1 ADVANCES IN HIGH-RESOLUTION SPECTRAL ESTIMATION

Christopher I. Byrnes,* Tryphon T. Georgiou**
and Anders Lindquist***

*Department of Systems Science and Mathematics
Washington University
St. Louis, Missouri 63130, USA
email: chrisbyrnes@seas.wustl.edu

**Department of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota 55455, USA
email: tryphon@ece.umn.edu

***Department of Mathematics
Royal Institute of Technology
100 44 Stockholm, Sweden
email: alq@math.kth.se

Abstract: We review a new approach to spectral estimation, based on the use of filter banks as a means of obtaining spectral interpolation data. This data replaces the standard covariance estimates used in traditional maximum entropy spectral estimation. The new method is based on our recent theory of analytic interpolation with degree constraint and produces suitable pole-zero (ARMA) models, for which the choice of the zeros (MA-part) of the model is completely arbitrary. By suitable choices of filter-bank poles and spectral zeros the estimator can be tuned to exhibit high resolution in targeted regions of the spectrum. A convex optimization approach is presented, which is based on a generalized concept of entropy.

To Sanjoy Mitter on the occasion of his 65^{th} birthday.

1.1 INTRODUCTION

In [11] we presented a new approach to spectral estimation, based on the use of filter banks as a means of obtaining spectral interpolation data. This approach relies on new results in analytic interpolation theory with degree constraint, developed in [10] and based on efforts by the authors over a number of years [2]–[10], [14]–[17].

The purpose of the bank of filters is to process, in parallel, the observation record in order to obtain estimates of the power spectrum at desired points. These points are related to the filter-bank poles and can be selected to give increased resolution over desired frequency bands. Our analytic interpolation theory implies that a second set of tunable parameters are given by so-called spectral zeros which determine the Moving-Average (MA) part of solutions. Consequently, we refer to the new approach as a Tunable High REsolution Estimator (THREE). The solutions turn out to be spectra of Auto-Regressive/Moving-Average (ARMA) filters of complexity at most equal to the dimension of the filter bank, and hence the method provides parametric spectral models.

Our computational procedure for obtaining suitable pole-zero (ARMA) models from filter-bank data is based on a convex optimization problem, the dual of a problem to maximize a generalized entropy gain. The theory for this was developed in [10], which generalizes a procedure in [9] for a similar problem.

For the default setting when the spectral zeros are chosen equal to the filter-bank poles, an alternative and particularly simple algorithm, based on the so-called central solution of the classical interpolation theory, is available; see, e.g., [11]. For any other setting, the corresponding convex optimization problem needs to be solved.

Typically, the resulting spectra show significantly higher resolution as compared to traditional linear predictive filtering. Moreover, they appear to be more robust than linear predictive filtering due to the fact that we use statistical estimates of only zeroth, or first order, covariance lags as opposed to high order lags. Therefore THREE appears to be especially suitable for being applied to short observation records.

1.2 BACKGROUND

Given a scalar, real-valued, zero-mean, stationary (Gaussian) stochastic process $\{y(t)\}_{t\in\mathbf{Z}}$, consider the basic problem of estimating its power spectral density $\Phi(e^{i\theta})$, $\theta \in [-\pi, \pi]$, from a finite observation record

$$y_0, y_1, y_2, \ldots, y_N. \tag{1.1}$$

Typically, modern spectral estimation techniques rely on estimates

$$\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n, \tag{1.2}$$

of the covariance lags $c_0, c_1, c_2, \ldots, c_n$ where $n << N$ and

$$c_k := \mathrm{E}\{y(t)y(t+k)\}. \tag{1.3}$$

Here $\mathrm{E}\{\cdot\}$ denotes mathematical expectation. We assume that the estimates (1.2) form a *bona fide* covariance sequence in the sense that the corresponding Toeplitz matrix is positive definite.

For simplicity, in this paper we shall assume that Φ is coercive, i.e., bounded away from zero on the unit circle. The covariance coefficients (1.3) are the Fourier coefficients of the spectral density Φ. In fact, the function

$$f(z) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{z + e^{-i\theta}}{z - e^{-i\theta}} \Phi(e^{i\theta}) d\theta \tag{1.4}$$

is the positive real part of Φ, i.e., the unique (strictly) positive real function satisfying

$$f(z) + f(z^{-1}) = \Phi(z) \tag{1.5}$$

on and in the neighborhood of the unit circle, and consequently

$$\Phi(e^{i\theta}) = 2\mathrm{Re}\{f(e^{i\theta})\}. \tag{1.6}$$

Moreover, f admits a series representation

$$f(z) = \frac{1}{2}c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \dots \tag{1.7}$$

for $|z| > 1$.

Given the estimates (1.3), the spectral estimation problem is thus reduced to finding a positive real function (1.7) satisfying the interpolation conditions

$$c_k = \hat{c}_k \quad k = 0, 1, \dots, n. \tag{1.8}$$

We also require that this function is rational of at most degree n. Then the unique (real) minimum-phase spectral factor g satisfying

$$g(z)g(z^{-1}) = f(z) + f(z^{-1}) \tag{1.9}$$

is also rational of degree less or equal to n, and we obtain a linear model

$$\text{white noise} \xrightarrow{\nu} \boxed{g(z)} \xrightarrow{\hat{y}} \tag{1.10}$$

of dimension at most n generating an approximant $\hat{y}$ of y in statistical steady state. We call the rational function g the *modeling filter* corresponding to the solution f.

Since this mathematical problem is equivalent to determining a covariance extension

$$c_{n+1}, c_{n+2}, c_{n+3}, \dots$$

so that the degree constraint is satisfied, we refer to it as the *rational covariance extension problem with degree constraint.* It is precisely the degree constraint that makes the parameterization of all solutions of this problem very challenging. Without this constraint, it is merely the classical Charathéodory extension problem, all meromorphic solutions of which are completely parameterized by the "free" Schur parameters [21]. In fact, choosing these Schur parameters to be all zero we obtain the well-known *maximum entropy solution*, which happens to satisfy the degree constraints.

However, the maximum entropy solution corresponds to a spectral factor g having all its zeros at the origin and therefore yields a very "flat" spectrum. This naturally raises the question whether the zeros could be chosen arbitrarily. In [14, 16] Georgiou proved that this is indeed the case. Moreover, he conjectured that the correspondence is injective so that the parameterization would be complete. The proof of existence was by degree theory, which is suitable for proving existence but cannot immediately be applied to uniqueness unless a very strong positivity condition can be shown to hold. Therefore the conjecture remained open for some time until it was proven by Byrnes, Lindquist, Gusev and Mateev [5]. In fact, in [5] somewhat more than Georgiou's conjecture was proven: To each stable[1] monic polynomial

$$\rho(z) = z^n + r_1 z^{n-1} + \ldots + r_{n-1} z + r_n \tag{1.11}$$

there is one, and only one, polynomial

$$\alpha(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \tag{1.12}$$

of degree n so that

$$g(z) = \frac{\rho(z)}{\alpha(z)} \tag{1.13}$$

is a modeling filter for the partial covariance sequence (1.2), and this bijection is a diffeomorphism. Hence the rational covariance extension problem with degree constraint is well-posed in a strong sense.

The proofs of [14, 16, 5] are not constructive, and do not provide an algorithm. A convex-optimization approach to determine the unique $\alpha(z)$ corresponding to any $\rho(z)$ is given in [9].

1.3 A NEW APPROACH TO SPECTRAL ESTIMATION

In the context of Section 1.2, traditional spectral estimation techniques amount to estimating the real part of $f(z)$ from estimates of its value at ∞ and on the values of finitely many of its derivatives at ∞, while we are interested in its values on the unit circle. Our new approach is based on the observation that the values of f at points other than ∞ can be estimated directly from the data (1.1). These interpolation points can then be chosen closer to the unit circle in the frequency band where high resolution is required.

In fact, given any self-conjugate set of distinct real or complex numbers $p_0, p_1, \ldots, p_n$ in the open unit disc and the corresponding transfer functions

$$G_k(z) = \frac{z}{z - p_k} \quad k = 0, 1, \ldots, n, \tag{1.14}$$

consider the bank of filters depicted in Figure 1. In this parallel connection, each filter is first-order if complex arithmetic is used, and always when p is real.

Otherwise, each complex pair $(p, \bar{p})$ corresponds to a second-order filter. Note that, if p_k is a complex number, then u_k is a complex stochastic process.

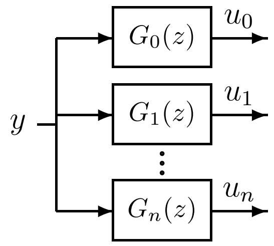

Figure 1: Bank of filters.

Then, as demonstrated in [11], it is easy to see that

$$f(p_k^{-1}) = \frac{1}{2}(1 - p_k^2)\mathrm{E}\{u_k(t)^2\}, \quad k = 0, 1, \dots, n, \tag{1.15}$$

i.e., the values of the positive real function f at the points $\{p_0^{-1}, p_1^{-1}, \dots, p_n^{-1}\}$ can be expressed in terms of the zero-lag covariances of the outputs $u_0, u_1, \dots, u_n$ of the filter bank. The idea is now to estimate these covariances from finite output data generated by the filter bank, thereby obtaining $n+1$ interpolation conditions.

The estimates (1.2), used in traditional approaches, are obtained either by suitable averaging of products $y_t y_{t+k}$, or by estimating the partial autocorrelation coefficients first, using averaging schemes such as Burg's algorithm [1]. In either case, the statistical reliability of such estimates decreases with the order k of the lag, due to the fact that averaging takes place over a shorter list of such cross products. In our new approach we only need to determine zero-lag estimates

$$\hat{c}_0(u_0), \hat{c}_0(u_1), \dots, \hat{c}_0(u_n)$$

based on the output data of the filter bank. However, as pointed out in [11], placing the filter poles too close to the unit circle decreases the accuracy of the covariance estimates, so there is a trade-off between high resolution and statistical accuracy.

In the following we assume that the filter-bank poles $p_0, p_1, \dots, p_n$ are distinct with $p_0 = 0$ and complex poles occurring in complex pairs. The condition $p_0 = 0$ implies that $G_0 \equiv 1$ so that the process y is itself one of the filter-bank outputs. Then, estimating the spectral density Φ from finite observation records of the outputs of the filter bank amounts to determining a positive real function f such that

$$f(p_k^{-1}) = w_k, \quad k = 0, 1, \dots, n, \tag{1.16}$$

where

$$w_k := \frac{1}{2}(1 - p_k^2)\hat{c}_0(u_k), \quad k = 0, 1, \dots, n. \tag{1.17}$$

Then (1.6) provides us with an estimate of the spectral density of y. Since we want this estimate to be rational of minimal complexity, we also require that

$$\deg f \leq n, \tag{1.18}$$

i.e., that f is a rational function of degree at most n.

For the moment ignoring the degree constraint (1.18), this is a classical Nevanlinna-Pick interpolation problem [23], for which there is a solution if and only if the Pick matrix

$$P_n := \left[\frac{w_k + \bar{w}_\ell}{1 - p_k \bar{p}_\ell}\right]_{k,\ell=0}^{n} \tag{1.19}$$

is non-negative definite. In the case that P_n is positive semi-definite but singular, the solution is unique. In the case $P_n > 0$, the complete set of solutions is given by a linear fractional transformation, which is constructed from the interpolation data, acting on a "free" parameter which is only required to have certain analytic properties, e.g., to be a positive-real function [23].

However, this parameterization of all solutions includes functions which may have very high degree, or even be nonrational, and provides no means of characterizing those solutions which satisfy the degree constraint (1.18). One particular such solution, the so-called *central solution* to be described below, is obtained by a trivial choice of the free parameter, but a complete parameterization of all solutions satisfying (1.18) requires a new paradigm. In fact, as in the covariance extension problem, the requirement that the degree of the interpolant f be at most n imposes (a highly nontrivial) nonlinear constraint on the class of solutions. The study of this constraint solution set has led to a rich theory, [2]–[12] and [14]–[17].

The complete parameterization described in Section 1.2 has the following counterpart in the present setting [15, 17, 12]: Suppose that $w_0, w_1, \dots, w_n$ is a self-conjugate set of values in the right half plane with the property that the Pick matrix (1.19) is positive definite. Then, to each real stable polynomial (1.11) there is is one, and only one, real stable polynomial (1.12) of degree n so that the positive-real part f of

$$\frac{\rho(z)\rho(z^{-1})}{\alpha(z)\alpha(z^{-1})}$$

satisfies the interpolation conditions (1.16), and this bijection is a diffeomorphism.

The roots of the polynomial $\rho(z)$ are called the *spectral zeros* of the corresponding interpolant f. As in the covariance extension problem, the minimum-phase spectral factor

$$g(z) = \frac{\rho(z)}{\alpha(z)}$$

of $f(z) + f(z^{-1})$ is the transfer function of a filter (1.10) which produces a statistical approximant of y when white noise is passed through it and reaches steady state, and which we call a THREE filter. The corresponding ARMA model is given by

$$a_0\hat{y}(t)+a_1\hat{y}(t-1)+\ldots+a_n\hat{y}(t-n) = \nu(t)+r_1\nu(t-1)+\ldots+r_n\nu(t-n), \quad (1.20)$$

and hence we refer to $r_1, r_2, \ldots, r_n$ as the *MA parameters* and to $a_0, a_1, \ldots, a_n$ as the *AR parameters*. Consequently, to any choice of MA parameters (such that $\rho(z)$ is a stable polynomial) there corresponds a unique choice of AR parameters (with $\alpha(z)$ likewise stable) so that the positive-real part of the spectral density satisfies the interpolation conditions (1.16). Hence the MA parameters can be chosen arbitrarily, while the same is not true for the AR parameters. In other words, an arbitrary choice of AR parameters may not have a matching selection of MA parameters so that together they meet the required constraints.

In this filter design there are two sets of design parameters, namely the filter-bank parameters $p_1, p_2, \ldots, p_n$ and the spectral zeros $\sigma_1, \sigma_2, \ldots, \sigma_n$. The choice $\sigma_k = p_k$ for $k = 1, 2, \ldots, n$, corresponds to the central solution mentioned above, for which there are simple algorithms; see, e.g., [11]. Next, we present an algorithm for determining the unique interpolant corresponding to an arbitrary choice of spectral zeros.

1.4 A CONVEX OPTIMIZATION APPROACH TO INTERPOLATION

Given the design parameters, i.e., the filter bank poles $p_1, p_2, \ldots, p_n$ and the spectral zeros $\sigma_1, \sigma_2, \ldots, \sigma_n$, form the rational function

$$\Psi(z) = \frac{\rho(z)\rho(z^{-1})}{\tau(z)\tau(z^{-1})}, \quad (1.21)$$

where $\tau(z)$ and $\rho(z)$ are the polynomials

$$\tau(z) := \prod_{k=1}^{n}(z - p_k) \quad = \quad z^n + \tau_1 z^{n-1} + \ldots + \tau_{n-1}z + \tau_n, \quad (1.22)$$

$$\rho(z) := \prod_{k=1}^{n}(z - \sigma_k) \quad = \quad z^n + r_1 z^{n-1} + \ldots + r_{n-1}z + r_n. \quad (1.23)$$

For each choice of design parameters we form the functional

$$\mathbf{I}_\Psi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[f(e^{i\theta}) + f(e^{-i\theta})]\Psi(e^{i\theta})d\theta, \quad (1.24)$$

on the set of positive real functions f. This functional is a generalized entropy gain. The ordinary entropy gain [19] is obtained by choosing $\Psi(z) \equiv 1$, i.e., $\tau(z) \equiv \sigma(z)$, which corresponds to the central solution.

Next, consider the optimization problem (P) to minimize $\mathbf{I}_\Psi(f)$ over all positive real (not necessarily rational) functions f subject to the (interpolation) constraints

$$f(p_k^{-1}) = w_k \quad k = 0, 1, \ldots, n. \tag{1.25}$$

In [10, Theorem 4.1] we proved the following result.

Theorem 1.4.1 *The constrained optimization problem (P) has a unique solution. This solution is strictly positive real and rational of the form*

$$f(z) = \frac{\beta(z)}{\alpha(z)}, \tag{1.26}$$

where $\alpha(z)$ and $\beta(z)$ are polynomials of degree n satisfying

$$\alpha(z)\beta(z^{-1}) + \beta(z)\alpha(z^{-1}) = \rho(z)\rho(z^{-1}). \tag{1.27}$$

Conversely, if f is a positive-real function which satisfies the interpolation conditions as well as (1.26) and (1.27), then it is the unique solution to (P).

Dividing (1.27) by $\alpha(z)\alpha(z^{-1})$, we obtain

$$g(z)g(z^{-1}) = f(z) + f(z^{-1}), \tag{1.28}$$

where g is given by

$$g(z) = \frac{\rho(z)}{\alpha(z)}, \tag{1.29}$$

which is the unique THREE filter with the spectral zeros specified by $\rho(z)$.

This optimization problem is infinite-dimensional and therefore not easy to solve. However, since the number of constraints (1.25) are finite, (P) has a dual with finitely many variables. In fact, let $w(z)$ be any real function which is analytic on and outside the unit circle and satisfies the interpolation conditions (1.25), and define the functional

$$\mathbf{J}_\Psi(Q) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\{Q(e^{i\theta})[w(e^{i\theta}) + w(e^{-i\theta})] - \log Q(e^{i\theta})\Psi(e^{i\theta})\}\theta \tag{1.30}$$

for all functions Q of the form

$$Q(z) = \frac{\alpha(z)\alpha(z^{-1})}{\tau(z)\tau(z^{-1})}, \tag{1.31}$$

where $\alpha(z)$ is a polynomial of degree n. A suitable w can be determined by solving a simple Vandermonde system [11]. The functional (1.30) does not depend on the particular choice of $w(z)$ but only on its values in the interpolation points. In fact, the part involving w is a quadratic form in the coefficients of $\alpha(z)$ whose parameters are precisely the entries of the Pick matrix (1.19).

Now, consider the convex optimization problem (D) to minimize $\mathbf{J}_\Psi(Q)$ over all Q in the class (1.31). The numerator of Q is a symmetric pseudo-polynomial of degree n, and hence (D) is an optimization problem in $n+1$ variables, while the requirement that Q be nonnegative on the unit circle corresponds to infinitely many constraints. The following result is proven in [10, Theorem 4.5].

Theorem 1.4.2 *The convex optimization problem (D) has a unique solution. The minimizing Q is positive on the unit circle, and the unique positive real function f satisfying*

$$\frac{\Psi(z)}{Q(z)} = f(z) + f(z^{-1}) \tag{1.32}$$

in the neighborhood of the unit circle also satisfies the interpolation conditions (1.25). The function f is precisely the maximizing function in Theorem 1.4.1. Conversely, any positive real function satisfying (1.32) and (1.25) is obtained in this way.

Since the minimizing Q is positive on the unit circle, the unique optimal solution lies in the interior of the feasible region. The condition that the gradient of $\mathbf{J}_\Psi(Q)$ be zero is equivalent to the interpolation condition (1.25). Given Q, we may determine $\alpha(z)$ in (1.31) by (minimum-phase) spectral factorization, and then (1.32) reduces to (1.28), and hence the filter g is given by (1.29), as required.

An algorithm based on the convex optimization problem (D) can be obtained by using Newton's method. Such an algorithm, formulated in state space, is described in detail in [11].

1.5 SIMULATIONS

We illustrate the algorithm by some simulations. We begin by estimating spectral lines in colored noise – a problem which is regarded as challenging [20, pages 285–286]. We consider a signal y comprised of two superimposed sinusoids in colored noise:

$$\begin{aligned} y(t) &= 0.5\sin(\omega_1 t + \phi_1) + 0.5\sin(\omega_2 t + \phi_2) + z(t) \quad t = 0, 1, 2, \ldots, \\ z(t) &= 0.8z(t-1) + 0.5\nu(t) + 0.25\nu(t-1) \end{aligned}$$

with ϕ_1, ϕ_2 and $\nu(t)$ independent normal random variables with zero mean and unit variance. The model is used to generate five sets of 300 data points in separate runs. This is done in order to investigate the statistical variability of the estimates and the robustness of the estimation methods.

The objective is to estimate the power spectrum in the vicinity of the spectral lines. In particular, it is desirable to be able to resolve the two distinct spectral peaks. We demonstrate the performance of a THREE filter of order 12 with the filter-bank poles chosen at $0, \pm .85, .9e^{\pm .42i}, .9e^{\pm .44i}, 0.9e^{\pm .46i}, 0.9e^{\pm .48i}, 0.9e^{\pm .50i}$ and the spectral zeros in the default setting of the central solution, i.e., with $\sigma_k = p_k$ for $k = 1, 2, \ldots, n$. Then we compare with a periodogram, computed with state-of-the-art windowing technology.

In Figure 2, the left column corresponds to $\omega_1 = 0.42$ and $\omega_2 = 0.53$, with the periodogram at the top and the THREE method at the bottom. The estimated spectra from the five separate data sets are superimposed, shown together with a smooth curve representing the true power spectrum of the colored noise and two vertical lines at the position of the spectral lines. Apparently both methods perform satisfactorily.

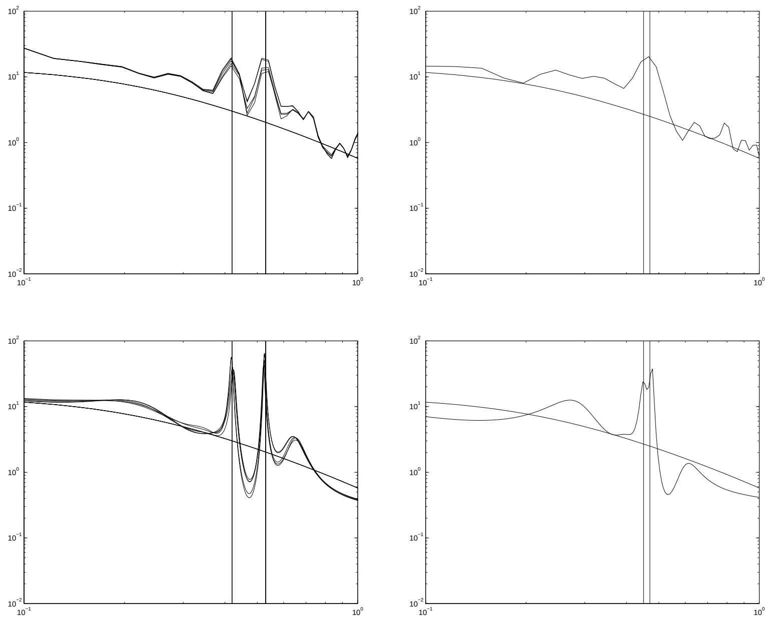

Figure 2: Spectral estimates of two sinusoids in colored noise.

However, if the spectral lines are moved closer so that $\omega_1 = 0.45$ and $\omega_2 = 0.47$, as depicted in the right column of Figure 2, only the THREE filter is capable of resolving the two sinusoids, clearly delineating their position by the presence of two peaks. In fact, the separation of the sinusoids is smaller than the theoretically possible distance that can be resolved by the periodogram using a 300 point record under ideal noise conditions, not satisfied here [22, page 33]. To achieve a better resolution (at the expense of some increased variability) the complex filter-bank poles were chosen slightly closer to the circle.

Secondly, we consider the effectiveness of THREE-based filtering in a case where the power spectrum has sharp transitions. More specifically, we consider data generated by passing white noise through a filter with the transfer function

$$T_\theta(z) = \frac{(z - .9e^{i\pi/3.2})(z - .9e^{-i\pi/3.2})}{(z - .9e^{i\theta})(z - .9e^{-i\theta})(z - .3e^{i\pi/3.5})(z - .3e^{-i\pi/3.5})}, \tag{1.33}$$

where θ takes the values $\theta = \pi/3$, $\theta = \pi/3.1$ and $\theta = \pi/2.9$. The spectrum of the output has sharp transitions due to the fact that poles and zeros are close to each other. In Figure 3 spectral estimates are depicted for each choice of θ

using periodograms (left) and fourth order THREE filter design (right) with filter-bank poles set at $0, .8e^{\pm .8i}, .8e^{\pm 1.3i}$ and spectral zeros at $0, -.8, .8e^{\pm i\pi/3.3}$. The true spectra are marked with dotted lines. In this set of experiments we have used a data record long enough to eliminate fluctuations of the estimated spectra, namely 2000 samples. Noting that *the order is only four*, the THREE estimates are remarkably good.

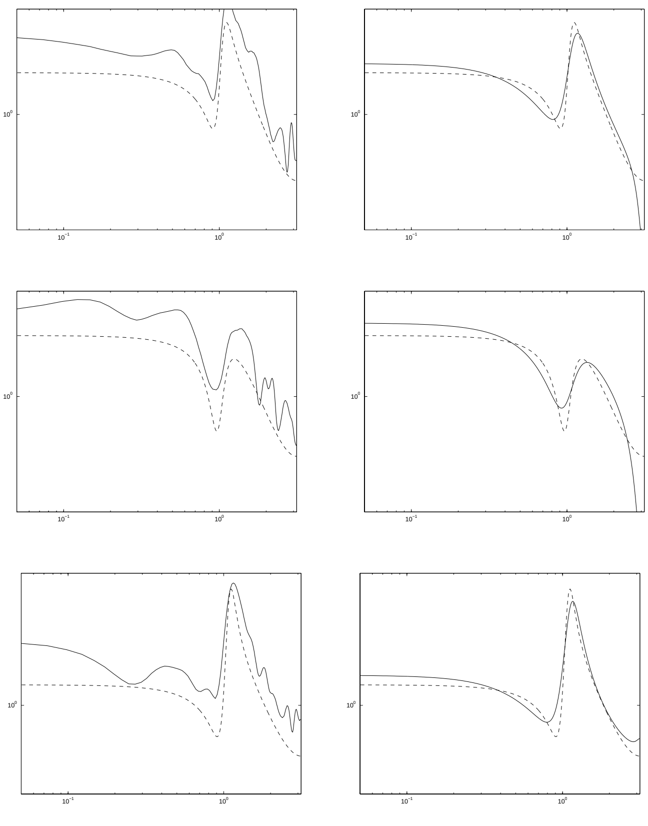

Figure 3: Spectral estimates for different choices of θ.

Acknowledgment

The authors gratefully acknowledge the support by AFOSR, NSF, TFR, the Göran Gustafsson Foundation, and Southwestern Bell.

Notes

1. A polynomial is called *stable* if all its root are located in the open unit disc $\{z \mid |z| < 1\}$.

References

[1] J.P. Burg, "A new analysis technique for time series data," NATO Advanced Institute on Signal Processing with Emphasis on Underwater Acoustics, 1968, reprinted in *Modern Spectrum Analysis*, D.G. Childers, ed., IEEE Press.

[2] C.I. Byrnes and A. Lindquist, "On the geometry of the Kimura-Georgiou parameterization of modelling filter," *Intern. J. Control*, **(50)**: 2301–2312, 1989.

[3] C.I. Byrnes, A. Lindquist and T. McGregor, "Predictability and Unpredictability in Kalman filtering," *IEEE Trans. on Aut. Control*, **AC-36**: 563–579, 1991.

[4] C.I. Byrnes, A. Lindquist and Y. Zhou, "On the nonlinear dynamics of fast filtering algorithms," *SIAM J. Control and Optimization*, **32**: 744–789, 1994.

[5] C. I. Byrnes, A. Lindquist, S.V. Gusev, and A. S. Matveev, "A complete parametrization of all positive rational extensions of a covariance sequence," *IEEE Trans. Autom, Control*, **AC-40:** 1841-1857, 1995.

[6] C. I. Byrnes and A. Lindquist, "On the Partial Stochastic Realization Problem," *IEEE Transactions on Automatic Control*, **AC-42:** 1049-1070, 1997.

[7] C. I. Byrnes, H. J. Landau, and A. Lindquist, "On the well-posedness of the rational covariance extension problem," in *Current and Future Directions in Applied Mathematics*, eds. M. Alber, B. Hu, J. Rosenthal, Birkhäuser, pp. 83-106, 1997.

[8] C. I. Byrnes and A. Lindquist, "On duality between filtering and control," in *Systems and Control in the Twenty-First Century*, C. I. Byrnes, B. N. Datta, D. S. Gilliam and C. F. Martin, editors, Birhäuser, 1997, pp. 101–136.

[9] C. I. Byrnes, S. V. Gusev, and A. Lindquist, "A convex optimization approach to the rational covariance extension problem," *SIAM J. Control and Opt.*,**37:** 211-229, 1999.

[10] C.I. Byrnes, T.T. Georgiou, and A. Lindquist, A generalized entropy criterion for Nevanlinna-Pick interpolation: A convex optimization approach to certain problems in systems and control," submitted to *IEEE Trans. on Aut. Control*.

[11] C.I. Byrnes, T.T. Georgiou, and A. Lindquist, "A new approach to spectral estimation: A tunable high-resolution spectral estimator," submitted to *IEEE Trans. on Signal Processing*.

[12] C. I. Byrnes and A. Lindquist, "On the duality between filtering and Nevanlinna-Pick interpolation," submitted to *SIAM J. Control and Opt.*

[13] Ph. Delsarte, Y. Genin, Y. Kamp and P. van Dooren, "Speech modelling and the trigonometric moment problem," *Philips J. Res.* **37** (1982), 277–292.

[14] T.T. Georgiou, *Partial Realization of Covariance Sequences*, Ph.D. thesis, CMST, University of Florida, Gainesville 1983.

[15] T.T. Georgiou, "A Topological approach to Nevanlinna-Pick Interpolation," *SIAM J. on Math. Analysis,* **18(5)**: 1248-1260, 1987.

[16] T.T. Georgiou, "Realization of power spectra from partial covariance sequences," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, **ASSP-35:** 438-449, 1987.

[17] T.T. Georgiou, The interpolation problem with a degree constraint," *IEEE Trans. on Automatic Control*, **44(3)**: 631-635, 1999.

[18] U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. California Press, 1958.

[19] S. A. Kassam and H. V. Poor, "Robust techniques for signal processing," *Proceedings IEEE* **73**: 433–481, 1985.

[20] B. Porat, *Digital Processing of Random Signals*, Prentice Hall, 1994.

[21] I. Schur, "On power series which are bounded in the interior of the unit circle" I and II, *Journal fur die reine und angewandte Mathematik* **148** (1918), 122–145.

[22] P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997.

[23] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math.Soc. Colloquium Publications, **20**, Providence, R. I., 1956.