur polynomial. This generates a time series

$$y_0, y_1, y_2, y_3, \ldots, y_N, \tag{44}$$

from which a covariance sequence is computed via the biased estimator

$$\hat{c}_k = \frac{1}{N} \sum_{t=k+1}^{N} y_t y_{t-k}, \tag{45}$$

which actually provides a sequences with positive Toepliz matrices. By setting $c_k := \hat{c}_k / \hat{c}_0$ we obtained a normalized covariance sequence

$$1, c_1, c_2, \ldots, c_n, \quad n \geq \hat{n}. \tag{46}$$

Example 1: Detecting the positive degree

Given a transfer function $w(z)$ of degree $\hat{n} = 2$ with zeros at $0.37e^{\pm i}$ and poles at $0.82e^{\pm 1.32i}$, estimate the covariance sequence (46) for $n = 2, 3, 4, 5$ and 6. Given these covariance sequences, we apply the algorithm of this paper to compute the $n \times n$ matrix P, using the zero polynomial $\sigma(z) = z^{n-\hat{n}}\hat{\sigma}(z)$, thus keeping the trigonometric polynomial $|\sigma(e^{i\theta})|^2$ constant. For each value of n, 100 Monte Carlo simulations are performed, and the average of the singular values of P are computed and shown in Table 1.

n=2	n=3	n=4	n=5	n=6
.42867	.42892	.42922	.42967	.43004
.25322	.25368	.25388	.25407	.25433
	$3.0409 \cdot 10^{-6}$	$2.5042 \cdot 10^{-5}$	$2.1045 \cdot 10^{-4}$	$4.3479 \cdot 10^{-4}$
		$2.6563 \cdot 10^{-7}$	$1.6027 \cdot 10^{-6}$	$1.0086 \cdot 10^{-4}$
			$3.6024 \cdot 10^{-7}$	$9.1628 \cdot 10^{-7}$
				$1.8882 \cdot 10^{-7}$

Table 1. Singular values of solution P of the CEE

For each $n > 2$, the first two singular values are considerably larger than the others. Indeed, for all practical purposes, the singular values below the line in Table 1 are zero. Therefore, as the dimension of P increases, its rank remains close to 2. This is to say that the positive degree [4] of the covariance sequence (46) is approximately 2 for all n. In Fig. 2 the spectral density for $n = 2$ is plotted together with those obtained by taking $n > 2$, showing no major difference.

Next, for $n = 4$, we compute the solution of the CEE with

$$\sigma(z) = \hat{\sigma}(z)(z - 0.6e^{1.78i})(z - 0.6e^{-1.78i}).$$

As expected, the rank of the 4×4 matrix solution P of the CEE, is approximately 2, and, as seen in Fig. 3 , $a(z)$ has roots that are very close to cancelling the zeros $0.6e^{\pm 1.78i}$ of $\sigma(z)$.

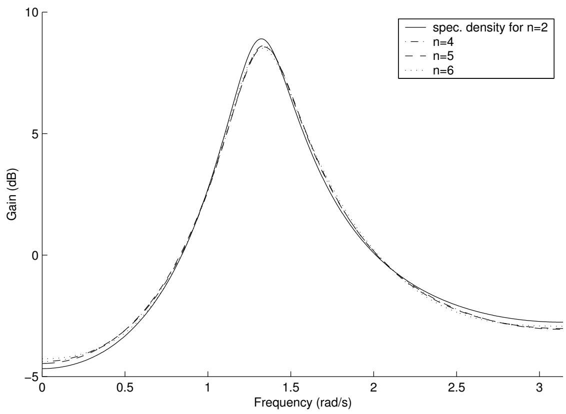

Fig. 1.
The given spectral density ($n = 2$) and the estimated one for $n = 4, 5, 6$.

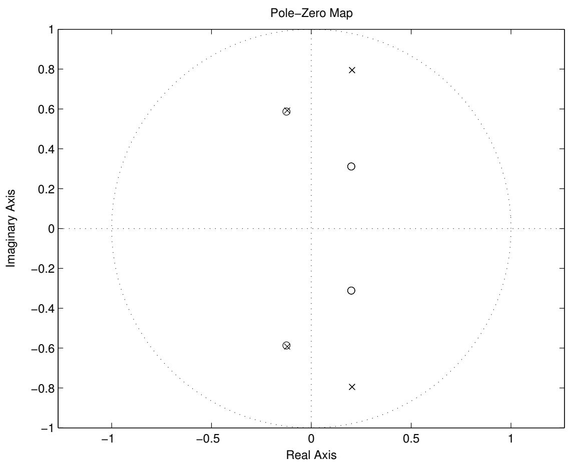

Fig. 2.
The spectral zeros (o) and the corresponding poles (x) for $n = 4$.

Example 2: Model reduction

Next, given a transfer function $w(z)$ of degree 10 with zeros

$$0.99e^{\pm 1.78i},\ 0.6e^{\pm 0.44i},\ 0.55e^{\pm 2i},\ 0.98e^{\pm i},\ 0.97e^{\pm 2.7i}$$

and poles

$$0.8e^{\pm 2.6i},\ 0.74e^{\pm 0.23i},\ 0.8e^{\pm 2.09i},\ 0.82e^{\pm 1.32i},\ 0.77e^{\pm 0.83i}$$

as in Fig. 3, we generate data (44) and a corresponding covariance sequence (46). Clearly, there is no zero-pole cancellation.

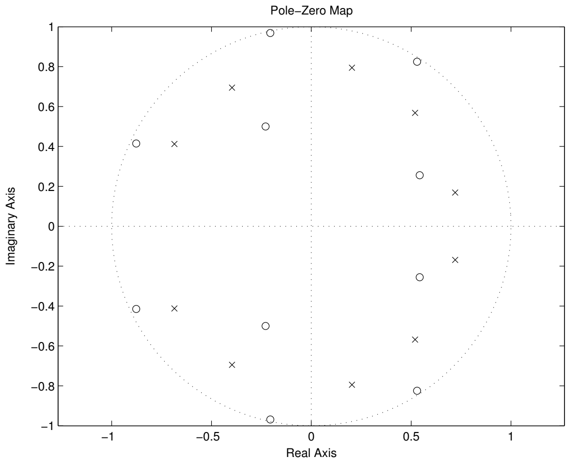

Fig. 3.
Zeros (o) and the corresponding poles (x) of $w(z)$.

Nevertheless, the rank of the 10×10 matrix solution P of CEE is close to 6. In fact, its singular values are equal to

$$\begin{array}{l} 1.1911\ 0.1079\ 0.0693\ 0.0627\ 0.0578\ 0.0434 \\ 0.0018\ 0.0012\ 0.0009\ 0.0008 \end{array}$$

The last four singular values are quite small, establishing an approximate rank of 6. The estimated spectral density ($n = 10$) is depicted in Fig. 4 together with the theoretical spectral density.

Clearly six zeros are dominant, namely

$$0.98e^{\pm i},\ 0.99e^{\pm 1.78i},\ 0.97e^{\pm 2.7i},$$

and these can be determined from the estimated spectral density in Fig. 4. Therefore applying our algorithm to the reduced covariance sequence $1, c_1, \ldots, c_6$ using the six dominant zeros to form $\sigma(z)$, we obtain a 6×6 matrix solution P of CEE and a corresponding reduced order system with poles and zeros as in Fig. 5. Comparing with Fig. 3, we see that the poles are located in quite different locations. Nevertheless, the corresponding reduced-order spectral estimate, depicted in Fig. 6, is quite accurate.

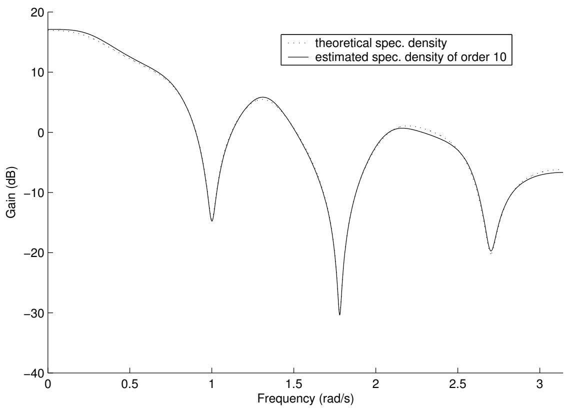

Fig. 4.
$n = 10$ estimate of spectral density together with the true spectral density.

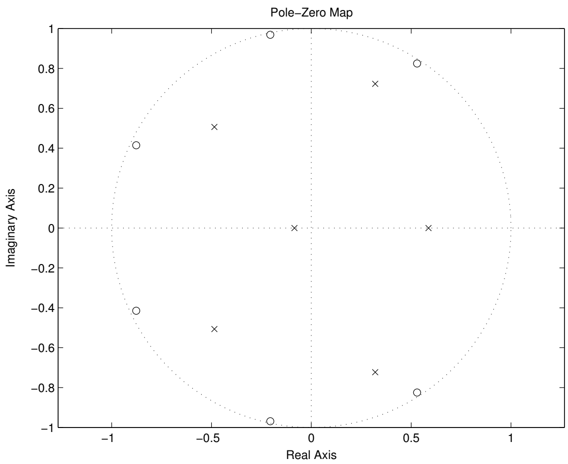

Fig. 5.
Zeros (o) and poles (x) of the reduced-order system.

References

1. J.C. Alexander, *The Topological Theory of an Embedding Method*, H. Wacker ed., Academic Press, N.Y., 1978.
2. Eugene L. Allgower and Kurt Georg, *Numerical Continuation Method, An Introduction*, Springer-Verlag, 1990.
3. A. Blomqvist, G.Fanizza and R. Nagamune , *Computation of bounded degree Nevanlinna-Pick interpolants by solving nonlinear equation.* The proceedings of the 42nd IEEE Conference of Decision and Control. (2003) 4511-4516.

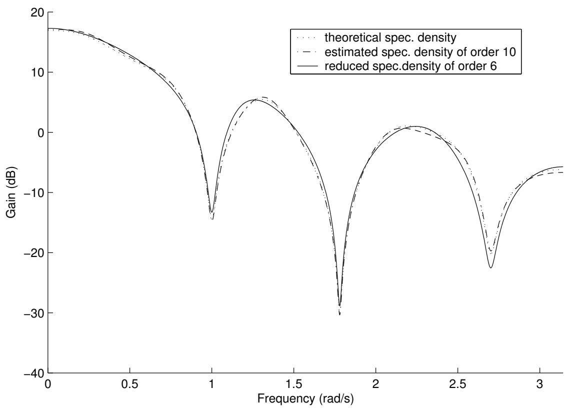

Fig. 6.
Reduced-order estimate of spectral density ($n = 6$) together with that of $n = 10$ and the true spectral density.

4. C. I. Byrnes and A. Lindquist, *On the partial stochastic realization problem*, IEEE Trans. Automatic Control **AC-42** (August, 1997), 1049–1069.
5. C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Matveev, A complete parameterization of all positive rational extensions of a covariance sequence, *IEEE Trans. Automat. Control*, **40** (1995), 1841–1857.
6. C. I. Byrnes, S.V. Gusev, and A. Lindquist, *From finite covariance windows to modeling filters: A convex optimization approach,* SIAM Review **43** (Dec. 2001), 645–675.
7. C. I. Byrnes, P.Enqvist and A. Lindquist, *Cepstral coefficients, covariance lags and pole-zero models for finite data strings*, IEEE Trans. Signal Processing **SP-50** (April, 2001), 677–693.
8. P.Enqvist, *A homotopy approach to rational covariance extension with degree constraint*, Int.J.Applied Mathematics and Computer Science **SP-11** (2001), 1173–1201.
9. T. T. Georgiou, *Partial realization of covariance sequences*, CMST, Univ. Florida, Gainesville, 1983.
10. T. T. Georgiou, *Realization of power spectra from partial covariance sequences*, IEEE Transactions Acoustics, Speech and Signal Processing **ASSP-35** (1987), 438–449.
11. A. Lindquist, *Some reduced-order non-Riccati equations for linear least-squares estimation: the stationary, single-output case*, Int. J. Control **24** (1976), 821–842.
12. A. Lindquist, *On Fredholm integral equations, Toeplitz equations and Kalman-Bucy filtering*, Applied mathematics and optimization **1** (1975), 355–373.