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AN ALGEBRAIC DESCRIPTION OF THE RATIONAL SOLUTIONS OF THE COVARIANCE EXTENSION PROBLEM

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1. Introduction.

A problem of great interest in signal and in speech processing is the design of shaping filters realizing stationary stochastic processes. That is, given a scalar stationary process $\{y_t\}$ we wish to find a rational function $w(z)$ such that y_t is the output of the stochastic linear system determined by $w(z)$ when driven by white noise. A problem of greater practical interest is that of determining suitable $w(z)$ from partial knowledge of the statistics of the process $\{y_t\}$. Explicitly, setting

$$c_i = E\{y(t+i)y(t)\}$$

one important version of this problem entails determining all shaping filters of a degree n which generate the data $c_0, c_1, \dots, c_n$ as the first $n+1$ correlation coefficients of the system output. The maximum entropy method is an important technique for generating one such solution, an "all-pole" shaping filter, but there is widespread interest in parameterizing all solutions.

This problem may be stated in several equivalent forms: finding a modelling filter $v(z)$ for the correlation coefficients, finding "positive" extensions $c_{n+1}, \dots, c_{n+k}, \dots$ of the correlation sequence, etc. As a starting point for understanding the analysis and geometry of the set of shaping filters, we present here semialgebraic criteria for describing which of the rational functions in the Kimura-Georgiou parameterization (see (1.4)) are modeling filters. These criteria are derived from an algorithm which contains, in particular, what we feel is a very much streamlined test for the positive real condition.

As we indicated we will study this problem through the equivalent formulation in terms of modelling filters. The following version of the partial stochastic realization problem is important in several areas of stochastic systems theory and statistical analysis, and has been the subject of much study [1,2,3]. Given a sequence $\{c_1, c_2, c_3, \dots, c_n\}$ of real numbers with the property that the Toeplitz matrix

$$T_n := \begin{bmatrix} 1 & c_1 & c_2 & \dots & c_n \\ c_1 & 1 & c_1 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ c_n & c_{n-1} & c_{n-2} & \dots & 1 \end{bmatrix} \tag{1.1}$$

is positive definite. Find an infinite extension $\{c_{n+1}, c_{n+2}, c_{n+3}, \cdots\}$ such that the function

$$v(z) = 1/2 + \sum_{k=1}^{\infty} c_k z^{-k}$$

is rational of degree at most n and <u>positive real</u>, i.e.

$$\text{(i)} \quad v(z) \text{ analytic on } |z| \geq 1 \tag{1.2}$$

$$\text{(ii)} \quad v(z) + v(1/z) > 0 \text{ on } |z| = 1 \tag{1.3}$$

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Let $\mathcal{C}_n$ denote the class of all such solutions. To each such solution there corresponds a rational spectral factor $w(z)$ of degree at most n, analytic in $|z| \geq 1$ and satisfying

$$w(z)w(1/z) = v(1/z) + v(z) \ .$$

Kimura [2] and Georgiou [3] have independently shown that to any solution of $\mathcal{C}_n$ there corresponds a sequence $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of real numbers such that

$$v(z) = \frac{1}{2}\frac{\psi_n(z) + \alpha_1\psi_{n-1}(z) + \cdots + \alpha_n\psi_0(z)}{\varphi_n(z) + \alpha_1\varphi_{n-1}(z) + \cdots + \alpha_n\varphi_0(z)} \tag{1.4}$$

where $\{\varphi_k(z)\}$ and $\{\psi_k(z)\}$ are the Szegö polynomials of first and second kind respectively. These are polynomial orthogonal on the unit circle, and they are defined by the recursions

$$\begin{cases} \varphi_{t+1}(z) = z\varphi_t(z) - \gamma_t\varphi_t^*(z) & \varphi_0(z) = 1 \\ \varphi_{t+1}^*(z) = \varphi_t^*(z) - \gamma_t z\varphi_t(z) & \varphi_0^*(z) = 1 \end{cases} \tag{1.5}$$

and

$$\begin{cases} \psi_{t+1}(z) = z\psi_t(z) + \gamma_t\psi^*(z) & \psi_t(z) = 1 \\ \psi_{t+1}^*(z) = \psi_t^*(z) + \gamma_t z\psi_t(z) & \psi_t^*(z) = 1 \end{cases} \tag{1.6}$$

respectively, where $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots\}$ are the *Schur parameters* defined by

$$\begin{cases} \gamma_t = & \frac{1}{r_t}\left[\sum_{i=0}^{t-1}\varphi_{t,t-i}c_{i+1} + c_{t+1}\right] \\ & r_{t+1} = (1-\gamma_t^2)r_t \ \ ; \ \ r_0 = 1 \end{cases} \tag{1.7}$$

Here $\{\varphi_{tk}\}$ are the coefficients of

$$\varphi_t(z) = z^t + \varphi_{t1}z^{t-1} + \cdots + \varphi_{tt} \tag{1.8}$$

and $\varphi_t^*(z) := z^t\varphi_t(1/z)$ is the reversed polynomial

$$\varphi_t^*(z) = \varphi_{tt}z^t + \cdots + \varphi_{t1}z + 1 \tag{1.9}$$

(Therefore, in each of equations (1.5) and (1.6), the second equation is equivalent to the first.)

The purpose of this paper is to algebraically characterize the set of all $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbf{R}^n$ for which $v(z)$ in (1.4) is positive real of degree at most n and provide an algorithm for determining this set, thereby solving a problem left open in [2,3].

2. The algorithm.

In this section we shall begin by giving explicit algebraic inequalities for describing the solution set of the problem presented above. These conditions are analogous to those provided by the classical stability test of Routh-Hurwitz, Schur-Cohn, Jury, etc.

To pin down the problem let us first note that if the two polynomials

$$\begin{cases} a(z) = z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n \\ b(z) = z^n + b_1z^{n-1} + b_2z^{n-2} + \cdots + b_n \end{cases} \tag{2.1}$$

are relatively prime, then the rational function

$$v(z) = \frac{1}{2}\frac{b(z)}{a(z)} \tag{2.2}$$

is positive real if and only if

$$\begin{aligned} &\text{(i)} \quad a(z)b(1/z) + a(1/z)b(z) > 0 \text{ on } |z| = 1 && (2.3)\\ &\text{(ii)} \quad a(z) \text{ have all its zeros in } |z| < 1 && (2.4)\\ &\text{(iii)} \quad b(z) \text{ have all its zeros in } |z| < 1 && (2.5) \end{aligned}$$

Clearly, (2.3) + (2.4) is equivalent to (1.2) + (1.3) and consequently conditions (i) and (ii) are sufficient. However, it is well-known (see e.g. [2]) that if $v(z)$ is positive real, then so is $1/v(z)$. In fact, (i) - (iii) can be exchanged for either (i) +(ii) or (i) + (iii).

Given a sequence $\{c_1, c_2, \cdots, c_n\}$ satisfying (1.1), the Szegö polynomials $\{\varphi_1, \varphi_2, \cdots \varphi_n, \psi_1, \psi_2, \cdots \psi_n\}$ are uniquely defined, and identifying (2.2) with (1.4) yields the following linear relationships between $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ respectively, namely

$$\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \varphi_{n1} & 1 & & & \\ \varphi_{n2} & \varphi_{n-1,1} & 1 & & \\ \varphi_{n3} & \varphi_{n-1,2} & \varphi_{n-2,1} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{nn} & \varphi_{n-1,n-1} & \varphi_{n-2,n-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{2.6}$$

and

$$\begin{bmatrix} 1 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \psi_{n1} & 1 & & & \\ \psi_{n2} & \psi_{n-1,1} & 1 & & \\ \psi_{n3} & \psi_{n-1,2} & \psi_{n-2,1} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{nn} & \psi_{n-1,n-1} & \psi_{n-2,n-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{2.7}$$

which define two bijective maps $\Phi : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ and $\Psi : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ sending $(1, \alpha)$ to $(1, a)$ and $(1, b)$ respectively.

Now, we would also like to allow for representations (1.4) which are of degree $< n$, i.e. for which $a(z)$ and $b(z)$ are not relatively prime. In such cases, (2.4) and (2.5) need not be satisfied, since the common zeros of $a(z)$ and $b(z)$ may be located outside the unit disc. However, since these zeros are being cancelled, there are many representations (1.4) representing the same $v(z)$ of degree $< n$, and this equivalence class contains elements for which (2.4) and (2.5) hold. Hence, to include all positive real rational extensions of $\{c_1, c_2, \cdots, c_n\}$ of degree at most n, it suffices to consider α such that the corresponding $a(z)$ and $b(z)$, defined through (2.6) and (2.7), satisfy (2.3) - (2.5). Let $\mathcal{A}_+$ denote this set of admissible α. Then

$$\mathcal{A}_+ \subset \mathcal{A} := \{\alpha \in \mathbf{R}^n \mid (1, \alpha) \in (\Phi^{-1}\mathcal{S}) \cap (\Psi^{-1}\mathcal{S})\} \tag{2.8}$$

where $\mathcal{S} \subset \mathbf{R}^{n+1}$ is the set of all $(1, a)$ such that $a(z)$ is stable, i.e. satisfies (2.4).

To test if $\alpha \in \mathcal{A}$ it suffices to check whether $a(z)$ and $b(z)$, determined via (2.6) and (2.7), have all its zeros in the open unit disc $\{|z| < 1\}$. This can be done by some known stability test such as the Jury conditions (see e.g.[4], p. 493), and therefore we restrict our attention to $\mathcal{A}$.

Therefore we shall now describe an algorithm which tests whether an $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathcal{A}$ also belongs to $\mathcal{A}_+$, providing the algebraic conditions which need to augmented to the Jury conditions in order that (1.4) be positive real. We know that $0 \in \mathcal{A}_+$ (maximum entropy realization), so in order to use the algorithm to delineate the set $\mathcal{A}_+$ we begin in a neighborhood of zero.

The algorithm goes as follows. The proofs will be given in Section 3.

STEP 0. Given α, solve (2.6) and (2.7) for a and b and test for stability. If $\alpha \notin \mathcal{A}$, stop.

STEP 1. *Determine* $(d_0, d_1, d_2, \cdots, d_n) \in \mathbf{R}^{n+1}$:

$\hat{a}_{00} := 1, \ \hat{b}_{00} := 1, \ \pi_{00} := 1$

for $k = 1, 2, \cdots, n$

$\pi_{k0} = (-1)^k$

$\pi_{kj} = \pi_{k-1,j-1} - \pi_{k-1,j} \quad (j = 1, 2, \cdots, k-1)$

$\pi_{kk} = 1$

$\hat{a}_{k0} = \hat{a}_{k-1,0} + a_k \pi_{k0}$

$\hat{a}_{kj} = \hat{a}_{k-1,j-1} + \hat{a}_{k-1,j} + a_k \pi_{kj} \quad (j = 1, 2, \cdots, k-1)$

$\hat{a}_{kk} = \hat{a}_{k-1,k-1} + a_k \pi_{kk}$

$\hat{b}_{k0} = \hat{b}_{k-1,0} + b_k \pi_{k0}$

$\hat{b}_{kj} = \hat{b}_{k-1,j-1} + \hat{b}_{k-1,j} + b_k \pi_{kj} \quad (j = 1, 2, \cdots, k-1)$

$\hat{b}_{kk} = \hat{b}_{k-1,k-1} + b_k \pi_{kk}$

Compute, for $k = 0, 1, 2, \cdots, n$,

$$d_k = \hat{a}_{nk}\hat{b}_{nk} + \sum_{j=1}^{\min(k,n-k)} (-1)^j \left[\hat{a}_{n,k+j}\hat{b}_{n,k-j} + \hat{a}_{n,k-j}\hat{b}_{n,k+j}\right] \tag{2.9}$$

STEP 2. *Determine* $(h_1, h_2, \cdots, h_{4n-1}) \in \mathbf{R}^{4n-1}$:

$\eta_0 := -1/d_0 \quad$ (by, Lemma 3.1, $d_0 > 0$)

$h_1 := n$

$d_k := 0$ for $k = n+1, n+2, \cdots, 2n-1$

for $k = 1, 2, 3 \cdots, 2n-1$

$h_{2k} = 0$

$$h_{2k+1} = k d_k \eta_0 + (k-1) d_{k-1} \eta_1 + (k-2) d_{k-2} \eta_2 + \cdots + d_1 \eta_{k-1} \tag{2.10}$$

$$\eta_k = (d_k \eta_0 + d_{k-1} \eta_1 + d_{k-2} \eta_2 + \cdots + d_1 \eta_{k-1}) \eta_0 \tag{2.11}$$

STEP 3. *Determine* σ_{4n-1}:

$$p_{0j} := 0 \quad \text{for} \quad j = 1, 2, 3, \cdots, 4n-1$$

$$q_{0j} := \begin{cases} 1 & \text{for } j = 1 \\ 0 & \text{for } j = 2, 3, \cdots, 4n-1 \end{cases}$$

$$\theta_0 := 1, \quad \nu_0 := 0, \quad \sigma_0 := 0$$

for $k = 1, 2, 3 \cdots, 4n-1$

$$\rho_k = (q_{k-1,1}h_1 + q_{k-1,2}h_2 + \cdots + q_{k-1,k}h_k)\,\theta_{k-1}$$

$$\nu_k = \nu_{k-1} + 1$$

if $\nu_k > 0$ and $\rho_k \neq 0$ then

$$\theta_k = \theta_{k-1}/\rho_k$$

$$\sigma_k = \sigma_{k-1} + (\nu_k \bmod 2)\ \mathrm{sign}\ \theta_k \qquad (2.12)$$

$$p_{k1} = 0$$

$$p_{kj} = q_{k-1,j-1} \quad j = 2, 3, \cdots, 4n-1$$

$$\nu_k \leftarrow -\nu_k$$

else

$$\theta_k = \theta_{k-1}$$

$$\sigma_k = \sigma_{k-1}$$

$$p_{kj} = p_{k-1,j} \quad j = 1, 2, \cdots, 4n-1$$

$$q_{k1} = -\rho_k p_{k-1,1}$$

$$q_{kj} = q_{k-1,j-1} - \rho_k p_{k-1,j} \quad j = 2, 3, \cdots, 4n-1$$

THEOREM. *Let $\alpha \in \mathcal{A}$. Then $\alpha \in \mathcal{A}_+$ if and only if*

$$\sigma_{4n-1} = 0\ .$$

To sum up, the algorithm defines a function $f : \mathcal{A} \to \mathbf{Z}$, sending α to σ_{4n-1}, and a criterion

$$f(\alpha) = 0$$

for positivity.

3. The proofs.

In Step 1 of the algorithm Conditions (2.3) - (2.5) are transformed from the unit circle to the imaginary axis. It is well-known and easy to check that the linear fractional transformation

$$z = \frac{1+s}{1-s} \qquad (3.1)$$

maps the closed unit disc onto the closed left complex half plane and that (2.4) and (2.5) are satisfied if and only if the polynomials

$$\hat{a}(s) := (1-s)^n a\left(\frac{1+s}{1-s}\right) \qquad (3.2)$$

and

$$\hat{b}(s) := (1-s)^n b\left(\frac{1+s}{1-s}\right) \qquad (3.3)$$

respectively have all their zeros in the open left half plane (see eg [5], p. 56). It is also easy to see (cf [6], p. 13) that $\hat{a}(s) = \hat{a}_n(s)$ and $\hat{b}(s) = \hat{b}_n(s)$ where the polynomials

$$\begin{cases} \hat{a}_k(s) & = \hat{a}_{k0}s^k + \hat{a}_{k1}s^{k-1} + \cdots + \hat{a}_{kk} \\ \hat{b}_k(s) & = \hat{b}_{k0}s^k + \hat{b}_{k1}s^{k-1} + \cdots + \hat{b}_{kk} \end{cases} \tag{3.4}$$

$(k = 0, 1, 2, \cdots, n)$ are determined via the polynomial recursions

$$\begin{cases} \hat{a}_k(s) & = (1+s)\hat{a}_{k-1}(s) + a_k\pi_k(s) & \hat{a}_0(s) = 1 \\ \hat{b}_k(s) & = (1+s)\hat{b}_{k-1}(s) + b_k\pi_k(s) & \hat{b}_0(s) = 1 \\ \pi_k(s) & = (1-s)\pi_{k-1}(s) & \pi_0(s) = 1 \end{cases} \tag{3.5}$$

These recursions constitute the first part of Step 1 of the algorithm.

It remains to transform Condition (2.3) from the unit circle to the imaginary axis. From (3.1) - (3.3) it is seen that

$$a(z)b(1/z) + a(1/z)b(z) = \frac{1}{(1-s^2)^n}\left[\hat{a}(s)\hat{b}(-s) + \hat{a}(-s)\hat{b}(s)\right] ,$$

and therefore, since $(1+\omega^2)^{-n} > 0$, for all $\omega \in \mathbf{R}$, Condition (2.3) holds if and only if

$$d(\omega) := \frac{1}{2}\left[\hat{a}(i\omega)\hat{b}(-i\omega) + \hat{a}(-i\omega)\hat{b}(i\omega)\right] \tag{3.6}$$

is positive for all $\omega \in \mathbf{R}$. It is now easy to see that $d(\omega)$ is an even polynomial of the form

$$d(\omega) = d_0\omega^{2n} + d_1\omega^{2(n-1)} + \cdots + d_{n-1}\omega^2 + d_n \tag{3.7}$$

where the coefficients $\{d_0, d_1, d_2, \cdots, d_n\}$ are given by (2.9).

LEMMA 3.1. *Condition (2.3) is equivalent to*

$$d(\omega) > 0 \quad \textit{for all} \quad \omega \in \mathbf{R} \tag{3.8}$$

where d is given by (3.6) or (3.7) + (2.9). If Conditions (2.4) and (2.5) hold, then $d_0 > 0$ and $d_n > 0$.

Proof. It remains to show that, if the zeros of $a(z)$ and $b(z)$ are located in the open unit disc, then $d_0 > 0$ and $d_n > 0$. From Step 1 of the algorithm it follows that

$$\hat{a}_{n0} = 1 - a_1 + a_2 - \cdots + (-1)^n a_n$$

and that

$$\hat{a}_{nn} = 1 + a_1 + a_2 + \cdots + a_n$$

and therefore, by the Jury conditions [4], Condition (2.4) implies that $\hat{a}_{n0} > 0$ and $\hat{a}_{nn} > 0$. Similarily, it follows from Condition (2.5) that $\hat{b}_{n0} > 0$ and $\hat{b}_{nn} > 0$. Consequently, $d_0 > 0$ and $d_n > 0$ as required. □

We have thus proved that, if $\alpha \in \mathcal{A}$, then the polynomial $d(\omega)$ is of exactly degree $2n$ and $d(0) > 0$. We may therefore proceed by testing (3.8) under these conditions. Such a test is provided by Steps 2 and 3 of the algorithm.

Since $d(0) > 0$, Condition (3.8) is equivalent to d having no real zeros. This, in turn is equivalent to

$$I_{-\infty}^{\infty}\left[d'(\omega)/d(\omega)\right] = 0 \tag{3.9}$$

where d' is the derivative of d and $I_{-\infty}^{\infty}R(\omega)$ denotes the *Cauchy index* of the real rational function R between the limits $(-\infty, \infty)$, i.e. the difference between the number of jumps in value of $R(\omega)$ from $-\infty$ to ∞ and the number of jumps between ∞ and $-\infty$ as the argument ω changes from $-\infty$ to ∞ (see [7], p. 174).

Consider the Laurent expansion

$$\frac{1}{2}\frac{d'(\omega)}{d(\omega)} = h_1\omega^{-1} + h_2\omega^{-2} + h_3\omega^{-3} + \cdots \tag{3.10}$$

of d'/d about $\omega = \infty$ and from the Hankel matrix

$$H := \begin{bmatrix} h_1 & h_2 & h_3 & \dots & h_{2n} \\ h_2 & h_3 & h_4 & \dots & h_{2n+1} \\ \dots & \dots & \dots & \dots & \dots \\ h_{2n} & h_{2n+1} & h_{2n+2} & \dots & h_{4n-1} \end{bmatrix} \tag{3.11}$$

Then it follows from a theorem of Hermite and Hurwitz (see [7], p. 210) that

$$I_{-\infty}^{\infty}\left[d'(\omega)/d(\omega)\right] = \sigma(H)\ , \tag{3.12}$$

where $\sigma(H)$ denotes the *signature* of H. (Note that the infinite Hankel matrix in the Theorem of Hermite and Hurwitz, as presented in [7], has the same signature as H, for they have the same rank, equal to the degree of the rational function (3.10).)

We sum up these conclusions in

LEMMA 3.2. *Let $d(\omega)$ be a polynomial (3.7) such that $d_n > 0$. Then Condition (3.8) is equivalent to*

$$\sigma(H) = 0 \tag{3.13}$$

where $\sigma(H)$ is the signature of the Hankel matrix (3.11).

Step 2 in the algorithm determines the Hankel elements $\{h_1, h_2, h_3, \cdots, h_{4n-1}\}$ and Step 3 computes the signature $\sigma(H)$.

LEMMA 3.3. *Let $d(\omega)$ be a polynomial (3.7) such that $d_0 > 0$. Then*

$$\frac{1}{2}\ \frac{d'(\omega)}{d(\omega)} = \omega\ c\ (\omega^2 I - A)^{-1} b \tag{3.14}$$

where (A, b, c) is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \dots & \beta_1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$c = [-\beta_{n-1}, -2\beta_{n-2}, -3\beta_{n-3}, \cdots, -(n-1)\beta_1, n]$$

and where

$$\beta_k = -d_k/d_0 \quad k = 1, 2, \cdots, n$$

Proof. A straightforward calculation yields

$$\frac{1}{2}\frac{d'(\omega)}{d(\omega)} = \omega\ G\left(\omega^2\right)$$

where G is the rational function

$$G(\omega) = \frac{n\omega^{n-1} - (n-1)\beta_1\omega^{n-2} - \cdots - \beta_{n-1}}{\omega^n - \beta_1\omega^{n-1} - \cdots - \beta_n}$$

which has the reachable canonical form (A, b, c); see e.g. [5]. □

Consequently, for $k = 0, 1, 2, 3, \cdots$,

$$\begin{cases} h_{2k} & = 0 \\ h_{2k+1} & = cA^k b \end{cases} \tag{3.15}$$

Now, for $k = 0, 1, 2, \cdots, n-1$, $A^k b$ has the structure

$$A^k b = -d_0\left(0, \cdots, 0, \eta_0, \eta_1, \cdots, \eta_k\right)^T \tag{3.16}$$

where T−superscript denotes transpose, $\eta_0 := -1/d_0$, and

$$\eta_k = \beta_k\eta_0 + \beta_{k-1}\eta_1 + \cdots + \beta_1\eta_{k-1} \tag{3.17}$$

Noting that $\beta_k = d_k\eta_0$, it is immediately seen that (3.17) is the same as (2.11). For $n \geq 0$,

$$A^k b = -d_0\left(\eta_{k-n+1}, \eta_{k-n+2}, \cdots, \eta_k\right) \tag{3.18}$$

and (2.11) remains valid if we set $d_k := 0$ for $k > 0$. Then, computing $cA^k b$ applying (2.11), (2.10) follows from (3.15).

Step 3, finally, is a modified version of the Berlekamp-Massey algorithm [8, 9, 10] introduced in [11] for the purpose of determining the parameter sequence in the partial realization problem. However, as pointed out in Remark 1 on page 286 in [11], we can also use this algorithm to determine the signature of a Hankel matrix. In fact, sticking to the notations of [11], H is congruent to a block diagonal matrix

$$D = \operatorname{diag}\left(\Pi_1, \Pi_2, \cdots, \Pi_m\right) \tag{3.19}$$

where $\Pi_1, \Pi_2, \cdots, \Pi_{m-1}$, are nonsingular lower triangular Hankel matrices and Π_m is either a matrix of the same type (if H is full rank) or a "northwest corner" of such a matrix (if H is singular). By Sylvester's law of inertia, $\sigma(H) = \sigma(D)$, and hence

$$\sigma(H) = \sum_{j=1}^{m}\sigma\left(\Pi_j\right) \tag{3.20}$$

Let us first consider the $m-1$ first blocks: If Π_j is $\nu_j \times \nu_j$ and has the element $\lambda_j \neq 0$ on the antidiagonal, $\sigma\left(\Pi_j\right)$ equals the sign of λ_j if ν_j is odd and zero if ν_j is even. Therefore,

$$\sigma(H) = \sum_{j=1}^{m-1}\left(\nu_j \bmod 2\right)\ \operatorname{sign}\lambda_j + \sigma\left(\Pi_m\right) \tag{3.21}$$

The block Π_m requires a special investigation. It is either a zero block or a matrix of the form

$$\Pi_m = \text{diag}(0, \hat{\Pi}_m) \tag{3.22}$$

where 0 is a zero matrix and $\hat{\Pi}_m$ is a nonsingular $\hat{\nu}_m \times \hat{\nu}_m$ lower triangular Hankel matrix with antidiagonal element $\lambda_m \neq 0$. Consequently,

$$\sigma(H) = \sum_{j=1}^{m} (\nu_j \bmod 2) \text{ sign } \lambda_j \tag{3.22}$$

where ν_m is either 0 or $\hat{\nu}_m$ depending on which case applies. The algorithm determines (3.22) recursively through equation (2.12), the index j in (3.22) being the number of times one has visited the subroutine containing (2.12); here θ_k is the inverse of λ_j (see equ (5.4) on page 315 in [11]). The total number of visits at (2.12) equals $m - 1$ if $\Pi_m \neq 0$ and m otherwise. In the latter case, $\hat{\Pi}_m$ is the "center" of a $\tilde{\nu}_m \times \tilde{\nu}_m$ block $\tilde{\Pi}_m$ which is also a lower triangular Hankel matrix. The ν_m in the algorithm is actually $\tilde{\nu}_m$, but this does not matter since ν_m is odd if and only if $\tilde{\nu}_m$ is.

Step 3 is merely a modification of the algorithm on page 316 in [11], the only thing having been added being (2.12). The reader is referred to Section 5 of said paper for a justification of the procedure. In this regard, note that p_k and q_k are vector representations of the polynomials ϕ_k and ψ_k in [11].

REFERENCES

1. R.E. Kalman, Realization of covariance sequences, *Proc. Toeplitz Memorial Conference,* Tel Aviv, Israel, 1981.

2. H. Kimura, Positive partial realization of covariance sequences, *Modeling, Identification and Robust Control,* C.I. Byrnes and A. Lindquist (editors), Elsevier Science Publishers B.V. (North Holland), 1986, 499-513.

3. T.T. Georgiou, Partial realization of covariance sequences, PhD dissertation, Dept. E.E., University of Florida, 1983.

4. L.A. Zadeh and C.A. Desoer, *Linear System Theory,* McGraw-Hill, 1963.

5. R.W. Brockett, *Finite Dimensional Linear Systems,* John Wiley and Sons, 1970.

6. G. Dahlquist, Positive functions and some applications to stability questions for numerical methods, *Recent Advances in Numerical Analysis,* C. de Boor and G. Golub (editors), Academic Press 1978.

7. F.R. Gantmacher, *The Theory of Matrices,* Vol.II, Chelsea 1959.

8. E.R. Berlekamp, *Algebraic Coding Theory,* McGraw Hill, 1968.

9. J.L. Massey, Shift register synthesis and BCH decoding, *IEEE Trans.Information Theory* IT - 15 (1969), 122-127.

10. N. Zierler, Linear recurring sequences and error correcting codes, in *Error Correcting Codes* (H.B. Mann, Ed.) 1968, 47-59.

11. W.B. Gragg and A. Lindquist, On the partial realization problem, *Linear Algebra* 50 (1983), 277-319.

12. D.D. Šiljak, New algebraic criteria for positive realness, *J. Franklin Institute* 291 (1971), 109-120.