Modeling of Low Rank Time Series

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***Abstract*— Rank-deficient stationary stochastic vector processes are present in many problems in network theory and dynamic factor analysis. In this paper we study hidden dynamical relations between the components of a discrete-time stochastic vector process and investigate their properties with respect to stability and causality. More specifically, we construct transfer functions with a full-rank input process formed from selected components of the given vector process and having a vector process of the remaining components as output. An important question, which we answer in the negative, is whether it is always possible to find such a deterministic relation that is stable. We also show how our results could be used to investigate the structure of dynamic network models and the latent low-rank stochastic process in a dynamic factor model.**

***Index Terms*— modeling, rank-deficient processes, dynamic factor models, stability, Granger causality**

I. Introduction

The basic topic of this paper is modeling of discrete-time stochastic vector processes with rank-deficient spectral densities, i.e., reduced-rank (or low rank) processes. In some literature such processes are also called sparse (or singular) signals. Processes of this kind are encountered in many practical applications where the data set is large with many correlated variables. In such applications available results for full-rank cases are no longer applicable, and therefore, as the theory of full-rank processes has become mature, many researchers have turned to rank-deficient processes in recent years, often in the form of different system realizations; see [1] for applications in navigation and [2].

Rank-deficiency processes may appear in networks where each component of the vector process is represented by a node [3]–[7]. Such dynamic network models are often needed in areas like econometrics, biology and engineering [8]–[10]. Rank-deficiency processes also play an important role in dynamic factor models [11]–[14] and generalized factor analysis (GFA) models [15], [16], where there are latent processes with singular spectral densities. Research on modeling and estimation of models [11], [13] with singular latent processes often aim at representing the latent process as a product of a minimal common factor and some gains, called factor loadings [15]. In econometrics singular AR and ARMA systems are important in the context of latent processes modeling [11], [13], systems stability [12], [14] and DSGE (dynamic stochastic general equilibrium) models. Singular processes are also studied for state space models to extract the dynamical relations between the correlated entries [17]–[19], [21] which relates to Granger causality [22], [23].

Granger causality frequently appears in the context of feedback and was originally proposed for economics, then applied in control and information [21], [24]–[26] and neurophysiological systems [27], revealing the causal relations between the entries. In addition, identification also needs to be considered when modeling the low rank processes, so that the approaches guarantee identifiability, rapidity, as well as the recovering of the real dynamical relations [18], [28]. Identification of low rank processes is considered in [19], [20].

Let $\{\zeta(t), t \in \mathbb{Z}\}$ be a stationary p-dimensional discrete-time stochastic vector process with a rational spectral density $\Phi(e^{i\theta})$ of rank $m < p$. Then by rearranging the components of ζ if necessary, there is a decomposition

$$\zeta = \begin{bmatrix} u \\ y \end{bmatrix} \tag{1}$$

with u an m-dimensional process with a spectral density $\Phi_u(e^{i\theta})$ of full rank m. As u and y are jointly stationary, there is a natural decomposition of $\Phi(z)$, namely

$$\Phi(z) = \begin{bmatrix} \Phi_u(z) & \Phi_{uy}(z) \\ \Phi_{yu}(z) & \Phi_y(z) \end{bmatrix}. \tag{2}$$

An important question then is whether there is a deterministic dynamic relation between u and the $p-m$-dimensional vector process y. More precisely, with z the time shift operator such that $zu(t) = u(t+1)$, is there a proper rational transfer function $F(z)$ taking the input u to the output y as in Figure 1, and, if so, how is it determined? Such deterministic relations

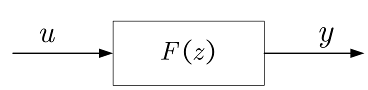

Fig. 1. Dynamical relation from $\boldsymbol{u}$ to $\boldsymbol{y}$

were studied in [17] in the context of sampling, which leads to increase of rank. In dynamic factor analysis u can play the role of a minimal static factor [11].

The $p \times p$ spectral density $\Phi(e^{i\theta})$ of rank m has a $p \times m$ stable spectral factor

$$W(z) = C(zI-A)^{-1}B + D, \tag{3}$$

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such that

$$W(z)W(z^{-1})' = \Phi(z). \tag{4}$$

Note that we do not require (3) to be minimum phase. In fact, we shall also cover the case when $D = 0$, in which case $W(z)$ has a zero at infinity. Then ζ has a minimal stochastic realization

$$x(t+1) = Ax(t) + Bw(t) \tag{5a}$$
$$\zeta(t) = Cx(t) + Dw(t) \tag{5b}$$

with $C \in \mathbb{R}^{p\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{p\times m}$ and $\{w(t), t \in \mathbb{Z}\}$ an m-dimensional normalized white noise process such that $\mathbb{E}\{w(t)w(t)'\} = I\delta_{ts}$. Moreover, A is a stability matrix, i.e., having all eigenvalues in the unit disc, and (C, A) and (A, B) are observable and reachable pairs, respectively [29].

For the case $D = 0$, the construction of $F(z)$ is quite straightforward. This was done in [17] and [18] for continuous-time systems and in [19] in the context of system identification. However, when $D \neq 0$, the situation is much more complicated, especially when the rank of D is smaller than m. In Section IV we shall present a realization for $F(z)$ of degree at most n.

It is important to understand that the transfer function $F(z)$ in Figure 1 need not be stable, although both u and y are stationary processes. In fact, u is in general affected by y through feedback. In general the components of ζ can be rearranged in several different ways, thus producing different u and y while upholding the requirement that u has full rank m. An important question is whether there is a choice for which $F(z)$ is a stability matrix. In this paper we shall demonstrate by way of counterexample that this is not always possible.

Section II will be devoted to feedback models, inserting the deterministic dynamic system in Figure 1 in a stochastic feedback environment. In Section III we explore the connections to dynamic network models. In Section IV we present the construction of $F(z)$ and investigate its properties. Section V deals with stability and causality, and in Section VI we consider the design of the feedback environment needed for internal stability, thus connecting to robust control [30]. In Section VII we show how our results could be used to investigate the structure of the latent low-rank stochastic process in a dynamic factor model. Section VIII provides some further examples to illustrate our results. Finally, in Section IX we give some conclusions.

II. Feedback representations

The two processes u and y in (1) are jointly stationary, and we can express both $y(t)$ and $u(t)$ as a sum of the best linear estimate based on the past of the other process plus an error term, i.e.,

$$y(t) = \mathbb{E}\{y(t) \mid \mathbf{H}_t^-(u)\} + v(t), \tag{6a}$$
$$u(t) = \mathbb{E}\{u(t) \mid \mathbf{H}_t^-(y)\} + r(t) \tag{6b}$$

where $\mathbf{H}_t^-(u)$ is the closed span of the past components $\{u_1(\tau), u_2(\tau), \dots, u_q(\tau)\} \mid \tau \le t\}$ of the vector process u in the Hilbert space of random variables, and let $\mathbf{H}_t^-(y)$ be defined likewise in terms of $\{y_1(\tau), y_2(\tau), \dots, y_p(\tau) \mid \tau \le t\}$. For future use, we shall also need the closed span $\mathbf{H}_t^+(u)$ of the future components $\{u_1(\tau), u_2(\tau), \dots, u_q(\tau)\} \mid \tau \ge t\}$ and the closed span $\mathbf{H}(u)$ of the complete (past and future) history of u, and similarly for y. Each linear projection in (6) can be represented by a linear filter which we write as

$$y = F(z)u + v, \tag{7a}$$
$$u = H(z)y + r, \tag{7b}$$

where $F(z)$ and $H(z)$ are proper rational transfer functions of dimensions $(p-m)\times m$ and $m\times(p-m)$, respectively. Hence we have the feedback configuration depicted in Figure 2.

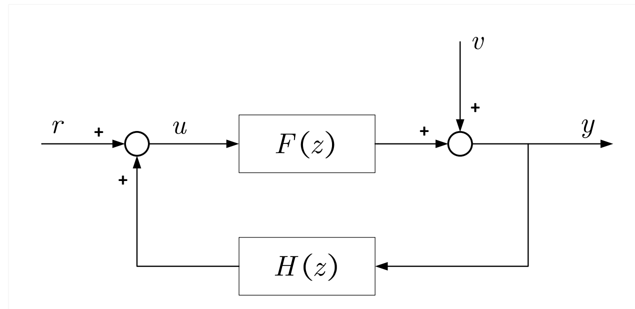

Fig. 2. Block diagram illustrating a feedback representation

The transfer functions $F(z)$ and $H(z)$ are in general not stable, but, since all processes are jointly stationary, the overall feedback configuration needs to be internally stable [30], [32]. The error processes v and r produced by (6) are in general correlated. However, there is always a feedback model (7) such that v and r are uncorrelated, and in the sequel we shall assume that this is so. To show that there is such a feedback model, we shall revisit some results in [24], where however a full-rank requirement to insure uniqueness was imposed. Here we want to allow for rank-deficient spectral densities.

Lemma 1: The transfer function matrix $T(z)$ from $\begin{bmatrix} r \\ v \end{bmatrix}$ to $\begin{bmatrix} u \\ y \end{bmatrix}$ in the feedback model (7) is given by

$$T(z) = \begin{bmatrix} P(z) & P(z)H(z) \\ Q(z)F(z) & Q(z) \end{bmatrix}, \tag{8a}$$

where

$$\begin{aligned} P(z) &= (I - H(z)F(z))^{-1}, \\ Q(z) &= (I - F(z)H(z))^{-1} \end{aligned} \tag{8b}$$

are strictly stable. Moreover,

$$Q(z)F(z) = F(z)P(z), \quad H(z)Q(z) = P(z)H(z), \tag{9}$$

and $T(z)$ is full rank and strictly stable.

Proof: Since the stationary processes v and r produce stationary processes y and u, the feedback model must be internally stable, and therefore $T(z)$ is (strictly) stable. From (7) we have

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & H(z) \\ F(z) & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} r \\ v \end{bmatrix} \tag{10}$$

and therefore

$$R(z)\begin{bmatrix}u\\y\end{bmatrix}=\begin{bmatrix}r\\v\end{bmatrix},$$

where

$$R(z):=\begin{bmatrix}I & -H(z)\\-F(z) & I\end{bmatrix}$$

is stable and full rank, being the transfer function from u, y to r, v. Then the Schur complements $(I-H(z)F(z))$ and $(I-F(z)H(z))$ are also full rank, and hence we can form the transfer functions $P(z)$ and $Q(z)$ as in (8b). Moreover,

$$\det P(z)=\det R(z)=\det Q(z),$$

and hence $P(z)$ and $Q(z)$ are also strictly stable. A straightforward calculation shows that $T(z)R(z)=I$, and hence $T(z)=R(z)^{-1}$, as claimed. It is immediately seen that $FP^{-1}=Q^{-1}F$ and that $P^{-1}H=HQ^{-1}$, and therefore (9) follows. ■

Remark 1: Note that $Q(z)$ is the output sensitivity function – generally merely called the sensitivity function – and $P(z)$ the input sensitivity function of the feedback system in Figure 2, and hence they must be strictly stable for this reason also [32, p.82], [30].

The white noise process w in (5) has the spectral representation

$$w(t)=\int_{-\pi}^{\pi}e^{it\theta}d\hat{w}\quad\text{where}\quad\mathbb{E}\{d\hat{w}d\hat{w}^*\}=\frac{d\theta}{2\pi}I$$

[29]. Then

$$\begin{bmatrix}u(t)\\y(t)\end{bmatrix}=\int_{-\pi}^{\pi}e^{it\theta}W(e^{i\theta})d\hat{w}.$$

Next define

$$\begin{bmatrix}r(t)\\v(t)\end{bmatrix}=\int_{-\pi}^{\pi}e^{it\theta}\begin{bmatrix}K(e^{i\theta}) & 0\\0 & G(e^{i\theta})\end{bmatrix}d\hat{w},\tag{11}$$

where $K(z)$ is $m\times m_1$ and $G(z)$ is $(p-m)\times m_2$ for some $m_1\le m$ and $m_2\le(p-m)$ such that $m_1+m_2=m$. Then v and r are uncorrelated as required. Moreover, it follows from Lemma 1 that

$$W=\begin{bmatrix}P & PH\\QF & Q\end{bmatrix}\begin{bmatrix}K & 0\\0 & G\end{bmatrix}=\begin{bmatrix}PK & PHG\\QFK & QG\end{bmatrix},$$

which in view of (9) can be written

$$W=\begin{bmatrix}W_{11} & W_{12}\\W_{21} & W_{22}\end{bmatrix}=\begin{bmatrix}PK & HQG\\FPK & QG\end{bmatrix}.\tag{12}$$

Consequently,

$$W_{21}=FW_{11}\quad\text{and}\quad W_{12}=HW_{22},\tag{13}$$

so, if W is full rank and thus $m=p$ as in [24], then $F=W_{21}W_{11}^{-1}$ and $H=W_{12}W_{22}^{-1}$. However, in our setting $m<p$ and hence F and H cannot both be uniquely determined from (13).

By (11) and Lemma 1, we have

$$\Phi(z)=T(z)\begin{bmatrix}\Phi_r(z) & 0\\0 & \Phi_v(z)\end{bmatrix}T(z)^*,\tag{14}$$

where $\Phi_v(z)=G(z)G(z)^*$ and $\Phi_r(z)=K(z)K(z)^*$ are the spectral densities of v and r, respectively, and * denotes transpose conjugate. Since T has full rank a.e., Φ is rank deficient if and only if at least one of Φ_v or Φ_r is.

Theorem 1: Suppose $(H\Phi_vH^*+\Phi_r)$ is positive definite a.e. on unit circle. Then

$$F=\Phi_{yu}\Phi_u^{-1}-\Phi_vH^*(H\Phi_vH^*+\Phi_r)^{-1}(I-HF),\tag{15}$$

that is

$$F=\Phi_{yu}\Phi_u^{-1}\tag{16}$$

if and only if $\Phi_vH^*\equiv 0$.

Proof: Given (2), (8) and (14), we have

$$\Phi_u=P(H\Phi_vH^*+\Phi_r)P^*=HQ\Phi_vH^*P^*+P\Phi_rP^*$$
$$\Phi_{yu}=Q(\Phi_vH^*+F\Phi_r)P^*=Q\Phi_vH^*P^*+FP\Phi_rP^*,$$

where we have used (9), i.e., $QF=FP$ and $HQ=PH$. Hence, in view of (9),

$$\Phi_{yu}-F\Phi_u=(I-FH)Q\Phi_vH^*P^*=\Phi_vH^*P^*,$$

which yields

$$\Phi_{yu}\Phi_u^{-1}-F=\Phi_vH^*(H\Phi_vH^*+\Phi_r)^{-1}P^{-1},$$

from which (15) follows by (8b). ■

III. Singular Dynamic Network Models

From (10) and (11) we have

$$\zeta(t)=M(z)\zeta(t)+N(z)w(t),\tag{17a}$$

where

$$M(z)=\begin{bmatrix}0 & H(z)\\F(z) & 0\end{bmatrix},\quad N(z)=\begin{bmatrix}K(z) & 0\\0 & G(z)\end{bmatrix}.\tag{17b}$$

We may choose the $p\times m$ matrix $N(z)$ to be stable and have a left stable inverse. Then since the diagonal elements of $M(z)$ are identically zero, (17) is a dynamic network model with no exogenous input but a noise term $N(z)w(t)$ [4]. Such models describe the dynamical dependencies between components of a multivariate stationary stochastic process in terms of a network whose links are dynamical relations. They play an important role in understanding the underlying mechanisms of complex systems in econometrics, biology and engineering [8]–[10].

Recovering or reconstructing the topology of a dynamic network is important when the prior information about the topology is scarce, or some nodes are not measurable. Related research evaluates network structure mainly from the aspect of graph theory [5], or dynamical information [7], [17], or both [6]. Determining $M(z)$ reveals both the direct relations between nodes and the topology of the network. Simpler topologic structures are often preferred for some degree of sparsity, which will affect calculations or further analysis. Compared with general dynamic network models, the matrices $M(z)$ and $N(z)$ in the model (17) are block matrices containing zero matrices. This stronger sparsity partitions the nodes in ζ into two groups, resulting in easier computation and a simpler topology structure with directions as in Figure 2. On the other hand, this 'natural' structure for singular process is more general than the existed research strictly requiring some matrices to be diagonal.

Dynamic network models have generally been studied for full-rank processes ζ, i.e., for $m = p$, when considering modeling, stability and identifiability [3] [4]. In this case, $M(z)$, $N(z)$ in (17) are square matrices, and the full rank process $w(t)$ has the same dimension as $\zeta(t)$. However, singular dynamic network models exist in many practice and theoretical scenarios, especially when there are dynamical dependencies, i.e., correlated nodes, in the network, leading us to consider the rank-deficient case $m < p$. Paper [17] deals with recovering dynamical dependencies from sampled data in such dynamic networks.

To connect the network model (17) to the the feedback representation of Section II we observe that

$$\zeta(t) = (I - M(z))^{-1} N(z) w(t)$$

and that

$$N(z)w(t) = \begin{bmatrix} r(t) \\ v(t) \end{bmatrix},$$

and consequently

$$(I - M(z))^{-1} = T(z), \tag{18}$$

which, by Lemma 1, is strictly stable and has a representation (8).

In Section IV we will take $v \equiv 0$. Then $\Phi_v = 0$, so, by (14), Φ_r must have full rank. Thus $m_1 = m$ and $m_2 = 0$ so that

$$N(z) = \begin{bmatrix} K(z) \\ 0 \end{bmatrix}.$$

In Sections IV and VI we show how to determine $F(z)$ and $H(z)$, respectively, and hence $M(z)$, however not uniquely.

IV. Deterministic dynamical relations

If $v = 0$, the feedback model (7) reduces to

$$y = F(z)u, \tag{19a}$$

$$u = H(z)y + r, \tag{19b}$$

where a nontrivial $H(z)$ will permit $F(z)$ to be unstable, as the feedback will stabilize the feedback loop. Then $\Phi_v = 0$, and there is a deterministic dynamical relation from u to y as depicted in Figure 1, where $F(z)$ is uniquely given by (16). Indeed, if $\Phi_v = 0$, then, by (14), Φ_r has full rank m, so the condition of Theorem 1 is satisfied.

However, the description (16) of $F(z)$ is not satisfactory. We would like to have system realisation of dimension at most n. Following [17], in [18] we gave such a description in the continuous-time case. However, then $D = 0$, which considerably simplifies the situation. The discrete-time case requires situations when D is nonzero and with a rank $\rho \le m$, which complicates the calculations considerably.

Let ρ be the rank of the $p \times m$ matrix D. To simplify calculations we shall first perform a singular value decomposition

$$UDV' = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \tag{20a}$$

where Σ is a diagonal $\rho \times \rho$ matrix consisting of the nonzero singular values, and U and V are orthogonal matrices of dimensions $p \times p$ and $m \times m$, respectively, i.e., $U'U = V'V = I$. We assume that the corresponding transformations

$$(\zeta, w) \to (U\zeta, Vw) \quad \text{and} \quad (B, C) \to (BV', UC) \tag{20b}$$

have already been performed in (5). Moreover,

$$\Phi(z) \to U\Phi(z)U', \qquad W(z) \to UW(z)V'. \tag{20c}$$

Next partition the new matrices C and B as

$$C = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} \qquad B = \begin{bmatrix} B_0 & B_1 \end{bmatrix}, \tag{21}$$

where C_0 C_1, C_2, B_0 and B_1 are $\rho \times n$, $(m-\rho) \times n$, $(p-m) \times n$, $n \times \rho$ and $n \times (m-\rho)$, respectively, after having changed, if necessary, the order of the component in ζ so that the square $(m-\rho) \times (m-\rho)$ matrix $C_1 B_1$ is full rank. As we shall see below, this can always be done and in general in several different ways. Then the partitioning of C leads to the representation

$$\zeta = \begin{bmatrix} u \\ y \end{bmatrix} \quad \text{where} \quad u = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \tag{22}$$

and the partitioning

$$\begin{bmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \\ W_{20} & W_{21} \end{bmatrix} \tag{23}$$

of the spectral factor (3). (Note that this does not correspond to the decomposition (12).) Consequently,

$$W_{00}(z) = C_0(zI - A)^{-1} B_0 + \Sigma \tag{24}$$

and

$$W_{jk}(z) = C_j(zI - A)^{-1} B_k \tag{25}$$

when $(j,k) \neq (0,0)$. Moreover, using Lemma 3 in the appendix, we have

$$W_{00}(z)^{-1} = \Sigma^{-1}\left[I - C_0(zI - \Gamma_0)^{-1} B_0 \Sigma^{-1}\right], \tag{26}$$

where

$$\Gamma_0 = A - B_0 \Sigma^{-1} C_0. \tag{27}$$

Theorem 2: Suppose that $C_1 B_1$ is nonsingular. Then the m-dimensional process u in (22) is full rank.

Proof: We need to show that the $m \times m$ spectral factor

$$\begin{bmatrix} W_{00}(z) & W_{01}(z) \\ W_{10}(z) & W_{11}(z) \end{bmatrix}$$

is full rank or, equivalently, that the Schur complement

$$S(z) = W_{11}(z) - W_{10}(z) W_{00}(z)^{-1} W_{01}(z) \tag{28}$$

is full rank. To this end, we first form

$$W_{10}(z) W_{00}(z)^{-1} = C_1(zI - A)^{-1} Q(z) B_0 \Sigma^{-1}, \tag{29}$$

where

$$\begin{aligned}Q(z) &= I - B_0\Sigma^{-1}C_0(zI-\Gamma_0)^{-1}\\ &= \left[zI-\Gamma_0-B_0\Sigma^{-1}C_0\right](zI-\Gamma_0)^{-1}\\ &= (zI-A)(zI-\Gamma_0)^{-1},\end{aligned}$$

which inserted into (29) yields

$$W_{10}(z)W_{00}(z)^{-1} = C_1(zI-\Gamma_0)^{-1}B_0\Sigma^{-1}, \tag{30}$$

and hence

$$\begin{aligned}&W_{10}(z)W_{00}(z)^{-1}W_{01}\\ &= C_1(zI-\Gamma_0)^{-1}B_0\Sigma^{-1}C_0(zI-A)^{-1}B_1\\ &= C_1(zI-A)^{-1}B_1 - C_1(zI-\Gamma_0)^{-1}B_1\\ &= W_{11}(z) - C_1(zI-\Gamma_0)^{-1}B_1,\end{aligned}$$

where we have used the fact that

$$B_0\Sigma^{-1}C_0 = A-\Gamma_0 = (zI-\Gamma_0)-(zI-A)^{-1}.$$

Consequently, the Schur complement (28) is given by

$$S(z) = C_1(zI-\Gamma_0)^{-1}B_1. \tag{31}$$

To see that $S(z)$ is full rank, first note thet

$$\begin{aligned}(zI-\Gamma_0)^{-1} =& z^{-1}(zI-\Gamma_0+\Gamma_0)(zI-\Gamma_0)^{-1}\\ =& z^{-1}I + z^{-1}\Gamma_0(zI-\Gamma_0)^{-1}.\end{aligned}$$

to obtain

$$S(z) = z^{-1}\left[C_1B_1 + C_1\Gamma_0(zI-\Gamma_0)^{-1}B_1\right], \tag{32}$$

which is clearly full rank whenever C_1B_1 is nonsingular. ∎

Theorem 3: Suppose that the order of the components in ζ is chosen so that C_1B_1 is nonsingular. Then the transfer function $F(z)$ mapping u to y is given by

$$F(z) = \left[F_0(z), F_1(z)\right], \tag{33}$$

where

$$F_0(z) = C_2(zI-\Gamma_1)^{-1}\left[I-B_1(C_1B_1)^{-1}C_1\right]B_0\Sigma^{-1} \tag{34a}$$

$$\begin{aligned}F_1(z) &= zC_2(zI-\Gamma_1)^{-1}B_1(C_1B_1)^{-1} &\text{(34b)}\\ &= C_2\Gamma_1(zI-\Gamma_1)^{-1}B_1(C_1B_1)^{-1} + C_2B_1(C_1B_1)^{-1} &\text{(34c)}\end{aligned}$$

with Γ_0 given by (27) and Γ_1 by

$$\Gamma_1 = \Gamma_0 - B_1(C_1B_1)^{-1}C_1\Gamma_0. \tag{35}$$

Proof: Since u is full rank (Theorem 2), $y = F(z)u$ is given by

$$y = \begin{bmatrix}W_{20} & W_{21}\end{bmatrix}\begin{bmatrix}W_{00} & W_{01}\\ W_{10} & W_{11}\end{bmatrix}^{-1}\begin{bmatrix}u_0\\ u_1\end{bmatrix},$$

and therefore

$$\begin{aligned}F_0W_{00} + F_1W_{10} &= W_{20}\\ F_0W_{01} + F_1W_{11} &= W_{21},\end{aligned}$$

from which we have

$$\begin{aligned}F_1(z) &= T(z)S(z)^{-1} &\text{(36a)}\\ F_0(z) &= W_{20}(z)W_{00}(z)^{-1} - F_1(z)W_{10}(z)W_{00}(z)^{-1} &\text{(36b)}\end{aligned}$$

where $S(z)$ is given by (31) or (32) and

$$T(z) = W_{21}(z) - W_{20}W_{00}(z)^{-1}W_{01}(z), \tag{37}$$

which clearly is obtained by exchanging C_1 by C_2 in the calculation leading to (31). Consequently,

$$T(z) = C_2(zI-\Gamma_0)^{-1}B_1. \tag{38}$$

To determine $F_1(z)$ we apply Lemma 3 in the appendix to obtain

$$\begin{aligned}&z^{-1}B_1S(z)^{-1}\\ &\quad= \left[I - B_1(C_1B_1)^{-1}C_1\Gamma_0(zI-\Gamma_1)^{-1}\right]B_1(C_1B_1)^{-1},\end{aligned}$$

where Γ_1 is given by (35). However,

$$\begin{aligned}&B_1(C_1B_1)^{-1}C_1\Gamma_0\\ &\quad= \Gamma_0-\Gamma_1 = (zI-\Gamma_1)-(zI-\Gamma_0)\end{aligned} \tag{39}$$

and therefore

$$z^{-1}B_1S(z)^{-1} = (zI-\Gamma_0)(zI-\Gamma_1)^{-1}B_1(C_1B_1)^{-1},$$

which together with (36a) and (38) yields (34b). To derive (34c) just insert

$$z(zI-\Gamma_1)^{-1} = \Gamma_1(zI-\Gamma_1)^{-1} + I \tag{40}$$

into (34b).

To determine F_0 from (36b) we first note that a calculation analogous to that leading to (29) yields

$$W_{20}(z)W_{00}(z)^{-1} = C_2(zI-\Gamma_0)^{-1}B_0\Sigma^{-1}. \tag{41}$$

Moreover, from (34b) and (29) we obtain

$$F_1(z)W_{10}(z)W_{00}(z)^{-1} = C_2(zI-\Gamma_1)^{-1}R(z)B_0\Sigma^{-1}, \tag{42}$$

where

$$R(z) = B_1(C_1B_1)^{-1}C_1z(zI-\Gamma_0)^{-1}$$

However, $z(zI-\Gamma_0)^{-1} = \Gamma_0(zI-\Gamma_0)^{-1} + I$, as in (40), so

$$R(z) = B_1(C_1B_1)^{-1}C_1\Gamma_0(zI-\Gamma_0)^{-1} + B_1(C_1B_1)^{-1}C_1,$$

which together with (39) yields

$$R(z) = (zI-\Gamma_1)(zI-\Gamma_0)^{-1} - I + B_1(C_1B_1)^{-1}C_1.$$

Inserting this expression for $R(z)$ into (42) we have

$$\begin{aligned}F_1(z)W_{10}(z)W_{00}(z)^{-1} &= C_2(zI-\Gamma_0)^{-1}B_0\Sigma^{-1}\\ &- C_2(zI-\Gamma_1)^{-1}\left[I-B_1(C_1B_1)^{-1}C_1\right]B_0\Sigma^{-1},\end{aligned} \tag{43}$$

Then (34a) follows directly from (36b), (41) and (43). ∎

Remark 2: In view of (16), $F(z)$ is uniquely determined by the decomposition (1) and the corresponding spectral density (2), so (33) does not depend on the particular choice of spectral factor $W(z)$ used in constructing it.

Corollary 1: The transfer function $F(z)$ given by (33) is (strictly) stable if and only if Γ_1 has all its eigenvalues in the (open) unit disc.

Proof: Since $\Gamma_1(zI-\Gamma_1)^{-1}=(zI-\Gamma_1)^{-1}\Gamma_1$, it follows from (34) that

$$F(z)=C_2(zI-\Gamma_1)^{-1}\hat{B}+\hat{D}, \tag{44a}$$

where

$$\hat{B}=\begin{bmatrix}(I-B_1(C_1B_1)^{-1}C_1)B_0\Sigma^{-1} & \Gamma_1B_1(C_1B_1)^{-1}\end{bmatrix} \tag{44b}$$

$$\hat{D}=\begin{bmatrix}0 & C_2B_1(C_1B_1)^{-1}\end{bmatrix}, \tag{44c}$$

and consequently the corollary follows. ∎

Note that, since $C_1\in\mathbb{R}^{(m-\rho)\times n}$ and $B_1\in\mathbb{R}^{n\times(m-\rho)}$, it is necessary that $m-\rho\le n$ for C_1B_1 to be nonsingular. Clearly, the McMillan degree of $F(z)$ is at most n. In special cases to be considered below, the McMillan degree will depend on the rank of the matrix Γ_1. To this end, we shall need the following lemma.

Lemma 2: Suppose that $\rho<m$ and that $B_1(C_1B_1)^{-1}C_1$ has n linearly independent eigenvectors. Then

$$\operatorname{rank}\Gamma_1\le n-(m-\rho).$$

Proof: By Lemma 4 in the appendix, the nonzero eigenvalues of $B_1(C_1B_1)^{-1}C_1$ are the same as those of $C_1B_1(C_1B_1)^{-1}=I_{m-\rho}$, so $B_1(C_1B_1)^{-1}C_1$ has $m-\rho$ nonzero eigenvalues all equal to 1. Then there is an $n\times n$ matrix T such that

$$T^{-1}B_1(C_1B_1)^{-1}C_1T=\begin{bmatrix}I_{(m-\rho)} & \\ & 0_{n-(m-\rho)}\end{bmatrix},$$

and therefore

$$T^{-1}(I-B_1(C_1B_1)^{-1}C_1)T=\begin{bmatrix}0_{(m-\rho)} & \\ & I_{n-(m-\rho)}\end{bmatrix}.$$

Then, since $\Gamma_1=(I-B_1(C_1B_1)^{-1}C_1)\Gamma_0$, the statement of the lemma follows. ∎

We summarize by formulating a procedure for calculating the matrix function $F(z)$ in the model (19) from a minimal realization of any minimal stable spectral factor $W(z)$ of the spectral density $\Phi(z)$ of $\zeta(t)$.

Procedure 1:

1) Perform the transformations (20).
2) Rearrange the last $p-\rho$ components of ζ so that the square $(m-\rho)\times(m-\rho)$ matrix C_1B_1 is nonsingular. In general this can be done in several different ways.
3) Determine $F(z)$ from (33), (34) or (44).

A. The special case $D=0$

When $D=0$, we have $\rho=0$, and hence $C_0=B_0=0$. More precisely,

$$C=\begin{bmatrix}C_1\\ C_2\end{bmatrix} \qquad B=B_1. \tag{45}$$

Then Step 1) in Procedure 1 is not required, and Step 3) is simplified. In fact, $F_0(z)=0$ and hence $F(z)=F_1(z)$. Moreover, $\Gamma_0=A$ and

$$\Gamma_1=\begin{bmatrix}I-B(C_1B)^{-1}C_1\end{bmatrix}A.$$

Consequently, (34b) yields

$$F(z)=zC_2(zI-\Gamma_1)^{-1}B(C_1B)^{-1} \tag{46a}$$

or alternatively from (34c)

$$F(z)=C_2\Gamma_1(zI-\Gamma_1)^{-1}B(C_1B)^{-1}+C_2B(C_1B)^{-1}. \tag{46b}$$

The realizations (46) are in general not minimal, as under the conditions of Lemma 2, Γ_1 has at least one zero eigenvalue, and hence a pole in zero which will cancel the factor z in (34b). In fact, by the next theorem, all zero eigenvalues will be cancelled, and the McMillan degree of $F(z)$ will be reduced accordingly.

Theorem 4: Suppose $B(C_1B)^{-1}C_1$ has n linearly independent eigenvectors. Then $F(z)$ has McMillan degree at most $n-m$.

Proof: The observability matrix of (46b) is

$$\begin{bmatrix}C_2\Gamma_1\\ C_2(\Gamma_1)^2\\ \vdots\\ C_2(\Gamma_1)^{(p-m)}\end{bmatrix}=\begin{bmatrix}C_2\\ C_2\Gamma_1\\ \vdots\\ C_2(\Gamma_1)^{p-m-1}\end{bmatrix}\Gamma_1,$$

which has at most the same rank as Γ_1. However, by Lemma 2, $\operatorname{rank}\Gamma_1<n-m$, so the realization (46b) is not observable and hence not minimal. In fact, the dimension of the unobservable subspace is at least m, so the dimension of $F(z)$ can be reduced reduced from n to $n-m$. ∎

If $m=n$, C_1 and B are both $m\times m$ matrices. Therefore, since $\operatorname{rank}(C_1B)=m$, they must both be invertible. Then $\Gamma_1=(I-B(C_1B)^{-1}C_1)\Gamma_0=0$, and consequently

$$F(z)=C_2C_1^{-1}$$

is constant and hence strictly stable.

B. The special case D full rank

When D has full rank, $\rho=m$ and $C_1=B_1=0$, and therefore

$$C=\begin{bmatrix}C_0\\ C_2\end{bmatrix} \qquad B=B_0.$$

Then Step 2) in Procedure 1 is not needed. Moreover, $F_1(z)=0$ and hence $F(z)=F_0(z)$,

$$\Gamma_1=\Gamma_0=A-B\Sigma^{-1}C_0,$$

and

$$F(z)=C_2(zI-\Gamma_0)^{-1}B\Sigma^{-1}. \tag{47}$$

V. Causality and Stability

The ordering of the element of ζ in the decomposition (1) is in general not unique, and different choices may create feedback models with different stability and causality properties.

A. Stability of $\mathsf{F(z)}$

For the process ζ to be stationary, the feedback configuration in Figure 2 needs to be internally stable [30]. However, $F(z)$ does not need to be stable, as the feedback model can be stabilized by feedback.

Manfred Deistler [31] has recently posed the question whether there is always a selection of the input u, such that corresponding transfer function is stable and causal. The following simple counterexample answers this question in the negative.

Let Φ be a spectral density with

$$A=\begin{bmatrix}\frac{3}{2} & 2\\ -1 & -\frac{3}{2}\end{bmatrix}, B=\begin{bmatrix}2\\ -3\end{bmatrix}, C=\begin{bmatrix}-4 & -2\\ -2 & 3\end{bmatrix}, D=\begin{bmatrix}0\\ 0\end{bmatrix},$$

which has rank $m=1$. This is a process of the type studied in subsection IV-A, where C is decomposed as (45). There are two choices of ordering of the components of ζ.

First choose $u=\zeta_1$ and $y=\zeta_2$. Then

$$C_1=\begin{bmatrix}-4 & -2\end{bmatrix}, \quad C_2=\begin{bmatrix}-2 & 3\end{bmatrix},$$

and

$$\Gamma_1=(I-B(C_1B)^{-1}C_1)A=\begin{bmatrix}-\frac{5}{2} & -3\\ 5 & 6\end{bmatrix},$$

which has rank 1 with eigenvalue 0 and $\frac{7}{2}$. Moreover,

$$F(z)=\frac{26z-27}{2(2z-7)}, \tag{48}$$

which is unstable.

Next, choose $u=\zeta_2$ and $y=\zeta_1$. Then

$$C_1=\begin{bmatrix}-2 & 3\end{bmatrix}, \quad C_2=\begin{bmatrix}-4 & -2\end{bmatrix},$$

and

$$\Gamma_1=(I-B(C_1B)^{-1}C_1)A=\begin{bmatrix}\frac{15}{26} & \frac{9}{13}\\ \frac{5}{13} & \frac{6}{13}\end{bmatrix},$$

which has rank 1 with eigenvalue 0, $\frac{27}{26}$. This yields

$$F(z)=\frac{2(2z-7)}{26z-27}, \tag{49}$$

which is again unstable.

Consequently, there is no selection of the order of the components in ζ admitting a stable $F(z)$. Indeed, $F(z)$ depends only on $\Phi(z)$ and not on the particular choice of spectral factor (3) (Remark 2), so we only have the two $F(z)$ obtained above.

B. Granger causality

If we want to predict the future of y given the past of y, would we get a better estimate if we also know the past of u? If so, we have *Granger causality from u to y* [22], [23], [25]–[27]. Let us consider the negative situation that there is no such advantage. In mathematical terms, we have non-causality if and only if

$$\mathbb{E}^{\mathbf{H}_t^-(y)\vee\mathbf{H}_t^-(u)}\lambda=\mathbb{E}^{\mathbf{H}_t^-(y)}\lambda \quad \text{for all } \lambda\in\mathbf{H}_t^+(y) \tag{50}$$

[23, Definition 1], where $\mathbb{E}^{\mathbf{A}}\lambda$ denotes the orthogonal projection of λ onto the subspace $\mathbf{A}$ and $\vee$ is vector sum, i.e., $\mathbf{A}\vee\mathbf{B}$ is the closure in the Hilbert space of stochastic variables of the sum of the subspaces $\mathbf{A}$ and $\mathbf{B}$; see, e.g., [29]. With $A\ominus B$ the orthogonal complement of $B\subset A$ in A, (50) can also be written

$$\mathbb{E}^{\mathbf{H}_t^-(y)}\lambda+\mathbb{E}^{[\mathbf{H}_t^-(y)\vee\mathbf{H}_t^-(u)]\ominus\mathbf{H}_t^-(y)}\lambda=\mathbb{E}^{\mathbf{H}_t^-(y)}\lambda$$

for all $\lambda\in\mathbf{H}_t^+(y)$, which is equivalent to

$$\left[\mathbf{H}_t^-(y)\vee\mathbf{H}_t^-(u)\right]\ominus\mathbf{H}_t^-(y)\perp\mathbf{H}_t^+(y),$$

where $\mathbf{A}\perp\mathbf{B}$ means that the subspaces $\mathbf{A}$ and $\mathbf{B}$ are orthogonal. Then, using the equivalence between properties (i) and (v) in [29, Proposition 2.4.2], we see that this in turn is equivalent to the following geometric condition for lack of Granger causality

$$\mathbf{H}_t^-(u)\perp\mathbf{H}_t^+(y)\mid\mathbf{H}_t^-(y), \tag{51}$$

i.e., $\mathbf{H}_t^-(u)$ and $\mathbf{H}_t^+(y)$ are conditionally orthogonal given $\mathbf{H}_t^-(y)$. Hence, if the past of y is known, the future of y is uncorrelated to the past of u, and therefore

$$\mathbb{E}\{y(t)\mid\mathbf{H}_t^-(u)\}=0,$$

so, in view of (6a), lack of Granger causality is equivalent to $F(z)\equiv 0$. Conversely, we have Granger causality from u to y if and only if $F(z)$ is nonzero.

An analogous argument applied to (6b) yields the the geometric condition

$$\mathbf{H}^-(y)\perp\mathbf{H}^+(u)\mid\mathbf{H}^-(u), \tag{52}$$

which is equivalent to $H(z)\equiv 0$. Then there is no feedback from y to u [29, p. 677]. Consequently, as stressed in [24], Granger causality and feedback are dual concepts. In the setting of Section IV we must have $H(z)$ nonzero if $F(z)$ is not strictly stable, because it is needed for stabilization of the feedback loop. Conversely, if $H(z)$ is zero, $F(z)$ must be strictly stable.

Theorem 5: Consider the feedback model (7), and in particular, (19). Then there is causality from u to y in the sense of Granger if and only if $F(z)$ is nonzero, and there is no feedback from y to u if and only if $H(z)$ is identically zero. In this case $F(z)$ is (strictly) stable.

It could be argued that a better (and stronger) definition of causality of $F(z)$ in the present setting is (52), namely that is there is no feedback from y to u [29, Section 2.6.5].

VI. How to determine $H(z)$

Given the transfer function $F(z)$ in Theorem 3, determining the corresponding $H(z)$ in (19) is a robust control problem [30], [32]. In fact, regarding $F(z)$ in Fig. 2 as a plant, we need to determine a compensator $H(z)$ so that the feedback system is internally stable. As pointed out in Remark 1, the sensitivity function is

$$Q(z)=[I-F(z)H(z)]^{-1},$$

i.e., the function $Q(z)$ in (8b). For the feedback system to be internally stable, $Q(z)$ needs to be analytic in the complement of the open unit disc (i.e., strictly stable) and satisfy certain interpolation conditions in unstable poles and non-minimum-phase zeros of $F(z)$ [30]. In addition there must be a bound

$$\|Q\|_\infty \le \gamma.$$

There is a minimum bound $\gamma_{\rm opt}$, but we choose $\gamma > \gamma_{\rm opt}$. Finally $Q(z)$ should be rational of small McMillan degree. This is an analytic (Nevanlinna-Pick) interpolation problem with rationality constraint [34]–[36], [39].

Given a solution $Q(z)$ of this analytic interpolation problem, we can determine $H(z)$ from $F(z)H(z) = I - Q(z)^{-1}$ provided that the pseudo-inverse $F^\dagger := (F^*F)^{-1}F^*$ exists. If $F(z)$ is a long rectangular matrix function and the pseudo-inverse $F^\dagger := F^*(FF^*)^{-1}$ exists, we may instead formulate an analytic interpolation problem for the input sensitivity function

$$P(z) = (I - H(z)F(z))^{-1},$$

(Remark 1) and solve for $H(z)$ from $H(z)F(z) = I - P(z)^{-1}$.

To explain the basic ideas of the procedure in simple terms we shall first consider the case that $F(z)$ and $H(z)$, and thus also $Q(z)$, are scalar, in which case the interpolation conditions are simple. Then $Q(z)$ must send the unstable poles of $F(z)$ to 0 and the non-minimum phase zeros to 1 [30]. Moreover, the function $f(z) := \gamma^{-1}Q(z^{-1})$ is a Schur function, i.e., a function that is analytic in the open unit disc and maps it into the open unit disc (Figure 3). Then,

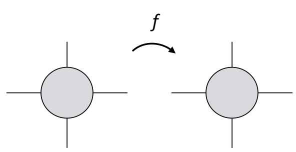

Fig. 3. Schur function

with the interpolation points $z_0, z_1 \dots, z_\nu$, the solutions of the analytic interpolation problem are completely parametrized by the polynomials in the class $\mathcal{S}$ of stable monic polynomials of degree ν. More precisely, to each $\sigma \in \mathcal{S}$, there is a unique pair of polynomials (α, β) with $\alpha \in \mathcal{S}$ and β a polynomial of at most degree ν such that $f(z) = \beta(z)/\alpha(z)$ satisfies the interpolation conditions and $|\alpha|^2 - |\beta|^2 = |\sigma|^2$ [37]. The solution corresponding to σ is obtained by maximizing

$$\int_{-\pi}^{\pi} \left|\frac{\sigma(e^{i\theta})}{\tau(e^{i\theta})}\right|^2 \log(1 - |f(e^{i\theta})|^2)d\theta, \tag{53}$$

where $\tau(z) = \Pi_{k=0}^{\nu-1}(1 - \bar{z}_k z)$, subject the interpolation conditions. Choosing $\sigma(z) = z^n\tau(z^{-1})$, we obtain the central or maximum-entropy solution maximizing

$$\int_{-\pi}^{\pi} \log(1 - |f(e^{i\theta})|^2)d\theta.$$

which is a linear problem [43].

As a simple example, let us consider (48), which has one unstable pole at $z = 7/2$ and one non-minimum phase zero at $z = 27/26$, yielding the interpolation conditions $f(\xi_0) = 0$ and $f(\xi_1) = \gamma^{-1}$, where $\xi_0 = 2/7$ and $\xi_1 = 26/27$. We may simplify the problem by moving the interpolation point ξ_0 to $z_0 = 0$, which can be done by the transformation

$$z = \frac{\xi - \xi_0}{1 - \xi_0\xi}$$

(for a real ξ_0), thus moving ξ_1 to $z_1 = (\xi_1 - \xi_0)(1 - \xi_0\xi_1)^{-1} = 128/137$.

Alternatively, we may solve an analytic interpolation for a Carthéodory function

$$\varphi(z) = \tfrac{1}{2}\frac{1 - f(z)}{1 + f(z)}, \tag{54}$$

mapping the open unit disc to the open right half plane, yielding the interpolation conditions $\varphi(z_0) = \frac{1}{2}$ and $\varphi(z_1) = \frac{1}{2}(\gamma - 1)(\gamma + 1)^{-1}$. Then we can use the optimization procedures in [34], [35] to determine an appropriate $Q(z)$. However it is simpler to apply the Riccati approach in [38], for which we have made the needed simple calculations in Appendix B. Choosing the central solution, we obtain from (87), (54) and (85)

$$f(z) = \frac{1 - 2\varphi(z)}{1 + 2\varphi(z)} = -uz = \gamma^{-1}z_1^{-1}z,$$

and hence, moving z back to ξ,

$$f(\xi) = f(z)\big|_{z=\frac{\xi-\xi_0}{1-\xi_0\xi}} = \gamma^{-1}z_1^{-1}\frac{\xi - \xi_0}{1 - \xi_0\xi},$$

then we have

$$Q(z) = \gamma f(\xi)\big|_{\xi=z^{-1}} = z_1^{-1}\frac{1 - \xi_0 z}{z - \xi_0} = \frac{137}{128}\frac{7 - 2z}{7z - 2},$$

which is analytic outside the closed unit circle. It is easy to check that Q satisfies $Q(7/2) = 0$ and $Q(27/26) = 1$, meaning that $Q(z)$ sends the unstable pole of $F(z)$ to 0 and the non-minimum phase zero to 1. So,

$$\begin{aligned} H(z) &= F(z)^{-1}(1 - Q(z)^{-1}) \\ &= \tfrac{2}{13}\frac{z - \xi_0^{-1}}{z - \xi_1^{-1}}\left(1 - \frac{\xi_1 - \xi_0}{1 - \xi_0\xi_1}\frac{z - \xi_0}{1 - \xi_0 z}\right), \\ &= \tfrac{2}{13}\frac{1 - \xi_0^2}{1 - \xi_0\xi_1}\frac{\xi_1}{\xi_0}, \end{aligned}$$

that is

$$H(z) = \tfrac{90}{137},$$

which is also stable.

The case when $Q(z)$ is matrix-valued is considerably more complicated, and we explain this in Section VIII-D in the context of a simple example, which we solve with the matrix version of the Riccati-type nonlinear matrix equation [38]. Moreover, we refer the reader to the literature, especially [39]–[42]. For the central solution, also see [43].

VII. Connections to Dynamic Factor Models

Suppose that we want to model a large dimensional stationary process $\{\eta(t), t \in \mathbb{Z}\}$, assumed zero-mean and of full rank by a *Dynamic Factor Analysis* model. This amounts to decomposing its spectral density, say $\Psi(z)$, into a sum of a low-rank spectral density $\Phi(z)$ and a diagonal full-rank spectral density $\Delta(z)$ which in principle should be diagonal, although this condition has been somewhat relaxed in the literature [44]–[46]. This corresponds to the decomposition

$$\eta(t) = \zeta(t) + \omega(t),$$

where $\{\omega(t), t \in \mathbb{Z}\}$ is a full rank noise process with uncorrelated components and $\{\zeta(t), t \in \mathbb{Z}\}$, called a *latent process*, has density $\Phi(z)$ having (hopefully very) low rank $m < n$. By possibly rearranging the components of $\zeta(t)$, this latent process can be decomposed (in several ways) into two components as in (1), i.e.,

$$\zeta(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix},$$

where $u(t)$ is chosen of full rank m. Then $\zeta(t)$ can be modeled by a feedback system of a special structure as (19). We shall study the latent process in this context.

Then

$$\zeta = \begin{bmatrix} I \\ F(z) \end{bmatrix} u, \tag{55}$$

so that u plays the role of a *factor process* of minimal dimension and (2) becomes

$$\Phi(z) = \begin{bmatrix} I \\ F(z) \end{bmatrix} \Phi_u(z) \begin{bmatrix} I \\ F(z^{-1}) \end{bmatrix}'. \tag{56}$$

where $\Phi_u(z)$ is full rank. Note that the factor process may not unique when actually doing estimation, and, even once the decomposition (1) is fixed, it is a priori not clear how it may be constructed from the data. There has been a widespread interest in estimating u from the observable data [47]. Now the feedback representation (19) provides a partial answer to this question as it shows that:

Corollary 2: Every minimal factor process u can be constructed by a noisy feedback

$$u(t) = H(z)y(t) + r(t)$$

on the "dependent" (or residual) variables $y(t)$.

Note again that there is non-uniqueness in this representation. In particular there are infinitely many pairs (F, H) which yield the same transfer function $r \to y$ of the feedback system (19) and hence the same spectral density $\Phi(z)$.

In view of (56) the spectral factor (3) of ζ can be written

$$W(z) = \begin{bmatrix} W_u(z) \\ W_{yu}(z) \end{bmatrix} = \begin{bmatrix} I \\ F(z) \end{bmatrix} W_u(z), \tag{57}$$

with

$$\Phi_u(z) = W_u(z) W_u(z^{-1})',$$

where $W_u(z)$ is a stable spectral factor and

$$F(z) = W_{yu}(z) W_u(z)^{-1}. \tag{58}$$

As mentioned above, u plays the role of a minimal dynamic factor [11], [13]. Moreover,

$$\begin{bmatrix} -F(z) & I \end{bmatrix} W(z) = 0, \tag{59}$$

so $\begin{bmatrix} -F & I \end{bmatrix}$ is the rational matrix function whose rows form a basis for the left kernel of W; cf. [11, Section 5]. This configuration is illustrated in Figure 4, where w is the generating white noise in the realization (5). More precisely, the transfer function $W_y(z)$ from w to y is a cascade of two transfer functions which we can compute.

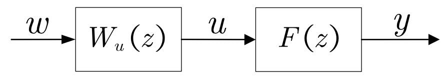

Fig. 4. Dynamical relation from $\boldsymbol{w}$ to $\boldsymbol{u}$ to $\boldsymbol{y}$

Introducing the decompositions

$$C = \begin{bmatrix} C_u \\ C_{yu} \end{bmatrix}, \quad D = \begin{bmatrix} D_u \\ D_{yu} \end{bmatrix}$$

in the format of (57), the latent process ζ has the representation (55) with the square $m \times m$ spectral factor $W_u(z)$ having the realization

$$W_u(z) = C_u(zI - A)^{-1}B + D_u, \tag{60}$$

where, in the notation of Section IV,

$$C_u = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}, \quad D_u = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}. \tag{61}$$

In particular,

$$W_u(z) = C_1(zI - A)^{-1}B \tag{62}$$

in the special case that $D = 0$, and

$$W_u(z) = C_0(zI - A)^{-1}B + \Sigma \tag{63}$$

when D is full rank.

Let us illustrate this with the simple example in subsection V-A, taking the first choice of ordering of the components in ζ, namely $u = \zeta_1$ and $y = \zeta_2$. Then $C_1 = \begin{bmatrix} -4 & -2 \end{bmatrix}$, so (62) yields

$$W_u(z) = \frac{28 - 8z}{4z^2 - 1},$$

which together with (48) yields the transfer functions in Figure 4. If instead we choose $u = \zeta_2$ and $y = \zeta_1$,

$$W_u(z) = \frac{54 - 52z}{4z^2 - 1},$$

and $F(z)$ given by (49).

VIII. Examples

We begin by giving examples illustrating the results in Section IV. We shall consider three different situations, namely that $D = 0$ and D is full rank, as well as the mixed case when $0 < \operatorname{rank} D < m$. Finally, we shall give an example for how to determine $H(z)$ when $Q(z)$ is a matrix.

A. *Example 1:* D = 0

Let $\Phi(z)$ be a spectral density with $D = 0$ and

$$A = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ \frac{7}{10} & \frac{1}{5} & -\frac{7}{5} \\ \frac{1}{2} & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -3 & -3 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has rank $m = 1$. First take C_1 to be the first row of C, i.e., $u = \zeta_1$, $y = (\zeta_2, \zeta_3, \zeta_4)'$, and

$$C_1 = \begin{bmatrix} 3 & -3 & -3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, $C_1 B = 9$, $B(C_1B)^{-1}C_1$ has $n = 3$ independent eigenvectors, and

$$\Gamma_1 = (I - B(C_1B)^{-1}C_1)A = \begin{bmatrix} \frac{16}{15} & \frac{1}{15} & -\frac{9}{5} \\ \frac{23}{30} & \frac{4}{15} & -\frac{17}{10} \\ \frac{3}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix},$$

which has rank two with eigenvalues $\frac{9}{10}$, $\frac{1}{3}$ and 0. However, by Theorem 4, the pole at zero will cancel, and we obtain

$$F(z) = \frac{1}{3(10z-9)(3z-1)} \begin{bmatrix} 5(2z+3)(5z-1) \\ 10z^2 + 49z - 13 \\ (7-6z)(5z-1) \end{bmatrix}$$

which is strictly stable of degree two rather than three.

Since the McMillan degree of $F(z)$ is two, it has a minimal realization of dimension two. One such realization is given by

$$F(z) = \tilde{C}(zI - \Gamma)^{-1}\tilde{B} + \tilde{D},$$

where

$$\Gamma = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{9}{10} \end{bmatrix}, \tilde{B} = \begin{bmatrix} -\frac{10}{459} \\ \frac{28}{17} \end{bmatrix}, \tilde{C} = \begin{bmatrix} 11 & 1 \\ 4 & \frac{7}{15} \\ 3 & \frac{1}{15} \end{bmatrix}, \tilde{D} = \begin{bmatrix} \frac{5}{9} \\ \frac{1}{9} \\ -\frac{1}{3} \end{bmatrix}.$$

Next, take C_1 to be the second row of C, i.e., $u = \zeta_2$, $y = (\zeta_1, \zeta_3, \zeta_4)'$, and

$$C_1 = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, $C_1 B = 5$, and

$$\Gamma_1 = (I - B(C_1B)^{-1}C_1)A = \begin{bmatrix} \frac{7}{10} & 0 & -\frac{11}{10} \\ \frac{2}{5} & \frac{1}{5} & -1 \\ \frac{7}{5} & 0 & -\frac{11}{5} \end{bmatrix},$$

which has rank two with eigenvalue $-\frac{3}{2}$, $\frac{1}{5}$ and 0. In this case, $B(C_1B)^{-1}C_1$ has only two independent eigenvalues, so we cannot apply Theorem 4. However, due to (46a), the zero pole will nevertheless cancel, and we obtain

$$F(z) = \frac{1}{5(2z+3)(5z-1)} \begin{bmatrix} 3(10z-9)(3z-1) \\ 10z^2 + 49z - 13 \\ (7-6z)(5z-1) \end{bmatrix}$$

which is unstable of degree two rather than three. The system

$$F(z) = \tilde{C}(zI - \Gamma)^{-1}\tilde{B} + \tilde{D},$$

with

$$\Gamma = \begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} \frac{8}{5} \\ \frac{14}{425} \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} -\frac{99}{34} & 3 \\ \frac{8}{17} & -1 \\ 1 & 0 \end{bmatrix}, \ \tilde{D} = \begin{bmatrix} \frac{9}{5} \\ \frac{1}{5} \\ -\frac{3}{5} \end{bmatrix}$$

is a minimal realization of $F(z)$.

B. *Example 2:* D *full rank*

1) *Example 2.1:*

Given the spectral density $\Phi(z)$, let

$$\bar{W}(z) = \bar{C}(zI - A)^{-1}\bar{B} + \bar{D}$$

be a spectral factor, where

$$A = \begin{bmatrix} -\frac{1}{5} & 0 & 0 \\ \frac{1}{2} & \frac{9}{20} & \frac{1}{20} \\ -\frac{13}{10} & \frac{1}{5} & \frac{3}{10} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} -3 & -1 & -1 \\ -2\sqrt{2} & -\frac{5}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{7}{\sqrt{2}} \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -1 \\ -1 & 1 \end{bmatrix},$$

with $\rho = m = 2$ and $\bar{D}$ full column rank.

Performing the transformations (20) on this system with

$$U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad V' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

we have a new system with

$$B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} -2 & -3 & 4 \\ -3 & -1 & -1 \\ -2 & -2 & -3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and hence

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, we have $B = B_0$,

$$C_0 = \begin{bmatrix} -2 & -3 & 4 \\ -3 & -1 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -2 & -2 & -3 \end{bmatrix},$$

and

$$\Gamma_0 = A - B\Sigma^{-1}C_0 = \begin{bmatrix} \frac{4}{5} & \frac{3}{2} & -2 \\ -\frac{1}{2} & \frac{49}{20} & -\frac{99}{20} \\ \frac{17}{10} & \frac{6}{5} & \frac{13}{10} \end{bmatrix},$$

which has full rank 3. Hence (47) is a minimal realization of $F(z)$, and

$$F(z) = \frac{-5}{\chi(z)} \begin{bmatrix} 240z^2 + 56z - 67, & 4(20z^2 - 521z - 161) \end{bmatrix},$$

where

$$\chi(z) = 400z^3 - 1820z^2 + 6510z - 2073.$$

By calculating the zeros of $\chi(z)$, we see thet $F(z)$ is unstable.

However, the following example shows that the McMillan degree of $F(z)$ may be strictly less than n even when D is full rank.

2) Example 2.2: Let $\Phi(z)$ be a spectral density with

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -\frac{3}{2} \\ 3 & -4 \\ 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Here D is already a tall diagonal matrix so Step 1) in Procedure 1 is not needed, and thus

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$C_0 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$\Gamma_0 = A - B\Sigma^{-1}C_0 = \begin{bmatrix} -5 & -\frac{9}{2} & 0 \\ -7 & -\frac{63}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix},$$

which has rank two. Hence $F(z)$ has a realization (47) with McMillan degree strictly less than $n = 3$. In fact, the observability and reachability matrices of this realization are

$$\mathcal{O} = \begin{bmatrix} C_2 \\ C_2\Gamma_0 \\ C_2\Gamma_0^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1/10 \\ 0 & 0 & 1/100 \end{bmatrix},$$

$$\begin{aligned} \mathcal{R} &= \begin{bmatrix} B\Sigma^{-1} & \Gamma_0 B\Sigma^{-1} & \Gamma_0^2 B\Sigma^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 3 & \frac{3}{2} & -\frac{57}{2} & -\frac{51}{2} & \frac{6441}{20} & \frac{5763}{20} \\ 3 & 4 & -\frac{399}{10} & -\frac{357}{10} & \frac{10370}{23} & \frac{15733}{39} \\ 1 & -1 & \frac{1}{10} & -\frac{1}{10} & \frac{1}{100} & -\frac{1}{100} \end{bmatrix}, \end{aligned}$$

so $\operatorname{rank}(\mathcal{O}\mathcal{R}) = 1$ and $F(z)$ has a minimal realization of dimension 1, namely

$$F(z) = C_2(zI - \Gamma_0)^{-1}B\Sigma^{-1} = \frac{1}{z - 1/10}\begin{bmatrix} 1 & -1 \end{bmatrix}$$

which is stable.

C. Example 3: mixed case

Let $\bar{\Phi}(z)$ be a spectral density with spectral factor

$$\bar{W}(z) = \bar{C}(zI - A)^{-1}\bar{B} + \bar{D},$$

where

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{4}{5} & \frac{26}{5} \\ 0 & \frac{7}{5} & \frac{18}{5} \\ 0 & -\frac{3}{10} & -\frac{7}{10} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{3}{\sqrt{2}} & 2 & -\frac{3}{\sqrt{2}} \\ -2\sqrt{2} & 1 & 0 \\ \frac{5}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} -1 & 4 & -1 \\ 1 & 3 & -2 \\ 3 & 0 & 0 \\ -4 & -1 & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{3}{\sqrt{2}} & 0 & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $p = 4$, $m = n = 3$, $\operatorname{rank}(\bar{D}) = \rho = 2$.

Performing the transformations (20) with

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad V' = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

we have the new system with

$$B = \begin{bmatrix} 2 & 0 & -3 \\ 1 & -2 & 2 \\ 0 & 2 & -3 \end{bmatrix}, C = \begin{bmatrix} -1 & 4 & -1 \\ 1 & 3 & -2 \\ 3 & 0 & 0 \\ -4 & -1 & 0 \end{bmatrix},$$

and

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently

$$\Sigma = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

and

$$B_0 = \begin{bmatrix} 2 & 0 \\ 1 & -2 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \quad C_0 = \begin{bmatrix} -1 & 4 & -1 \\ 1 & 3 & -2 \end{bmatrix}.$$

Choose C_1 to be the first row among the last $p - \rho = 2$ rows of C, i.e., the third row of C,

$$C_1 = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$$

with $C_1B_1 = -9$ full rank. Then by (27) and (35),

$$\Gamma_0 = A - B_0\Sigma^{-1}C_0 = \begin{bmatrix} -\frac{5}{2} & \frac{44}{5} & \frac{16}{5} \\ -\frac{5}{3} & \frac{17}{5} & \frac{59}{15} \\ \frac{2}{3} & \frac{17}{10} & -\frac{61}{30} \end{bmatrix},$$

$$\Gamma_1 = \Gamma_0 - B_1(C_1B_1)^{-1}C_1\Gamma_0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{10}{3} & \frac{139}{15} & \frac{91}{15} \\ \frac{19}{6} & -\frac{71}{10} & -\frac{157}{30} \end{bmatrix},$$

with rank 3 and 2, respectively. Therefore we have $F(z) = \begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix}$ with

$$F_0 = \frac{1}{90z^2 - 363z - 488}\begin{bmatrix} 7(30z+1) & 10(5-6z) \end{bmatrix},$$

$$F_1 = \frac{-300z^2 + 1520z + 1793}{3(90z^2 - 363z - 488)}.$$

Hence $F(z)$ is unstable with two poles, namely $(121 - \sqrt{34161})/60$ and $(121 + \sqrt{34161})/60$.

D. Example 4: determining $\mathrm{H}(z)$ in the matrix case

In this example, we first use coprime factorization (see, e.g., [33]) to obtain the interpolation conditions as in [39, p. 2174], and then solve the Nevanlinna-Pick interpolation problem by the approach in [38].

Suppose that Theorem 3 has given us

$$F(z) = \begin{bmatrix} \frac{z+3}{z+2} & 0 \\ 0 & \frac{z-4}{z-2} \end{bmatrix}.$$

Then, by the bilinear Tustin transformation, we have the corresponding function

$$G(s) = F\left(\frac{1+s}{1-s}\right) = \begin{bmatrix} \frac{2s-4}{s-3} & 0 \\ 0 & \frac{5s-3}{3s-1} \end{bmatrix} \tag{64}$$

in the s-domain. Suppose we want to find a function $K(s)$ so that the sensitive function

$$S(s) = (I - GK)^{-1} \tag{65}$$

is stable. Then the discrete-time sensitivity function $Q(z)$ can be obtained by performing the Tustin transformation. If one wants to use the input sensitive function $P(z)$ instead, $S(s) = (I - KG)^{-1}$ should be used instead.

By [33, Lemma1, p. 23],

$$G(s) = N(s)M(s)^{-1} = \tilde{M}(s)^{-1}\tilde{N}(s) \tag{66}$$

with

$$\tilde{X}M - \tilde{Y}N = I \tag{67a}$$

$$\tilde{M}X - \tilde{N}Y = I, \tag{67b}$$

where (67a) is the condition for M and N to be right coprime and (67b) is the condition for $\tilde{M}$ and $\tilde{N}$ to be left coprime. By [33, Theorem 1, p. 38], the internally stable controllers are given by

$$K = (Y - ML)(X - NL)^{-1}, \tag{68}$$

where $L \in RH_\infty$ (i.e., the space stable proper rational matrix function) is arbitrary. This is a classical parametrization that however puts no limit on the degree of K, and uniqueness is not established. Choosing an L, we may directly obtain a sensitivity function $S(s)$.

From the above, we may write S as

$$S(s) = T_1(s) - T_2(s)L(s)T_3(s), \tag{69}$$

where

$$T_1 = X\tilde{M}, \quad T_2 = N, \quad T_3 = \tilde{M}. \tag{70}$$

Hence the (transmission) zeros of T_2 and T_3 are respectively the zeros and poles of $G(s)$. Performing inner-outer factorizations of T_2 and T_3, we obtain $T_2 = \Theta_2\tilde{T}_2$ and $T_3 = \tilde{T}_3\Theta_3$, where Θ_2 and Θ_3 are inner functions containing the unstable poles and nonminimum-phase zeros of G. Denote $\phi := \det\Theta_2 \det\Theta_3$; then our interpolation points are the zeros of ϕ with the same multiplicities.

Now proceeding along the lines of [33, pages 23-25], we determine X, $\tilde{M}$ and N from (64). Then inner-outer factorization of the corresponding matrices (70) yields

$$\Theta_2 = \begin{bmatrix} \frac{s-2}{s+2} & 0 \\ 0 & \frac{5s-3}{5s+3} \end{bmatrix}, \quad \Theta_3 = \begin{bmatrix} \frac{s-3}{s+3} & 0 \\ 0 & \frac{3s-1}{3s+1} \end{bmatrix}. \tag{71}$$

$$\phi = \frac{(3s-1)(5s-3)(s-2)(s-3)}{(3s+1)(5s+3)(s+2)(s+3)} \tag{72}$$

with interpolation points

$$s_0 = 0.3333, \quad s_1 = 0.6, \quad s_2 = 2, \quad s_3 = 3,$$

all of multiplicity 1. Define $\tilde{S} := \phi\Theta_2^* S\Theta_3^*$ and

$$\begin{aligned}\tilde{T}_1 &:= \phi\Theta_2^* T_1\Theta_3^* \\ &= (s-0.3333)(s-3) \\ &\begin{bmatrix} \frac{(s-0.6)(s-11)}{(s+0.3333)(s+0.6)(s+1)^2} & \frac{12(s-0.6)(s-2)}{(s+0.6)(s+1)^2(s+3)(s+4.1111)} \\ 0 & \frac{(s-2)(s+21.7778)}{(s+0.6667)(s+2)(s+3)(s+4.1111)} \end{bmatrix}.\end{aligned} \tag{73}$$

Then we have the interpolation conditions $\tilde{S}(s_k) = \tilde{T}_1(s_k)$ for $k = 0, 1, 2, 3$ or more specifically,

$$\begin{aligned}\tilde{S}(s_0) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{S}(s_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0.3590 \end{bmatrix}, \\ \tilde{S}(s_2) &= \begin{bmatrix} 0.3846 & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{S}(s_3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned} \tag{74}$$

Denote

$$f(\xi) := \gamma^{-1}\tilde{S}\left(\frac{1-\xi}{1+\xi}\right) \tag{75}$$

with $\xi = \frac{1-s}{1+s}$. Then $f(\xi)$ is analytic outside the unit circle and hence stable. With

$$z = \frac{\xi - \xi_0}{1 - \xi_0\xi}, \tag{76}$$

we have the Carthéodory function

$$\varphi(z) := \frac{1}{2}(I - f(\xi))^{-1}(I + f(\xi)) \tag{77}$$

with interpolation conditions

$$z_0 = 0, \ z_1 = -0.2857, \ z_2 = -0.7143, \ z_3 = -0.8000, \tag{78a}$$

$$\begin{aligned}\varphi(z_0) &= \tfrac{1}{2}I, \quad \varphi(z_1) = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & \frac{\gamma+0.3590}{\gamma-0.3590} \end{bmatrix}, \\ \varphi(z_2) &= \tfrac{1}{2}\begin{bmatrix} \frac{\gamma+0.3846}{\gamma-0.3846} & 0 \\ 0 & 1 \end{bmatrix}, \quad \varphi(z_3) = \tfrac{1}{2}I.\end{aligned} \tag{78b}$$

Next we choose $\gamma = 10$ and apply the interpolation approach in [38]. Analogously with the scalar case, there is a complete parameterization of all solutions with degree constraint, and here we choose the central solution, which takes the form

$$\varphi(z) = \begin{bmatrix} \varphi_{11}(z) & 0 \\ 0 & \varphi_{22}(z) \end{bmatrix}, \tag{79}$$

where

$$\varphi_{11} = -\frac{0.5(z+1.1840)(z^2+0.0842z+1.0581)}{(z-1.0121)(z^2+1.9154z+1.2379)}, \tag{80a}$$

$$\varphi_{22} = -\frac{0.5(z-1.0486)(z^2+2.2808z+1.9319)}{(z+1.5899)(z^2+0.2064z+1.2743)}. \tag{80b}$$

Then we go back from $\varphi(z)$ to $f(\xi)$, and then to $S(s)$. Finally we have

$$Q(z) = S\left(\frac{z-1}{z+1}\right) = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}. \tag{81}$$

where

$$Q_{11} = \frac{3.0838(z-0.25)(z+2)}{z^2 - 0.3283z + 0.0372}, \tag{82a}$$

$$Q_{22} = \frac{1.2199(z-2)(z+0.3333)(z+0.5)}{(z-0.5)(z^2-0.6446z+0.1727)}, \tag{82b}$$

so $Q(z)$ is stable, as required. Moreover

$$H(z) = F(z)^{-1}(I - Q(z)^{-1}) = \begin{bmatrix} H_{11}(z) & 0 \\ 0 & H_{22}(z) \end{bmatrix}, \tag{83}$$

where

$$H_{11} = \frac{0.6757(z-0.2526)}{z+0.25}, \tag{84a}$$

$$H_{22} = \frac{0.1803(z+0.1404)(z+2.5933)}{(z+0.5)(z+0.3333)}, \tag{84b}$$

which is also stable.

IX. Conclusion

This paper has been devoted to modeling of rank-deficient stationary vector processes, present in singular dynamic network models, dynamic factor models, etc. This is done by rearranging the components of the process as in (1) to obtain two vector processes, a full rank factor process $u(t)$ and a residual process $y(t)$. This is then analyzed in the context of the feedback representation illustrated in Figure 2 by choosing $v(t) = 0$, thus obtaining the model (19), providing a deterministic relation between $u(t)$ and $y(t)$. We then show that the transfer function $F(z)$ is uniquely defined and given by Theorem 1. However, different choices of $u(t)$ and $y(t)$ give different $F(z)$. In general $F(z)$ is not stable, so since all processes are stationary, the complete feedback configuration needs to be internally stable, leading us to robust control theory to determine $H(z)$ and thus stabilizing the feedback loop.

We are presently woking on a paper [20] applying these principles to system identification of low rank vector processes.

Appendix

A. Some lemmas

Lemma 3: Let $G(z)$ be the proper rational transfer function

$$G(z) = C(zI - A)^{-1}B + D$$

where D is square and nonsingular. Then

$$G(z)^{-1} = D^{-1} - D^{-1}C\left[zI - (A - BD^{-1}C)\right]^{-1}BD^{-1}.$$

Proof: The rational matrix function $G(z)$ is the transfer function of the control system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Inserting $u(t) = D^{-1}[y(t) - Cx(t)]$ in the first equation yields the inverse system

$$\begin{aligned} x(t+1) &= (A - BD^{-1}C)x(t) + BD^{-1}y(t) \\ u(t) &= -D^{-1}Cx(t) + D^{-1}y(t) \end{aligned}$$

with transfer function $G(z)^{-1}$. ■

Lemma 4: If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, then the nonzero eigenvalues of AB and BA are the same.

Proof: The two matrices

$$T_1 := \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \quad \text{and} \quad T_2 := \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

are similar. In fact, $S^{-1}T_1S = T_2$, where S is the $(m+n)\times(m+n)$ matrix

$$S := \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}.$$

Therefore T_1 and T_2 have the same characteristic polynomials, i.e.,

$$\det(\lambda I_m - AB)\lambda^n = \lambda^m \det(\lambda I_n - BA).$$

Therefore each nonzero eigenvalue $\tilde{\lambda}$ of AB is also an eigenvalue of BA and vice versa. ■

B. Determinig $\mathsf{H}(z)$ in the scalar case

Given a scalar transfer function $F(z)$ with an unstable pole in $z = \xi_0^{-1}$ and a nonminimum-phase zero in $z = \xi_1^{-1}$, we want to determine a scalar $H(z)$ so that the feedback loop in Figure 2 is internally stable. As explained in Section VI, this amounts to determining a Carathéodory function φ which satisfies the interpolation conditions $\varphi(0) = \frac{1}{2}$ and $\varphi(z_1) = w_1$, where $z_1 = (\xi_1 - \xi_0)(1 - \xi_0\xi_1)^{-1}$ and $w_1 = \frac{1}{2}(\gamma - 1)(\gamma + 1)^{-1}$.

To this end, we shall use the Riccati-type approach of [38]. In the problem formulation in the introduction of that paper $m = 1$, $n_0 = n_1 = 1$ and $n = 1$, and from Section III-B in the same paper, we have the matrices

$$W = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & w_1 \end{bmatrix},\ Z = \begin{bmatrix} 0 & 0 \\ 0 & z_1 \end{bmatrix},\ e = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ V = \begin{bmatrix} 1 & 0 \\ 1 & z_1 \end{bmatrix},$$

which yields

$$T = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^{-1} \end{bmatrix}$$

and thus, continuing along the lines of [38, Section III-B], we have the interpolation data

$$u = -\gamma^{-1}z_1^{-1}, \quad U = -\gamma^{-1}. \tag{85}$$

With σ an arbitrary parameter in the interval $(-1, 1)$, the Riccati-type equation then becomes

$$p = \sigma^2(p - p^2) + (u + U\sigma - U\sigma p)^2, \tag{86}$$

which, by [38, Theorem 9], has a unique solution $0 < p < 1$. Then the corresponding solution of the interpolation problem is

$$\varphi(z) = \tfrac{1}{2}\frac{1 + bz}{1 + az},$$

where

$$\begin{aligned} a &= (1 - U)\sigma(1 - p) - u \\ b &= (1 + U)\sigma(1 - p) + u. \end{aligned}$$

The central solution is obtained by setting $\sigma = 0$, yielding

$$\varphi(z) = \tfrac{1}{2}\frac{1 + uz}{1 - uz}. \tag{87}$$

To obtain a general solution for a nonzero σ we need to solve the nonlinear equation (86), which can be done by the homotopy continuation method in subsection III-E of [38]. The parameter γ has to be chosen so that the Pick condition in [38, Proposition 3] is satisfied.

REFERENCES

[1] J. Jonker, A. Y. Aravkin, J. V. Burke, G. Pillonetto, , S. Webster, "Fast robust methods for singular state-space models," *Automatica*, vol. 105, pp. 399-405, 2019.

[2] V. Ciccone, Y. Chen, T. T. Georgiou, M. Pavon, "Regularized transport between singular covariance matrices," *ArXiv*: 2006.10000, 2020.

[3] G. Bottegal, A. Chiuso, P. M. Van den Hof, "On dynamic network modeling of stationary multivariate processes" in*18th IFAC Symposium on System Identification (SYSID)*, Stockholm, Sweden, 2018, pp. 850-855.

[4] H.H. Weerts, P. M. Van den Hof, and A.G. Dankers, "Identifiability of linear dynamic networks," *Automatica*, vol. 89, pp. 247-258, 2018.

[5] A. S. Bazanella, M. Gevers, J. M. Hendrickx, A. Parraga, "Identifiability of dynamical networks: which nodes need be measured?" in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, 2017, pp. 5870-5875.

[6] D. Materassi, M. V.Salapaka, "On the problem of reconstructing an unknown topology via locality properties of the wiener filter," *IEEE Trans. Autom. Control*, vol. 57, no. 7, pp. 1765-1777, 2012.

[7] M. Józas, M. Petreczky, and M. K. Camlibel, "Relationship between Granger noncausality and network graph of state-space representations," *IEEE Trans. Autom. Control*, vol. 64, no.3, pp. 912-927, Mar. 2019.

[8] O. Shy, "A Short Survey of Network Economics," *Social Science Electronic Publishing*, vol. 38, no. 2, pp. 119-149, 2010.

[9] C. A. Hidalgo, N. Blumm, A-L. Barabási and N.A. Christakis, "A Dynamic Network Approach for the Study of Human Phenotypes," *PLoS Computational Biology*, vol. 5, no. 4, e1000353, 2009.

[10] M. Ben-Akiva, A. D. Palma, K. Isam, "Dynamic network models and driver information systems," *Transportation Research Part A General*, vol. 25, no. 5, pp. 251-266, 1991.

[11] M. Deistler, "Singular ARMA systems: a structural theory," *Numerical Algebra, Control and Optimization*, vol. 9, no. 3, pp. 383-391, Sept. 2019.

[12] B. D. O. Anderson, M. Deistler, W. Chen and A. Filler, "Autoregressive models of singular spectral matrices," *Automatica*, vol. 48, no. 11, pp. 2843-2849, 2012.

[13] M. Deistler, B.D.O. Anderson, A. Filler and W. Chen, "Generalized linear dynamic factor models: an approach via singular autoregressions," *European Journal of Control*, vol. 16, no. 3, pp. 211-224, 2010.

[14] W. Chen, B. D. O. Anderson, M. Deistler and A. Filler, "Solutions of Yule-Walker equations for singular AR processes," *J. Time Ser. Anal.* vol. 32, no. 5, pp. 531-538, 2010.

[15] G. Bottegal and G. Picci, "Modeling complex systems by generalized factor analysis", *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 759-774, 2015.

[16] G. Picci, L. Falconi, A. Ferrante, M. Zorzi, "Hidden Factor estimation in Dynamic GeneralizedFactor Analysis Models", *Unpublished,* 2021.

[17] T. T. Georgiou and A. Lindquist, "Dynamic relations in sampled processes," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 144-149, Jan. 2019.

[18] W. Cao, A. Lindquist and G. Picci, "Spectral rank, feedback, causality and the indirect method for CARMA identification", in *59th IEEE Conference on Decision and Control (CDC)*, Jeju, Korea(South), 2020, pp. 4299-4305.

[19] G. Picci, W. Cao and a. Lindquist, "Modeling and identification of low rank vector processes", in *19th IFAC Symposium on System Identification (SYSID): Learning Models for Decision and Control*, Padova, Italy.

[20] W. Cao, G. Picci and A. Lindquist, "Identification of Low Rank Vector Processes", to appear.

[21] L. Barnett and A. K. Seth, "Granger causality for state space models", *Physical Review E*, 2015.

[22] C. W. J. Granger, "Economic processes involving feedback," *Inf. Control*, vol. 6, no. 1, pp. 28-48, 1963.

[23] C. Granger, "Investigating causal relations by econometric models and cross-spectral methods," *Econometrica*, vol. 37, no. 3, pp. 424-438, 1969.

[24] P. E. Caines, *Linear Stochastic Systems*, New York, USA: John Wiley and Sons, 1988.

[25] L. Petrović and S. Dimitrijević, "Causality with finite horizon of the past in continuous time," *Statistic and Probability Letters*, vol. 82, no. 7, pp. 1219 - 1223, 2012.

[26] B. D. O. Anderson, M. Deistler, and J. M. Dufour, "On the sensitivity of Granger causality to errors-in-variables, linear transformations and subsampling," *J. Time Ser. Anal.*, vol. 40, pp. 102-123, 2019.

[27] L. Barnett and A. K. Seth, "Detectability of Granger causality for subsample continuous-time neurophysiological processes," *J. Neurosci. Meth.*, vol. 275, pp. 93-121, Jan. 2017.

[28] S. Basu, X.Li, and G. Mochailidis, "Low rank and structured modeling of high-dimensional vector autoregressions", *IEEE Trans. on Singnal Processing*, vol. 67, no. 5, pp. 1207-1222, 2019.

[29] A. Lindquist and G. Picci, *Linear Stochastic Systems: A Geometric Approach to Modeling, Estimation and Identification*, Berlin Heidelberg: Springer, 2015.

[30] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*, MacMillan, 1992.

[31] M. Deistler, private communication.

[32] K. Zhou, Essentials of Robust Control, Prentice Hall, 1998.

[33] B. A. Francis, *A Couse in H_∞ Control Theory*, Lecture Notes in Control and Information Sciences, Springer-Verlag, 1987.

[34] C.I. Byrnes, T.T. Georgiou, and A. Lindquist. A generalized entropy criterion for Nevanlinna-Pick interpolation with degree constraint. *IEEE Trans. Automatic Control*, 46(6):822-839, 2001.

[35] C.I. Byrnes, T.T. Georgiou, and A. Lindquist, Analytic interpolation with degree constraint: A constructive theory with applications to control and signal processing, *Proceedings of the 38th IEEE Conference on Decision and Control* 1999, pp. 982–988.

[36] T.T. Georgiou, and A. Lindquist, Remarks on control design with degree constraint, *IEEE Trans.Automatic Control* vol. 51, no. 7, July 2006, pp. 1150-1156.

[37] C. I. Byrnes, T. T. Georgiou, A. Lindquist and A. Megretski, Generalized interpolation in H-infinity with a complexity constraint, *Transactions of the American Mathematical Society* **358** (2006), no. 3, pp. 965-987.

[38] Cui, Y. and Lindquist, A, The covariance extension equation: A Riccati-type approach to analytic interpolation, *IEEE Trans.Automatic Control*, 2021, to be published.

[39] Blomqvist A, Lindquist A, and Nagamune R, Matrix-valued Nevanlinna-Pick interpolation with complexity constraint: An optimization approach, *IEEE Trans. Automatic Control*, 2003, 48(12): 2172-2190.

[40] Ferrante, A, Pavon, M, and Zorzi, M, Application of a global inverse function theorem of Byrnes and Lindquist to a multivariable moment problem with complexity constraint, in *Three Decades of Progress in Control Sciences*, X. Hu et al. (Eds.), 2010, Springer, pp. 153-167.

[41] Ramponi, F, Ferrante, A and Pavon, M, A globally convergent matrical algorithm for multivariate spectral estimation, *IEEE Trans. Automatic Control*, 2009, 54(10): 2376–2388.

[42] Zhu; B and Baggio, G, On the existence of a solution to a spectral estimation problem á la Byrnes-Georgiou-Lindquist, *IEEE Transactions on Automatic Control*, 2019.

[43] D. Mustafa and K. Glover, Minimum entropy H-infinity control, *SIAM Rev.*, vol. 35, pp. 652-654, 1993.

[44] M. Forni and M. Lippi, The generalized dynamic factor model: representation theory, *Econometric Theory*,vol 17, 2001, pp 113-1141.

[45] M. Forni, M. Hallin, M. Lippi and L. Reichlin, The generalized dynamic factor model: identification and estimation, *The review of Economic and Statistics*, vol 65, 2000, pp 453-473.

[46] J. H.Stock and M. W.Watson, Dynamic Factor Analysis Models, *Oxford Handbook of Economic Forecasting*, Oxford University Press, 2010.

[47] C. Doz, D. Giannone and L. Reichlin, A two-step estimator for large approximate dynamic factor models based on Kalman filtering, *Journal of Econometrics*, vol 164:1, 2011, pp 188-205.

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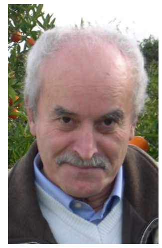

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