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Integral Equations and Operator Theory

The Generalized Moment Problem with Complexity Constraint

Christopher I. Byrnes and Anders Lindquist

Abstract. In this paper, we present a synthesis of our differentiable approach to the generalized moment problem, an approach which begins with a reformulation in terms of differential forms and which ultimately ends up with a canonically derived, strictly convex optimization problem. Engineering applications typically demand a solution that is the ratio of functions in certain finite dimensional vector space of functions, usually the same vector space that is prescribed in the generalized moment problem. Solutions of this type are hinted at in the classical text by Krein and Nudelman and stated in the vast generalization of interpolation problems by Sarason. In this paper, formulated as generalized moment problems with complexity constraint, we give a complete parameterization of such solutions, in harmony with the above mentioned results and the engineering applications. While our previously announced results required some differentiability hypotheses, this paper uses a weak form involving integrability and measurability hypotheses that are more in the spirit of the classical treatment of the generalized moment problem. Because of this generality, we can extend the existence and well-posedness of solutions to this problem to nonnegative, rather than positive, initial data in the complexity constraint. This has nontrivial implications in the engineering applications of this theory. We also extend this more general result to the case where the numerator can be an arbitrary positive absolutely integrable function that determines a unique denominator in this finite-dimensional vector space. Finally, we conclude with four examples illustrating our results.

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1. Introduction and main results

With so many profound uses of the generalized moment problem in pure and applied mathematics, and engineering and science, it almost seems unnecessary to state this classical inverse problem. Nonetheless, we refer the reader to the texts (see, e.g., [1, 2, 22, 23]) and provide the standard definitions below.

Consider a sequence of complex numbers $c_1, c_2, \dots, c_n$ and a sequence of continuous, linearly independent complex-valued continuous functions $\alpha_1, \alpha_2, \dots, \alpha_n$ defined on the real interval $[a, b]$. The moment problem is then to find all monotone, nondecreasing functions μ of bounded variation such that

$$\int_a^b \alpha_k(t) d\mu(t) = c_k, \quad k = 1, 2, \dots, n. \tag{1.1}$$

In order for the moment equations (1.1) to hold, it is necessary that c_k be real whenever α_k is real, with a similar statement holding for the case that α_k is purely imaginary. Indeed, a purely imaginary moment condition can always be reduced to a real one, and henceforth we shall assume that this is the case. In fact, we assume that $\alpha_0, \dots, \alpha_{r-1}$ are real functions and $\alpha_r, \dots, \alpha_n$ are complex-valued functions whose real and imaginary parts, taking together with $\alpha_0, \dots, \alpha_{r-1}$, are linearly independent over $\mathbb{R}$.

Let $\mathfrak{P}$ be the real vector space that is the sum of the real span of $\alpha_0, \dots, \alpha_{r-1}$ and the complex span of $\alpha_r, \dots, \alpha_n$. Then, in particular, the real dimension of $\mathfrak{P}$ is $2n - r + 2$. Since the formulation of the generalized moment problem is defined in terms of a choice of basis of $\mathfrak{P}$, we will use the notation for a vector in $\mathfrak{P}$ interchangeably with the coefficients of this vector with respect to the given basis.

Let $\mathfrak{P}_+$ be the cone of all functions in $\mathfrak{P}$ that have positive real part, and let $\overline{\mathfrak{P}}_+$ be its closure. Since

$$\mathrm{Re} \sum_{k=0}^{n} c_k q_k = \int_a^b \mathrm{Re} \left\{ \sum_{k=0}^{n} q_k \alpha_k \right\} d\mu, \tag{1.2}$$

a necessary condition for the existence of a solution to the moment problem is that the sequence $c := (c_1, c_2, \dots, c_n)$ is nonnegative[1] in the sense that

$$\langle c, q \rangle := \mathrm{Re} \sum_{k=0}^{n} c_k q_k \geq 0 \tag{1.3}$$

for all $(q_0, q_1, \dots, q_n) \in \mathbb{R}^r \times \mathbb{C}^{n-r+1}$ such that

$$q := \sum_{k=0}^{n} q_k \alpha_k \in \overline{\mathfrak{P}}_+. \tag{1.4}$$

In this paper, we consider only solutions of the moment problem for which the measure $d\mu$ is positive. For such a solution to exist, it is necessary that c satisfies

[1] In [23] such a sequence is referred to as positive. What we shall refer to as positive throughout this paper is referred to as strictly positive in [23].

the condition

$$\langle c, q\rangle > 0, \quad \text{for all } q \in \overline{\mathfrak{P}}_+ \setminus \{0\}, \tag{1.5}$$

as is immediately seen from (1.2). Such sequences c are called *positive*. Denote by $\mathfrak{C}_+$ the cone of positive sequences. We shall assume that $\mathfrak{P}_+$ is nonempty. Then, since it is also open and convex, it is diffeomorphic to $\mathbb{R}^{2n-r+2}$; see, e.g., [9, p.771]. It follows from the classical theory [23] that $\mathfrak{C}_+$ is also nonempty, convex and open (see also Corollary 2.2), and, as such, it is diffeomorphic to a Euclidean space of the same dimension as $\mathfrak{P}_+$.

We note that, with the choice of basis for $\mathfrak{P}$ we have made, $Q := \operatorname{Re}\{q\}$ determines q. Indeed, taking the real part of (1.4), we obtain

$$Q = \sum_{k=0}^{r-1} q_k\alpha_k + \sum_{k=r}^{n} a_k\beta_k - \sum_{k=r}^{n} b_k\gamma_k,$$

where $\alpha_k = \beta_k + i\gamma_k$ and $q_k = a_k + ib_k$ for $k = r, r+1, \dots, n$. In particular, since these vectors are linearly independent, we can uniquely recover the coefficients of q with respect to this basis.

In this context, the most basic result in the classical literature on the moment problem [23] is that for every positive sequence there is a positive measure which solves the corresponding moment problem. This has fundamental consequences for classical moment problems such as the trigonometric moment problem studied by Schur and Caratheodory or the Nevanlinna-Pick interpolation problem, each of which arise in engineering applications. In these applications, however, it is important that the solution of the moment problem be expressible as a certain rational function or, more explicitly, as a ratio of functions in $\mathfrak{P}_+$. More generally, in a seminal paper Sarason interpreted Nevanlinna-Pick interpolation as a problem relating H^∞ of the disc and operator theory. In the case of a partial isometry, the corresponding H^∞ interpolant was also a ratio of two functions in a particular (coinvariant) subspace of H^2. A parameterization of all such interpolants corresponding to strict contractions has recently been given in [13].

In this paper, we first study the generalized moment problem in finite dimensions while retaining the complexity constraint

$$\frac{d\mu}{dt} = \frac{P(t)}{Q(t)}, \tag{1.6}$$

with $P = \operatorname{Re}\{p\}$ and $Q = \operatorname{Re}\{q\}$ where $p, q \in \mathfrak{P}_+$, and with p being preassigned. Later, in Sections 2 and 3 we will extend the range of P. We have previously shown [11, 12] that for each $c \in \mathfrak{C}_+$ and $p \in \mathfrak{P}_+$ there exists a unique $q \in \mathfrak{P}_+$ so that the generalized moment problem with the complexity constraint (1.6) is solvable. In this paper our first contribution is to show that this problem is well-posed in the sense of Hadamard, i.e., that the solution not only exists and is unique but is also continuous (in fact, smooth, where appropriate) with respect to the initial conditions. We previously have demonstrated this under the hypothesis that $\mathfrak{P} \subset C^2[a, b]$ [11, 12]. Recently, using the results obtained in [21], it is possible to prove this in the case $\mathfrak{P} \subset C^1[a,b]$. More generally, we make the following hypothesis concerning the cone $\mathfrak{P}$, which turns out to be a necessary condition; see Section 4.

(**H1**) *For each $q \in \partial\mathfrak{P}_+$, the integral*

$$\int_a^b \frac{dt}{Q}$$

diverges.

Remark 1.1. If $\mathfrak{P}$ consists of Lipschitz continuous functions, then $\mathfrak{P}$ satisfies hypothesis **H1**, generalizing our previously announced result for smooth functions.

Remark 1.2. If $\mathfrak{P}$ is spanned by a Chebyshev system (or T-system) [23] and contains a constant function, then, after a reparameterization, $\mathfrak{P}$ consists of Lipschitz continuous functions [23, p. 37], and thus satisfies hypothesis **H1**.

Theorem 1.3. *Suppose $\mathfrak{P}$ satisfies hypothesis* **H1**. *Let $p \in \mathfrak{P}_+$, and set $P := \text{Re}\{p\}$. Then the map $f^p : \mathfrak{P}_+ \to \mathfrak{C}_+$ sending $q \in \mathfrak{P}_+$ to $c := (c_1, c_2, \dots, c_n) \in \mathfrak{C}_+$, where*

$$c_k = \int_a^b \alpha_k(t) \frac{P(t)}{Q(t)} dt, \quad k = 0, 1, \dots, n, \tag{1.7}$$

and $Q := \text{Re}\{q\}$, is a diffeomorphism.

Theorem 1.4. *Suppose $\mathfrak{P}$ satisfies hypothesis* **H1**. *Let $c \in \mathfrak{C}_+$. Then the map $g^c : \mathfrak{P}_+ \to \mathfrak{P}_+$ sending $p \in \mathfrak{P}_+$ to $q = (f^p)^{-1}(c)$ is a diffeomorphism onto its image $\mathfrak{Q}_+$.*

Our approach follows from a differentiable viewpoint, so to speak, of the generalized moment problem. Indeed, parameterizing q via $q = \sum_{k=0}^n q_k \alpha_k$, we construct the 1-form

$$\omega = \text{Re}\left\{ \sum_{k=0}^n \left[c_k - f_k^p(q)\right] dq_k \right\},$$

on $\mathfrak{P}_+$. Explicitly, we have

$$\begin{aligned} \omega &= \text{Re}\left\{ \sum_{k=0}^n c_k dq_k - \int_a^b \sum_{k=0}^n \alpha_k \frac{P}{Q} dq_k dt \right\} \\ &= \text{Re} \sum_{k=0}^n c_k dq_k - \int_a^b \frac{P}{Q} dQ dt \end{aligned}$$

so taking the exterior derivative (on $\mathfrak{P}_+$) we obtain

$$d\omega = \int_a^b \frac{P}{Q^2} dQ \wedge dQ dt = 0,$$

establishing that the 1-form ω_c is closed.

Therefore, by the Poincaré Lemma, there exist a smooth function $\mathbb{J}$ such that, modulo a constant of integration,

$$\mathbb{J} = \int \omega = \int \left(\mathrm{Re} \sum_{k=0}^{n} c_k dq_k - \int_a^b \frac{P}{Q} dQ dt \right),$$

with the integral being independent of the path between two endpoints. Computing the path integral, one finds that

$$\mathbb{J}(q) = \langle c, q \rangle - \int_a^b P \log Q \, dt, \tag{1.8}$$

which is strictly convex and bounded from below for positive sequences $c_0, c_1, \ldots, c_n$ (Proposition 2.1). The functional $\mathbb{J}$ has an interior critical point precisely at the solution of the generalized moment problem. To see this, on the second factor of $\mathbb{R}^r \times \mathbb{C}^{n-r+1}$, we decompose the exterior differential as the sum $d = \partial + \bar{\partial}$, where $\bar{\partial}$ is the Cauchy-Riemann operator. Since $\mathbb{J}$ is real, to say that $d\mathbb{J} = 0$ is to say that $\partial\mathbb{J} = 0$ or, equivalently, that $\bar{\partial}\mathbb{J} = 0$. Finally, by inspection we see that $\partial\mathbb{J} = 0$ is the set of defining equations of the generalized moment problem.

Theorem 1.5. *Suppose $\mathfrak{P}$ satisfies hypothesis* **H1**. *Let $(p, c) \in \mathfrak{P}_+ \times \mathfrak{C}_+$, and set $P := \mathrm{Re}\{p\}$. Then the functional*

$$\mathbb{J} : \overline{\mathfrak{P}}_+ \to \mathbb{R} \cup \{\infty\},$$

given by (1.8)*, has a unique minimizer $\hat{q} \in \mathfrak{P}_+$, and $\hat{q} = (f^p)^{-1}(c)$, where $f^p : \mathfrak{P}_+ \to \mathfrak{C}_+$ is the map defined in Theorem* 1.3.

Remark 1.6. Modulo the technical discussions below, from the above discussion and results, we can see that, fixing a positive sequence c and a $p \in \mathfrak{P}_+$, there will always exist a $q \in \overline{\mathfrak{P}}_+$ that minimizes $\mathbb{J}$. One should even hope that this q solves the moment problem. The point of the above results is that q is actually an interior point i.e., $q \in \mathfrak{P}_+$. In fact, hypothesis **H1** is a necessary condition for the solution of the moment problem to be an interior point, as discussed in Section 4.

Remark 1.7. If we denote by $L^1_+[a, b]$ the set of absolutely integrable functions which are positive a.e., then Theorems 1.3 and 1.5 hold for $P \in L^1_+[a, b]$ and $q \in \mathfrak{P}_+$, *mutatis mutandis*, as we show in Section 3.

For several decades, it has been known that the rational covariance extension problem arising in spectral estimation and stochastic systems theory can be recast as the trigonometric moment problem [14]. In fact, the rational covariance extension problem is equivalent to this moment problem with the complexity constraint we have introduced. In this context, there are well-known designs, such as the Pisarenko filter, that p lies on the boundary of $\mathfrak{P}_+$. For such interpolation problems, Georgiou has shown [20] that, for each $p \in \partial\mathfrak{P}_+$, there exists a unique $q \in \overline{\mathfrak{P}}_+$ such that Q can only vanish at points where P vanishes to at least as high an order (so that P/Q is integrable). In [3] it was shown that such extended interpolation problems are well-posed (in the sense of Hadamard).

As we shall see in Section 3, the relationship between the moment problem and the optimization problem of Theorem 1.5 continues to hold even for $p \in \overline{\mathfrak{P}}_+ \setminus \{0\}$. Therefore, the function g^c can be extended as a function $g^c : \overline{\mathfrak{P}}_+ \setminus \{0\} \to \overline{\mathfrak{Q}}_+ \setminus \{0\}$ sending p to the corresponding solution q to the generalized moment problem with complexity constraint, a solution that is also a minimizer. For $p = 0$, the generalized moment problem with complexity constraint does not make sense, but the optimization problem does, reducing to the minimimization of a linear functional on a convex set. By definition, this problem has a unique solution at $q = 0$. Therefore, we define $g^c(0) = 0$. The assertion that this problem is well-posed depends on the zero structure of elements of $\mathfrak{P}_+ \setminus \{0\}$.

(**H2**) *For each $p \in \overline{\mathfrak{P}}_+ \setminus \{0\}$, the zero locus of $P := \mathrm{Re}\{p\}$ has measure zero.*

Remark 1.8. Every T-system satisfies hypothesis **H2**. In particular, this applies to the power moment problem and the trigonometric moment problem of odd degree.

Remark 1.9. The cones $\mathfrak{P}$ corresponding to the trigonometric moment problem of all degrees and the Nevanlinna-Pick interpolation problem satisfy hypothesis **H2**. More generally, finite-dimensional spaces of analytic functions satisfy hypothesis **H2**.

Theorem 1.10. *Suppose $\mathfrak{P}$ satisfies hypotheses* **H1** *and* **H2***, and define $g^c(0)$ to be zero. Let $c \in \mathfrak{C}_+$. Then, the extended map $g^c : \overline{\mathfrak{P}}_+ \to \overline{\mathfrak{Q}}_+$ is a homeomorphism. In fact, the moment problem* (1.1) *with the complexity constraint* (1.6) *has a unique solution $q \in \overline{\mathfrak{Q}}_+ \setminus \{0\}$ for each $p \in \overline{\mathfrak{P}}_+ \setminus \{0\}$ with the property that P/Q is integrable. For all $p \in \overline{\mathfrak{P}}_+$, the corresponding q is also the unique minimizer of the of the functional $\mathbb{J}$.*

2. Well-posedness of the generalized moment problem on $\mathfrak{P}_+$

Fix $c \in \mathfrak{C}_+$ and $p \in \mathfrak{P}_+$, and consider the strictly convex functional (1.8) defined on the closed convex set $\overline{\mathfrak{P}}_+$. We first note that $\mathbb{J}$ is bounded from below.

Proposition 2.1. *There exists an $\varepsilon_c > 0$ such that, for all nonzero $(p, q) \in \overline{\mathfrak{P}}_+ \times \overline{\mathfrak{P}}_+$,*

$$\mathbb{J}(q) \geq \varepsilon_c \|Q\|_\infty - \|P\|_1 \log \|Q\|_\infty, \tag{2.1}$$

where $P = \mathrm{Re}\{p\}$ and $Q = \mathrm{Re}\{q\}$.

Proof. The linear form $\langle c, q\rangle$ has a minimum, m_c, in the compact set $\{q \in \overline{\mathfrak{P}}_+ \mid \|q\|_\infty = 1\}$. Since $c \in \mathfrak{C}_+$, $m_c > 0$. Then, for an arbitrary $q \in \mathfrak{P}_+$,

$$\langle c, q\rangle = \langle c, \frac{q}{\|q\|_\infty}\rangle \|q\|_\infty \geq m_c \|q\|_\infty \geq \varepsilon_c \|Q\|_\infty$$

for a positive constant ε_c. Therefore,

$$\mathbb{J}(q) \geq \varepsilon_c \|Q\|_\infty - \int_a^b P \log\left(\frac{Q}{\|Q\|_\infty}\right) dt - \|P\|_1 \log \|Q\|_\infty.$$

Since the second term is nonnegative, (2.1) follows. □

Corollary 2.2. *The cone* $\mathfrak{C}_+$ *is open in* $\mathbb{R}^{2n-r+2}$.

Proof. The corollary follow immediately from $\langle c, q\rangle \geq m_c\|q\|$. ☐

Corollary 2.3. *For all* $r \in \mathbb{R}$ *the sublevel sets of* $\mathbb{J}^{-1}(-\infty, r]$ *are compact.*

Proof. Comparing linear to logarithmic growth in

$$r \geq \varepsilon_c\|Q\|_\infty - \|P\|_1 \log \|Q\|_\infty,$$

we see that the sublevel sets are bounded both from above and below. They are closed because they are the sublevel sets of a function. ☐

In particular, $\mathbb{J}$ has a unique minimum $\hat{q}$. We claim that $\hat{q} \in \mathfrak{P}_+$. From the theory of convex optimization it follows that to say $\hat{q}$ is the minimum is to say that

$$d\mathbb{J}_{\hat{q}}(q - \hat{q}) \geq 0, \quad \text{for all } q \in \overline{\mathfrak{P}}_+. \tag{2.2}$$

(See, e.g., [26, p. 264].) Next, choose $q - \hat{q} \in \mathfrak{P}_+$ and denote by $d\mu$ the positive measure $P(Q - \hat{Q})dt$. Then

$$d\mathbb{J}_{\hat{q}}(q - \hat{q}) = \langle c, q - \hat{q}\rangle - \int_a^b \frac{d\mu}{\hat{Q}}. \tag{2.3}$$

If $\hat{q} \in \partial\mathfrak{P}_+$, then, by hypothesis **H1**, the positive integral in (2.3) diverges to infinity, contradicting (2.2). Therefore $\hat{q} \in \mathfrak{P}_+$ and the stronger critical point condition

$$d\mathbb{J}_{\hat{q}} = 0 \tag{2.4}$$

is satisfied. Since this is the set of moment equations (1.1), Theorem 1.5 has been established.

We now turn to Theorem 1.3. By Theorem 1.5, f^p is a surjection. Because $\mathbb{J}$ is strictly convex, f^p is an injection. Moreover,

$$df_k^p(\alpha_j) = -\int_a^b \alpha_j \frac{P}{Q^2}\alpha_k dt. \tag{2.5}$$

Therefore, the Jacobian of f^p is a negative-definite, symmetric matrix, and, by the Implicit Function Theorem, it follows that f^c is a local diffeomorphism. Hence, since f^p is bijective, f^p is a diffeomorphism, thus proving Theorem 1.3.

Finally, fix $c \in \mathfrak{C}_+$ and consider the map g^c sending p to q. By Theorem 1.5, g^c is well-defined and, by definition, surjective.

Lemma 2.4. *The map* $g^c : \mathfrak{P}_+ \to \mathfrak{Q}_+$ *is injective.*

Proof. Suppose $q = g^c(p_1)$ and $q = g^c(p_2)$ for some $q \in \mathfrak{Q}_+$. Then

$$\int_a^b \alpha_k \frac{P_1 - P_2}{Q} dt = 0 \quad k = 1, 2, \ldots, n. \tag{2.6}$$

Now

$$P_i = \operatorname{Re}\left\{\sum_{k=1}^n p_k^{(i)}\alpha_k\right\}, \quad i = 1, 2.$$

Therefore, (2.6) yields

$$\operatorname{Re}\sum_{k=1}^{n}\left\{\left[p_k^{(1)}-p_k^{(2)}\right]\int_a^b \alpha_k \frac{P_1-P_2}{Q}dt\right\} = \int_a^b \frac{(P_1-P_2)^2}{Q}dt = 0,$$

which holds if and only if $P_1 = P_2$; i.e., if and only if $p_1 = p_2$. □

Theorem 2.5. *The map g^c is a diffeomorphism between smooth manifolds.*

Proof. Consider all pairs $(p,q) \in \mathfrak{P}_+ \times \mathfrak{P}_+$ satisfying $\varphi(p,q) = 0$, where the function $\varphi : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathbb{C}^n$ is given by

$$\varphi_k(p,q) = c_k - \int_a^b \alpha_k \frac{P}{Q}dt, \quad k = 0,1,\dots,n.$$

It is easy to see that

$$\frac{\partial \varphi_k}{\partial q_j} = \int_a^b \alpha_j \frac{P}{Q^2}\alpha_k dt$$

is the gramian of a positive definite quadratic form and therefore is positive definite. Consequently, by the Implicit Function Theorem, $g^c(p) = q$ is smooth, and its image $\mathfrak{Q}_+$ is an open smooth submanifold. Likewise,

$$\frac{\partial \varphi_k}{\partial p_j} = -\int_a^b \alpha_j \frac{1}{Q}\alpha_k dt$$

is negative definite, and hence p is locally a smooth function of q. Therefore, g^c is a local diffeomorphism. Since it is an injection, $g^c : \mathfrak{P}_+ \to \mathfrak{Q}_+$ is a diffeomorphism. □

This proves Theorem 1.4.

3. Continuous extension to the boundary of $\mathfrak{P}_+$

We now turn to the proof of Theorem 1.10. Fix $c \in \mathfrak{C}_+$. For $p \in \partial\mathfrak{P}_+ \setminus \{0\}$, we will construct a solution q to the generalized moment problem with complexitiy constraint by approximating p by a sequence (p_n) lying in $\mathfrak{P}_+$. The fact that q is independent of the sequence (p_n) is implied by the following result.

Main Lemma 3.1. *Suppose $\mathfrak{P}$ satisfies hypothesis* **H2**. *Let $(p_k, q_k) \in \overline{\mathfrak{P}}_+ \times \overline{\mathfrak{P}}_+$ be sequence of pairs that solve the generalized moment problem with complexity constraint. If $p_k \to p \neq 0$, then there exists a $q \neq 0$ such that $q_k \to q$ and*

1. *(p,q) solves the generalized moment problem with complexity constraint;*
2. *q is the (unique) minimizer of the corresponding functional $\mathbb{J}$.*

Proof. Computing the directional derivative of $\mathbb{J}$ at q in the direction h, we obtain

$$d\mathbb{J}_q(h) = \langle c,h\rangle - \operatorname{Re}\left\{\sum_{k=0}^{n} h_k \int_a^b \alpha_k \frac{P}{Q}dt\right\}.$$

Therefore, if $(\hat{p}, \hat{q})$ solves the generalized moment problem with complexity constraint, then

$$d\mathbb{J}_{\hat{q}}(q - \hat{q}) = 0, \quad \text{for all } q \in \overline{\mathfrak{P}}_+ \tag{3.1}$$

so that (2.2) is satisfied, and therefore $\hat{q}$ is the minimizer of $\mathbb{J}$. In particular, condition (1) implies condition (2).

Now, suppose that (p_k, q_k) is a sequence satisfying the hypotheses of the Main Lemma. We claim that

$$\|q_k\| \le M \tag{3.2}$$

for some $M > 0$. For this we need some notation. Let

$$\mathbb{J}_k(q) = \langle c, q\rangle - \int_a^b P_k \log Q\ dt$$

be the functionals corresponding to p_k, and let $\mathbb{J}$ be the functional corresponding to p. Suppose the sequence $(\|q_k\|)$ is unbounded. Then there is a subsequence, which we shall also denote by (q_k), for which $\|Q_k\| > 1$ and $\|Q_k\| \to \infty$.

Choose an arbitrary, but fixed, $\tilde{q} \in \mathfrak{P}_+$. By optimality, $\mathbb{J}_k(\tilde{q}) \ge \mathbb{J}_k(q_k)$. Since $\log \tilde{Q}$ is continuous on the interval $[a, b]$, and since $p_k \to p$,

$$\mathbb{J}_k(\tilde{q}) \to \mathbb{J}(\tilde{q}).$$

Therefore, there exists a positive constant L such that $L \ge \mathbb{J}_k(\tilde{q})$ for all k. Similarly, there is a positive constant N such that $\|P_k\|_1 \le N$. Combining these inequalities with (2.1), we obtain the inequality

$$L \ge \mathbb{J}_k(q_k) \ge \varepsilon_c \|Q_k\|_\infty - N \log \|Q_k\|_\infty. \tag{3.3}$$

Comparing linear and logarithmic growth in (3.3), we see that $\|q_k\|$ is bounded from above, contrary to hypothesis.

Suppose that (q_{k_j}) is a convergent subsequence with limit q^*. Since $p_k \to p \ne 0$ and (p_k, q_k) satisfy the moment equations with a fixed $c \in \mathfrak{C}_+$, $q^* \ne 0$. Choosing a $v \in \mathfrak{P}_+$ such that

$$\rho := \operatorname{Re} \sum v_\ell \alpha_\ell > 0,$$

the integral

$$\int_a^b \rho \frac{P_{k_j}}{Q_{k_j}} dt = \langle c, v\rangle$$

is bounded, and hence P_{k_j}/Q_{k_j} is integrable. Except on a set of measure zero, because of hypothesis **H2** we have

$$\liminf_{k_j \to \infty} \frac{P_{k_j}}{Q_{k_j}} = \lim_{k_j \to \infty} \frac{P_{k_j}}{Q_{k_j}} = \frac{P}{Q^*}$$

so that, by Fatou's Lemma,

$$\int_a^b \alpha_\ell \frac{P}{Q^*} dt \le c_\ell = \liminf_{k_j \to \infty} \int_a^b \alpha_\ell \frac{P_{k_j}}{Q_{k_j}} dt.$$

Choosing a $v \in \mathfrak{P}_+$ as before, we obtain

$$\int_a^b \rho \frac{P}{Q^*} dt \leq \langle c, v \rangle,$$

and hence P/Q^* is integrable. By the Dominated Convergence Theorem, P_{k_j}/Q_{k_j} converges to P/Q^* in $L_1[a,b]$. Moreover, since each α_ℓ is continuous, and hence bounded on $[a,b]$, we have

$$c_\ell = \lim_{k_j \to \infty} \int_a^b \alpha_\ell \frac{P_{k_j}}{Q_{k_j}} dt = \int_a^b \alpha_\ell \frac{P}{Q^*} dt.$$

In particular, P/Q^* satisfies the moment problem, and hence q^* is the minimizer of $\mathbb{J}$. Therefore, the bounded sequence (q_k) has a unique cluster point $q = q^*$. □

Suppose $p \in \partial\mathfrak{P}_+ \setminus \{0\}$ and that (p_k) is a sequence in $\mathfrak{P}_+$ that tends to p. Moreover, suppose that $\mathfrak{P}$ satisfies hypotheses **H1** and **H2**. Then, by Theorem 1.5, there is a sequence (q_k) in $\mathfrak{P}_+$ such that each pair (p_k, q_k) satisifies the generalized moment problem with complexity constraint, and, by the Main Lemma, there exists a unique q so that (p,q) solves the moment problem. Since q is the unique minimizer of the optimization problem corresponding to p, it follows that q is independent of the approximating sequence (p_k). Therefore, the map g^c extends to a well-defined mapping on $\overline{\mathfrak{P}}_+$.

Now, suppose $p_k \to p$ in $\overline{\mathfrak{P}}_+ \setminus \{0\}$, where (p_k) may have infinitely many terms in $\partial\mathfrak{P}_+$. We know that to each p_k and to p corresponds a q_k and a q, which together form a solution to the moment problem. Therefore, by the Main Lemma, $q_k \to q$ so that g^c extends to continuous map on $\overline{\mathfrak{P}}_+ \setminus \{0\}$.

The proof of Lemma 2.4 shows that g^c extends to an injection on $\overline{\mathfrak{P}}_+ \setminus \{0\}$. To show that g^c is a homeomorphism it remains to prove that it is surjective and that $(g^c)^{-1}$ is continuous. This will follow if we can establish that g^c is proper; i.e., $(g^c)^{-1}(K)$ is compact for all compact $K \subset \overline{\mathfrak{Q}}_+$.

Lemma 3.2. *The map* $g^c : \overline{\mathfrak{P}}_+ \setminus \{0\} \to \overline{\mathfrak{Q}}_+ \setminus \{0\}$ *is proper.*

Proof. We first observe that the optimization problem can be rescaled. In fact, the functional

$$\mathbb{J}^\lambda(q) = \langle c, \lambda q \rangle - \int_a^b \lambda P \log \lambda Q \; dt$$

has the same minimizers as (1.8). Hence we can restrict our attention to $p \in \overline{\mathfrak{P}}_+$ such that $\|p\|_\infty = 1$. Let $M_1 \subset \overline{\mathfrak{P}}_+$ be the space of such p. Hence the diagram

$$\begin{array}{ccc}
\overline{\mathfrak{P}}_+ \setminus \{0\} & \xrightarrow{g^c} & \overline{\mathfrak{Q}}_+ \setminus \{0\} \\
{\scriptstyle \frac{p}{\|p\|}} \downarrow & & \downarrow {\scriptstyle \frac{g^c(p)}{\|p\|}} \\
M_1 & \xrightarrow{g^c} & g^c(M_1)
\end{array}$$

commutes. Here the restriction $g^c : M_1 \to \overline{\mathfrak{Q}}_+$ is continuous, 1-1 and onto its image $g^c(M_1)$. The space M_1 is compact, and, since it is the continuous image of a compact set, so is $g^c(M_1)$, Hence, $g^c : M_1 \to g(M_1)$ is also proper, implying that it is a homeomorphism.

Let $q \in \overline{\mathfrak{Q}}_+ \setminus \{0\}$ be arbitrary, and let (q_n) be a sequence in $g^c(\overline{\mathfrak{P}}_+ \setminus \{0\})$ that converges to q. It is then bounded. We want to show that the sequence (p_n), defined by $p_n := (g^c)^{-1}(q_n)$, cannot tend to infinity. Now, by compactness, the sequence $(\tilde{q}_n)$, defined by

$$\tilde{q}_n := \frac{q_n}{\|p_n\|_\infty},$$

tends to a limit $\tilde{q}$. Since (q_n) is bounded, $\tilde{q}$ would be zero, if $\|p_n\|_\infty \to \infty$. However, if $\tilde{q}$ were zero, then so is $\tilde{p} := (g^c)^{-1}(\tilde{q})$, which is a contradiction. Hence, $g^c : \overline{\mathfrak{P}}_+ \setminus \{0\} \to \overline{\mathfrak{Q}}_+ \setminus \{0\}$ is proper, as claimed. □

Lemma 3.2 implies that the map g^c is surjective. Indeed, if $q \in \overline{\mathfrak{Q}}_+ \setminus \{0\}$, then q is the limit of the sequence (q_k) in $\mathfrak{Q}_+$. This sequence is the image of a sequence (p_k) in $\mathfrak{P}_+$. Moreover, the preimage of (q_k) and q is compact and contains (p_k), which therefore has a convergent subsequence $(p_{k_j}) \to p$ for some $p \in \overline{\mathfrak{P}}_+ \setminus \{0\}$. This implies

$$q_{k_j} = g^c(p_{k_j}) \to g^c(p) = q$$

so that g^c is surjective and therefore has an inverse $(g^c)^{-1}$. Since g^c is proper, it is a closed mapping, and therefore $(g^c)^{-1}$ is continuous.

We have shown that $g^c : \overline{\mathfrak{P}}_+ \setminus \{0\} \to \overline{\mathfrak{Q}}_+ \setminus \{0\}$ is a homeomorphism. Suppose (p_k, q_k) solves the same moment problem. Then $p_k \to 0$ if and only if $q_k \to 0$. In particular, with the convention $g^c(0) = 0$, in harmony with the optimization problem, g^c extends to a homeomorhism of $\overline{\mathfrak{P}}_+$ with $\overline{\mathfrak{Q}}_+$. This concludes the proof of Theorem 1.10.

Theorem 1.10 can be generalized along the lines of of Remark 1.7 at the price of giving up well-posedness. In fact, the assumption that $p \in \overline{\mathfrak{P}}_+ \setminus \{0\}$ is only used to show injectivity. Therefore we have the following result, which will be used in Section 5.

Theorem 3.3. *Suppose* $\mathfrak{P}$ *satisfies hypotheses* **H1** *and* **H2**, *and let* $c \in \mathfrak{C}_+$. *Then, for all* $P \in \overline{L^1_+[a,b]}$, *the moment problem* (1.1) *with the complexity constraint* (1.6) *has a unique solution* $q \in \overline{\mathfrak{P}}_+ \setminus \{0\}$ *with the property that* P/Q *is integrable. For all* $P \in \overline{L^1_+[a,b]}$, *the corresponding* q *is also the unique minimizer of* $\mathbb{J}$.

Theorem 3.3 follows, *mutatis mutandis*, from the first half of the proof of Theorem 1.10. To see this, first note that the extensions of Theorems 1.3 and 1.5, announced in Remark 1.7, to the case where $P \in L^1_+[a,b]$ reposes on the observation that the proof of Proposition 2.1 extends to this case. Then, the Main Lemma (as well as the subsequent two paragraphs) is modified by considering a sequence (P_k) in $\overline{L^1_+[a,b]}$ converging to P in L^1.

4. Necessity of hypothesis H1

For simplicity, set $[a, b] = [-1, 1]$. The heart of the construction is as follows. Set $\alpha_0 = 1$ (or, more generally, any positive element in the function space $\mathfrak{P}$) and choose α_1 to be a nonnegative function that vanishes at zero, but with its reciprocal being integrable with finite integral value v. By choice, α_0 and α_1 are linearly independent. Let $\mathfrak{P} = \text{span}_{\mathbb{R}}\{\alpha_0, \alpha_1\}$. Of course, $\alpha_0 \in \mathfrak{P}_+$ and $\alpha_1 \in \partial\mathfrak{P}_+$. Now choose $p = 1$ and $q = \alpha_1$. Then define $c = (c_0, c_1) = (v, 2)$ via (1.7), which, by construction, is a positive sequence. For this c, however, although $p \in \mathfrak{P}_+$, the corresponding $q \in \partial\mathfrak{P}_+$. This proves necessity of hypothesis **H1**.

Example. Consider $\mathfrak{P} = \text{span}\{1, |t|^{1/2}\}$. In this case, positive functions correspond to to values of q_0, q_1 for which $q_0 + q_1|t|^{1/2} > 0$ for all $t \in [-1, 1]$. That is, $\mathfrak{P}_+$ is the open convex set defined by the inequalities $q_0 > 0$ and $q_1 > -q_0$. The sequence $c = (4, 2)$ is positive because $4q_0 + 2q_1 > 0$ on $\mathfrak{P}_+$. For the choice $p = 1$ and this positive sequence, the generalized moment problem with complexity constraint is solved uniquely by $q(t) = |t|^{1/2}$, which lies on the boundary of $\mathfrak{P}_+$.

5. The primal problem

In this section, generalizing the results described in [12], we use Theorem 3.3 to analyse a primal optimization problem which has as its dual the minimization of the functional $\mathbb{J}$. It is worth noting that the solution to the primal optimization problem automatically satisfies the complexity constraint (1.6).

Theorem 5.1. *Suppose that $c \in \mathfrak{C}_+$ and that $\mathfrak{P}$ satisfies hypotheses* **H1** *and* **H2**. *For any $P \in \overline{L^1_+[a, b]} \setminus \{0\}$, the constrained optimization problem to minimize the functional*

$$\mathbb{I}(\Phi) = \int_a^b P(t) \log \frac{P(t)}{\Phi(t)} dt, \tag{5.1}$$

over $\overline{L^1_+[a, b]}$ subject to the constraints

$$\int_a^b \alpha_k(t)\Phi(t)dt = c_k, \quad k = 0, 1, \ldots, n, \tag{5.2}$$

has a unique solution, and it has the form

$$\Phi = \frac{P}{Q}, \quad Q := \text{Re}\{q\},$$

where $q \in \overline{\mathfrak{P}}_+$ is the unique minimum of (1.8). *If $P \in L^1_+[a, b]$, hypothesis* **H2** *is not needed, and the unique minimum of* (1.8) *lies in $\mathfrak{P}_+$.*

Proof. By Jensen's inequality, $\mathbb{I}(\Phi) \geq -\|P\|_1 \log (\|\Phi\|_1/\|P\|_1)$, and hence the functional is bounded from below. Form the Lagrangian

$$L(\Phi, q) = -\mathbb{I}(\Phi) + \text{Re} \sum_{k=0}^{n} q_k \left[c_k - \int_a^b \alpha_k \Phi dt \right],$$

where $(q_0, q_1, \dots, q_n) \in \mathbb{R}^r \times \mathbb{C}^{n-r+1}$ are Lagrange multipliers. Then, defining

$$Q = \mathrm{Re}\{q\}, \quad \text{where } q = \sum_{k=0}^{n} q_k \alpha_k,$$

we obtain

$$L(\Phi, q) = \int_a^b P \log \frac{\Phi}{P}\, dt + \langle c, q \rangle - \int_a^b Q\Phi dt.$$

Clearly, the dual functional

$$\psi(q) = \sup_{\Phi \in \overline{L^1_+[a,b]}} L(\Phi, q)$$

takes finite values only if $q \in \overline{\mathfrak{P}}_+$, so we restrict our attention to such Lagrange multipliers.

First, consider the case that $P \in L^1_+[a,b]$. For any $q \in \overline{\mathfrak{P}}_+$ and any $\Phi \in L^1_+[a,b]$ such that P/Φ is integrable, the directional derivative

$$d_{(\Phi,q)} L(h) = \int_a^b \left[\frac{P}{\Phi} - Q \right] h dt = 0$$

for all $h \in L^1[a,b]$ if and only if

$$\Phi = \frac{P}{Q} \in L^1_+[a,b],$$

which inserted into the dual functional yields

$$\psi(q) = \langle c, q \rangle - \int_a^b P \log Q dt - \int_a^b P dt.$$

Since the last term is constant, the dual problem to minimize $\psi(q)$ over $\overline{\mathfrak{P}}_+$ is equivalent to the optimization problem

$$\min_{q \in \overline{\mathfrak{P}}_+} \mathbb{J}(q),$$

which, by Theorem 1.5 generalized as in Remark 1.7, has a unique minimizer $\hat{q} \in \mathfrak{P}_+$ satisfying the moment conditions (5.2) with Φ given by

$$\hat{\Phi} := \frac{P}{\hat{Q}} \in L^1_+[a,b]. \tag{5.3}$$

Since the function $\Phi \mapsto L(\Phi, \hat{q})$ is strictly concave and

$$dL_{(\hat{\Phi},\hat{q})}(h) = \int_a^b \left[\frac{P}{\hat{\Phi}} - \hat{Q} \right] h\, dt = 0, \tag{5.4}$$

for all $h \in L^1[a,b]$, we have

$$L(\Phi, \hat{q}) \le L(\hat{\Phi}, \hat{q}) \tag{5.5}$$

for all $\Phi \in L^1_+[a,b]$, with equality if and only if $\Phi = \hat{\Phi}$.

However, $L(\Phi,\hat{q}) = -\mathbb{I}(\Phi)$ for all Φ satisfying the moment conditions (5.2). In particular, since (5.2) holds with $\Phi = \hat{\Phi}$, $L(\hat{\Phi},\hat{q}) = -\mathbb{I}(\hat{\Phi})$. Consequently, (5.5) implies that, for all $\Phi \in L^1_+[a,b]$ satisfying the moment conditions,

$$\mathbb{I}(\Phi) \geq \mathbb{I}(\hat{\Phi}), \tag{5.6}$$

with equality if and only if $\Phi = \hat{\Phi}$. Hence, $\mathbb{I}$ has a unique minimum in the space of $\Phi \in L^1_+[a,b]$ satisfying the constraints (5.2), and it is given by (5.3).

Next, consider the case that $P \in \overline{L^1_+[a,b]} \setminus \{0\}$. By Theorem 3.3, the functional $\mathbb{J}$ has a unique minimizer $\hat{q} \in \overline{\mathfrak{P}}_+ \setminus \{0\}$ such that

$$\hat{\Phi} := \frac{P}{\hat{Q}} \in \overline{L^1_+[a,b]} \tag{5.7}$$

satisfies the moment condition (5.2), and thus $L(\hat{\Phi},\hat{q}) = -\mathbb{I}(\hat{\Phi})$. Then, (5.4) holds for all $h := \Phi - \hat{\Phi}$ such that $\Phi \in \overline{L^1_+[a,b]}$, and hence (5.5) holds for all $\Phi \in \overline{L^1_+[a,b]}$, with equality if and only if $\Phi = \hat{\Phi}$. Consequently, (5.6) is satisfied for all for all $\Phi \in \overline{L^1_+[a,b]}$ satisfying the moment conditions (5.2), with equality if and only if $\Phi = \hat{\Phi}$. □

6. Examples

We illustrate our results with a number of examples.

Example. The trigonometric moment problem is a basic moment problem that corresponds to the interval $[a,b] = [-\pi,\pi]$ and the choice of basis

$$\alpha_k(\theta) = e^{ik\theta}, \quad k = 0,1,\dots,n.$$

It is easy to see that, with this basis, $\mathfrak{P}$ satisfies **H1** and **H2**. Also the moment sequence $c = (c_0, c_1, \dots, c_n) \in \mathfrak{C}_+$ if and only if the Toeplitz matrix

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ \bar{c}_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \cdots & c_0 \end{bmatrix}$$

is positive definite [23]. In many applications (see, e.g., [10]), we want to consider only solutions that are rational functions of degree at most n. In our present setting, this degree constraint is enforced by imposing the complexity constraint (1.6) with $P \in \overline{\mathfrak{P}}_+$.

Example. A Carathéodory function is an analytic function in the open unit disc that maps points there into the open left half-plane. Given $n+1$ distinct points $z_0, z_1, \dots, z_n$ in the open unit disc, consider the problem to determine the rational Carathéodory functions f of degree at most n satisfying the interpolation condition

$$f(z_k) = c_k, \quad k = 0,1,\dots,n, \tag{6.1}$$

where $c_0, c_1, \dots, c_n$ are prescribed values in the open right half of the complex plane with c_0 real. This Nevanlinna-Pick interpolation problem differs from the classical one in that a degree constraint on the interpolant f has been introduced, a restriction motivated by applications [10, 8]. In fact, many problems in systems and control can be reduced to Nevanlinna-Pick interpolation (see, e.g., [15, 16]), and, as the interpolant generally can be interpreted as a transfer function, the bound on the degree is a natural complexity constraint.

To reformulate this interpolation problem as a generalized moment problem, we note that, by the Herglotz Theorem,

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\Phi(e^{i\theta})d\theta, \quad \Phi(e^{i\theta}) = \mathrm{Re}\{f(e^{i\theta})\}, \tag{6.2}$$

and hence $[a, b] = [-\pi, \pi]$ and, for $k = 0, 1, \dots, n$,

$$\alpha_k(\theta) = \frac{1}{2\pi}\frac{e^{i\theta}+z_k}{e^{i\theta}-z_k}. \tag{6.3}$$

The corresponding vector space $\mathfrak{P}$ satisfies **H1** and **H2**. Moreover, $c \in \mathfrak{C}_+$ if and only if the Pick matrix

$$\left[\frac{c_k+\bar{c}_\ell}{1-z_k\bar{z}_\ell}\right]_{k,\ell=0}^{n}$$

is positive definite [23]. To consider only solutions satisfying the nonclassical degree constraint $\deg(f) \leq n$, we impose the complexity constraint (1.6) with the restriction that $p \in \overline{\mathfrak{P}}_+$.

If the interpolation points $z_0, z_1, \dots, z_n$ are not distinct, the interpolation conditions are modified in the following way. If $z_k = z_{k+1} = \cdots = z_{k+m-1}$, the corresponding interpolation conditions are replaced by

$$f(z_k) = c_k, \quad f'(z_k) = c_{k+1}, \quad \dots, \quad \frac{1}{(m-1)!}f^{(m-1)}(z_k) = c_{k+m-1}$$

Differentiating (6.2), we obtain the corresponding basis functions, namely (6.3) and

$$\alpha_{k+1}(\theta) = \frac{1}{2\pi}\frac{2e^{i\theta}}{(e^{i\theta}-z_k)^2}, \quad \dots, \quad \alpha_{k+m-1}(\theta) = \frac{1}{2\pi}\frac{2e^{i\theta}}{(e^{i\theta}-z_k)^m}.$$

As before, the degree constraint corresponds to $p \in \overline{\mathfrak{P}}_+$.

Example. A well-known method in systems identification amounts to estimating the first $n+1$ coefficients in an orthogonal basis function expansion

$$G(z) = \frac{1}{2}c_0 f_0(z) + \sum_{k=1}^{\infty} c_k f_k(z)$$

of a transfer function $G(z)$ [28], where the functions $f_0, f_1, f_2, \dots$ are orthonormal on the unit circle. Given the estimated coefficients $c_0, c_1, \dots, c_n$, the usual problem considered in the literature [27] is to find a rational function G of smallest degree which match these coefficients. Here, however, we consider the corresponding problem where G is a Carathéodory function of degree at most n. This problem remained open for a long time but has recently been resolved using the methods that we shall describe next [4, 5].

Defining

$$\alpha_k(\theta) = f_k(e^{i\theta}), \quad k = 0, 1, \dots, n,$$

this problem can be reformulated as a generalized moment problem with complexity constraint by observing that

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_k(e^{i\theta}) \Phi(e^{i\theta})\, d\theta, \quad k = 0, 1, \dots, n,$$

where $\Phi(e^{i\theta}) = 2\mathrm{Re}\{G(e^{i\theta})\}$. As in Example 6, the degree constraint is enforced by choosing $P \in \mathfrak{P}_+$. The vector space $\mathfrak{P}$ satisfies **H1** and **H2** for any of the usual choices of orthogonal basis.

Example. Finally, consider the power moment problem obtained by choosing

$$\alpha_k(t) = t^k, \quad k = 0, 1, \dots, n,$$

which again defines a space $\mathfrak{P}$ satisfiying **H1** and **H2**, and let $P \in L^1_+[a, b]$ be a probability density. Then, the function

$$\mathbb{S}(\Phi, P) = \mathbb{I}(\Phi)$$

is the Kullback-Leibler distance between Φ and P [24]. Then the optimization problem of Theorem 5.1 is equivalent to minimizing $\mathbb{S}(\Phi, P)$ subject to the moment conditions (1.1). This gives an interesting interpretation to the present problem: Given an a priori probability density P, we want to find another probability density Φ that has prescribed moments up to order n and that minimizes the Kullback-Leibler distance to P, generalizing maximum entropy methods.

References

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publishing, New York, 1965.

[2] N.I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments*, American Mathematical Society, Providence, Rhode Island, 1962.

[3] A. Blomqvist, G. Fanizza and R. Nagamune, Computation of bounded degree Nevanlinna-Pick interpolants by solving nonlinear equations, *Proc. 42nd IEEE Conf. Decision and Control* (2003), 4511–4516.

[4] A. Blomqvist and G. Fanizza, Identification of rational spectral densities using orthonormal basis functions, *Proc. 2003 Symposium on System Identification*, 2003.

[5] A. Blomqvist and B. Wahlberg, A data driven orthonormal parameterization of the generalized entropy maximization problem, *Proc. 16th International Symposium on Mathematical Theory of Networks and Systems*, 2004.

[6] C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Matveev, A complete parameterization of all positive rational extensions of a covariance sequence, *IEEE Trans. Automat. Control*, **40** (1995), 1841–1857.

[7] C. I. Byrnes, S. V. Gusev, and A. Lindquist, A convex optimization approach to the rational covariance extension problem, *SIAM J. Contr. and Optimiz.* **37** (1998) 211–229.

[8] C. I. Byrnes, T. T. Georgiou, and A. Lindquist, A generalized entropy criterion for Nevanlinna-Pick interpolation with degree constraint, *IEEE Trans. Automat. Control* **46** (2001), 822–839.

[9] C. I. Byrnes and A. Lindquist, On the duality between filtering and Nevanlinna-Pick interpolation, *SIAM J. Contr. and Optimiz.* **39** (2000), 757–775.

[10] C. I. Byrnes, S. V. Gusev, and A. Lindquist, From finite covariance windows to modeling filters: A convex optimization approach, *SIAM Review* **43** (2001), 645–675.

[11] C. I. Byrnes and A. Lindquist. Interior point solutions of variational problems and global inverse function theorems. Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden, Report TRITA/MAT-01-OS13, 2001.

[12] C. I. Byrnes and A. Lindquist, A convex optimization approach to generalized moment problems, *Control and Modeling of Complex Systems: Cybernetics in the 21st Century: Festschrift in Honor of Hidenori Kimura on the Occasion of his 60th Birthday*, K. Hashimoto, Y. Oishi and Y. Yamamoto, Editors, Birkhäuser, 2003, 3–21.

[13] C.I. Byrnes, T.T. Georgiou, A. Lindquist, and A. Megretski, Generalized interpolation in H^∞ with a complexity constraint, *Trans. of the American Math. Society*, to to appear (electronically published on December 9, 2004).

[14] P. Delsarte, Y. Genin, Y. Kamp, and P. van Dooren, Speech modelling and the trigonometric moment problem, *Philips J. Res.* **37** (1982), 277–292.

[15] Ph. Delsarte, Y. Genin and Y. Kamp, *On the role of the Nevanlinna-Pick problem in circuits and system theory*, Circuit Theory and Applications **9** (1981), 177–187.

[16] J. C. Doyle, B. A. Frances and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan Publ. Co., New York, 1992.

[17] T.T. Georgiou, *Partial Realization of Covariance Sequences*, Ph.D. thesis, CMST, University of Florida, Gainesville 1983.

[18] T. T. Georgiou, Realization of power spectra from partial covariance sequences, *IEEE Trans. Acoustics, Speech and Signal Processing* **35** (1987), 438–449.

[19] T. T. Georgiou, A topological approach to Nevanlinna-Pick interpolation, *SIAM J. Math. and Anal.* **18** (1987), 1248–1260.

[20] T. T. Georgiou, The interpolation problem with a degree constraint, *IEEE Trans. Automat. Control* **44** (1999), 631–635.

[21] T.T. Georgiou, Solution of the general moment problem via a one-parameter imbedding, *IEEE Trans. on Automatic Control*, to be published.

[22] U. Grenander and G. Szegö, *Toeplitz Forms and their Applications*, Univ. California Press, 1958.

[23] M.G. Krein and A.A. Nudelman, *The Markov Moment Problem and Extremal Problems*, American Mathematical Society, Providence, Rhode Island, 1977.

[24] S. Kullback, *Information Theory and Statistics*, John Wiley, New York, 1959.

[25] J. W. Milnor, *Topology from Differentiable Viewpoint*, Revised Edition, Princeton University Press, Princeton, New Jersey, 1997.

[26] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[27] Z. Szabó, P. Heuberger, J. Bokar and P. An den Hof, Extended Ho-Kalman algorithm for systems represented in generalized orthonormal bases, *Automatica* **36** (2000), 1809–1818.

[28] B. Wahlberg, Systems identification using Laguerre models, *IEEE Trans. Automatic Control* **AC-36** (1991), 551–562.

Christopher I. Byrnes
Department of Electrical and Systems Engineering
Washington University
St. Louis, Missouri 63130
USA
e-mail: chrisbyrnes@seas.wustl.edu

Anders Lindquist
Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm
Sweden
e-mail: alq@math.kth.se

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