Identification of Low Rank Vector Processes

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Abstract

We study modeling and identification of stationary processes with a spectral density matrix of low rank. Equivalently, we consider processes having an innovation of reduced dimension for which Prediction Error Methods (PEM) algorithms are not directly applicable. We show that these processes admit a special feedback structure with a deterministic feedback channel which can be used to split the identification in two steps, one of which can be based on standard algorithms while the other is based on a deterministic least squares fit. Identifiability of the feedback system is analyzed and a unique identifiable structure is characterized. Simulations show that the proposed procedure works well in some simple examples.

Key words: Multivariable system identification, low-rank process identification, feedback representation, rank-reduced output noise.

1 Introduction

Quite often in the identification of large-scale time series one has to deal with *low rank* signals which in general, have a rank deficient spectral density. Such low rank time series may arise in diverse areas such as control systems, economics, networked systems, biology and other fields. Suppose we want to identify an $(m + p)$-dimensional vector time series y by modeling it as a weakly stationary zero-mean purely non deterministic (p.n.d.) process $y \equiv \{y(t) \,;\, t \in \mathbb{Z}\}$, having a rank deficient rational spectral density $\Phi(z)$ of rank m. This spectral density can always be written in factorized form

$$\Phi(e^{i\theta}) = W(e^{i\theta})W(e^{-i\theta})^\top, \qquad (1)$$

with W an $(m+p) \times m$ full rank stable rational spectral factor. This rank deficiency of the spectrum Φ and consequently of the process y appears in models used in a variety of applications and is discussed in the literature from different points of view.

The identification of such singular models has recently

been addressed in [1], [8], [11], [19], [20], [7] and [10]. Some papers, like [8],[11] propose adaptations of the Prediction Error Method (PEM) identification even if direct application of this principle is not possible due to the reduced-rank output noise. Singular autoregressive (AR) or autoregressive moving average (ARMA) models are an approach discussed in [19], [16]. These models make contact with factor models and dynamic factor analysis representations see [17], [9]. Applications to biological networks reconstruction are discussed in [3], [25]. Specific engineering examples where identification of rank-deficient processes is involved are discussed in [13],[18].

Let the process y be partitioned as

$$y(t) := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \qquad (2)$$

where $y_1(t)$, $y_2(t)$ are jointly stationary of dimension m and p. By properly rearranging the components of y, we may assume that $y_1(t)$ is a process of full rank m. The spectral density can also be partitioned as

$$\Phi(z) = \begin{bmatrix} \Phi_{11}(z) & \Phi_{12}(z) \\ \Phi_{21}(z) & \Phi_{22}(z) \end{bmatrix}, \qquad (3)$$

where $\Phi_{11}(z)$ is full rank.

In this paper we follow ideas first presented in [21], [23], [24] and especially in [10]. In [21] it is shown that there must exist a, in general non-causal, deterministic relation between the components of a singular vector process $y(t)$ while in [23], [24] and in [10] these deterministic relations are elucidated and specified by a special feedback model. In this paper we shall elaborate further on this special representation and on its use in identification. We shall show that the low rank structure implies a deterministic relation between the variables $y_1(t)$ and $y_2(t)$ which is in a sense "dual" of that introduced in [23] and [24]. We show that this structure is natural and is instrumental in the identification of low rank vector processes.

The structure of this paper is as follows. In Section 2 we introduce feedback models for low-rank processes, and prove the existence of a deterministic dynamical relation which reveals the special structure of these processes. In Section 3 we exploit the special feedback structure for identification of the deterministic relation and the transfer functions of the white noise representation models. In Section 4 we study the identifiability of the special feedback model. This structure is in general not identifiable and a characterization of all equivalent forward loop transfer functions is provided based on classical result of stabilization theory in robust control.

The equivalence of transfer functions in the feedback loop is discussed in Section 5. There are infinitely many stable forward transfer functions for a fixed feedback channel. However a unique stable forward loop representation with a white noise error process exists, which is a causal Wiener filter plus an orthogonal error term. Identification of this model is briefly discussed. The identification of processes with an external measurable input is considered in Section 6. Several simulation examples are reported in Section 7. Finally, in Section 8 we come to some conclusions.

Notations: All random processes in this paper are discrete-time ($t \in \mathbb{Z}$), wide sense stationary with zero mean and finite variance. Most notations comply with those used in the book [4] and are quite standard in the system identification literature,. In particular, multiplication by z is the one step ahead shift operator acting as: $zy(t) = y(t+1)$ and $y(t) = W(z)u(t)$ designates the response of a linear system with transfer function $W(z)$ to an input function $u \equiv \{u(t); t \in \mathbb{Z}\}$.

2 Feedback models of stationary processes

In this section, we shall first review the definition and some properties of general feedback models. Then we will derive a special feedback model for low-rank processes and prove the existence of a deterministic relation between $y_1(t)$ and $y_2(t)$.

Definition 1 (Feedback Model) *A* Feedback model *of the process* $y(t) := \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^\top$ *of dimension* $m+p$,

is a pair of equations

$$y_1(t) = F(z)y_2(t) + v(t), \tag{4a}$$
$$y_2(t) = H(z)y_1(t) + r(t), \quad t \in \mathbb{Z} \tag{4b}$$

satisfying the following conditions:

- *v and r are jointly stationary uncorrelated processes called the* modeling error *and the* input noise*;*
- *$F(z)$ and $H(z)$ are $m \times p$, $p \times m$ causal transfer function matrices, one of which is strictly causal, i.e., has at least one delay;*
- *the closed loop system mapping $[v, r]^\top$ to $[y_1, y_2]^\top$ is well-posed and* internally stable *;*

The block diagram illustrating a feedback representation is shown in Fig. 1. Note that the transfer functions $F(z)$ and $H(z)$ are in general not stable, but the overall feedback configuration needs to be internally stable [5, Chap. 3.2]. In the sequel, we shall often suppress the argument z whenever there is no risk of misunderstanding. The following construction shows that feedback repre-
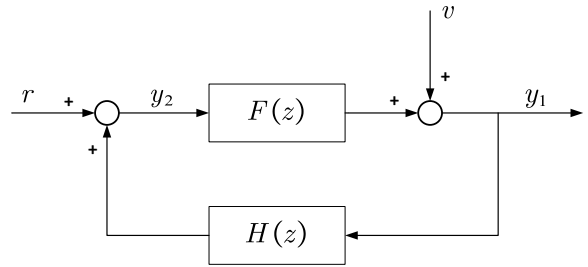

Fig. 1. Block diagram illustrating a feedback model

sentations of p.n.d. jointly stationary processes always exist. Let $\mathbf{H}_t^-(y_1)$ be the closed span of the past components $\{y_{11}(\tau), \ldots, y_{1m}(\tau) \mid \tau \leq t\}$ of the vector process y_1 in an ambient Hilbert space of second order zero-mean random variables [4] and let $\mathbf{H}_t^-(y_2)$ be defined likewise in terms of $\{y_{21}(\tau), y_{22}(\tau), \ldots, y_{2p}(\tau) \mid \tau \leq t\}$. A representation similar to (4) may be gotten from the formulas for causal Wiener filters expressing both $y_1(t)$ and $y_2(t)$ as the sum of the best linear estimate based on the past of the other process plus an error term

$$y_1(t) = \mathbb{E}\{y_1(t) \mid \mathbf{H}_{t-1}^-(y_2)\} + v(t), \tag{5a}$$
$$y_2(t) = \mathbb{E}\{y_2(t) \mid \mathbf{H}_t^-(y_1)\} + r(t). \tag{5b}$$

For a processes with a rational spectral density the Wiener predictors can be expressed in terms of causal rational transfer functions $F(z)$ and $H(z)$ as in Fig 1. Here we choose $F(z)$ to be strictly causal. An alternative representation with $H(z)$ strictly causal can also be given, to guarantee well-posedness of the feedback system. Although the errors v and r obtained by the procedure (5) may be correlated, in Appendix A we will

show that there exist feedback model representations where they are uncorrelated. The following result describes basic properties of feedback representations. It has been proven in [23], [10] and is also reported in the companion paper [24], therefore its proof is omitted.

Theorem 2 *The transfer function matrix $T(z)$ from $\begin{bmatrix} v \\ r \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ of the feedback model is given by*

$$T(z) = \begin{bmatrix} P(z) & P(z)F(z) \\ Q(z)H(z) & Q(z) \end{bmatrix}, \qquad (6a)$$

with

$$\begin{aligned} P(z) &= (I - F(z)H(z))^{-1}, \\ Q(z) &= (I - H(z)F(z))^{-1} \end{aligned} \qquad (6b)$$

where the inverses exist. Moreover, $T(z)$ is a full rank (invertible a.e.) and (strictly) stable function which yields

$$\Phi(z) = T(z) \begin{bmatrix} \Phi_v(z) & 0 \\ 0 & \Phi_r(z) \end{bmatrix} T(z)^*, \qquad (7)$$

where $\Phi_v(z)$ and $\Phi_r(z)$ are the spectral densities of v and r, respectively, and * denotes transpose conjugate.

Since $T(e^{i\theta})$ has full rank a.e., Φ is rank deficient if and only if at least one of Φ_v or Φ_r is. Thus the rank of Φ is equal to the sum of the ranks of Φ_v and Φ_r.

Lemma 3 *Suppose $(F\Phi_r F^* + \Phi_v)$ is positive definite a.e. on the unit circle. Then*

$$H = \Phi_{21}\Phi_{11}^{-1} - \Phi_r F^*(\Phi_v + F\Phi_r F^*)^{-1}(I - FH), \quad (8)$$

that is

$$H = \Phi_{21}\Phi_{11}^{-1} \qquad (9)$$

if and only if $\Phi_r \equiv 0$.

PROOF. From (6) and (7), we have

$$\begin{aligned} \Phi_{21} &= Q(H\Phi_v + \Phi_r F^*)P^* = QH\Phi_v P^* + Q\Phi_r F^* P^*, \\ \Phi_{11} &= P(\Phi_v + F\Phi_r F^*)P^*, \end{aligned}$$

and using the easily verified relations

$$PF = FQ, \quad HP = QH.$$

we get

$$\Phi_{21} = HP\Phi_v P^* + Q\Phi_r F^* P^*.$$

Adding and subtracting the term $HPF\Phi_r F^* P^*$ we end up with

$$\begin{aligned} \Phi_{21} &= H\Phi_{11} + (Q - QHF)\Phi_r F^* P^* \\ &= H\Phi_{11} + \Phi_r F^* P^* \end{aligned}$$

since $Q - QHF = I$. Then (9) follows if and only if $\Phi_r = 0$ since P is invertible and F times a spectral density can be identically zero only if the spectral density is zero as otherwise this would imply that the output process of a filter with stochastic input would have to be orthogonal to the input. □

In the following we specialize to feedback models of rank deficient processes. We shall show that there are feedback model representations where the feedback channel is described by a *deterministic relation* between y_1 and y_2.

Theorem 4 *Let y be an $(m+p)$-dimensional process of rank m. Any full rank m-dimensional subvector process y_1 of y can be represented by a feedback scheme of the form*

$$\begin{aligned} y_1 &= F(z)y_2 + v, & (10a) \\ y_2 &= H(z)y_1. & (10b) \end{aligned}$$

where the input noise v is of full rank m.

PROOF. Recall that n-tuples of real rational functions form a vector space $\mathbb{R}^n(z)$ where the rank of a rational matrix is the rank almost everywhere.

The claim is equivalent to the two statements
1. If we have the structure (10), i.e. $\Phi_r \equiv 0$; then y_1 is of full rank $m = \text{rank}(\Phi)$.
2. Conversely if y_1 is of full rank $m = \text{rank}(\Phi)$ then $\Phi_r \equiv 0$.

Part 1 follows from Lemma 3 since because of (7) then Φ_v must have rank $m(= \text{rank}(\Phi))$.
Part 2 is not so immediate. One way to show it could be as follows.

Since $\Phi(z)$ has rank m a.e. there must be a full rank $p \times (m+p)$ rational matrix which we write in partitioned form, such that

$$[A(z)B(z)]\Phi(z) = 0 \iff [A(z)B(z)] \begin{bmatrix} \Phi_{11}(z) \\ \Phi_{21}(z) \end{bmatrix} = 0 \quad (11)$$

$$\iff [A(z)B(z)] \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 0$$

where A, B are $p \times m$, $p \times p$ matrices and the last formula has the usual interpretation.

We claim that $B(z)$ must be of full rank p. One can prove this using the invertibility of $\Phi_{11}(z)$. For, suppose $B(z)$ is singular, then pick a p-dimensional non-zero row vector $a(z)$ in the left null space of $B(z)$ and multiply from the left the second relation by $a(z)$. This would imply that also $a(z)A(z)\Phi_{11}(z) = 0$ which in turn implies $a(z)A(z) = 0$ since Φ_{11} is full rank. However $a(z)[A(z) \ B(z)]$ cannot be zero for the matrix $[A(z) \ B(z)]$ is full rank p and hence $a(z)$ must be zero. So $B(z)$ must be full rank.

Now take any nonsingular $p \times p$ rational matrix $M(z)$ and consider instead $M(z)[A(z) \ B(z)]$, which provides an equivalent relation to (11). By choosing $M(z) = B(z)^{-1}$ we can reduce $B(z)$ to the identity to get

$$\begin{bmatrix} -H(z) & I \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 0$$

where $H(z)$ is a rational matrix function, so that one gets the deterministic dynamical relation

$$y_2(t) = H(z)y_1(t).$$

Substituting in the general feedback model one concludes that $y_2(t)$ must then be a functional of only the noise v since $y_1(t)$ is such. Therefore by the uncorrelation of v and r one must conclude that in the second equation of (4) r must be the zero process i.e. $\Phi_r = 0$. Hence a representation like (10) must hold. $\qquad\square$

3 Identification of low rank processes

Suppose we want to identify by a PEM method a model of an $(m+p)$-dimensional time series y of rank m. To this purpose, the model class should be selected to guarantee identifiablility (i.e. uniqueness) and it is specific of the PEM method that it should actually be an *innovation representation* of y which is well known to be essentially unique. This representation involves a *minimum phase* spectral factor $W(z)$ satisying (1) whereby

$$y(t) = W(z)e(t), \qquad (12)$$

where $e(t)$ is the m-dimensional normalized innovation process of y, a white noise of covariance I_m.

Consider then the model (12) block-partitioned as in (2),

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} := \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix} e(t), \qquad (13)$$

where y_1 and y_2 are described by the special feedback model (10). From the defining property of y_1 and y_2 in

our partition, $W_1(z)$ must be square $m \times m$, stable, causal and non singular (invertible a.e.) and $W_2(z)$ stable and causal.

Proposition 5 *The transfer function of the feedback channel in model* (10) *is given by the expression*

$$H(z) = W_2(z)W_1(z)^{-1} \qquad (14)$$

and is unique. In fact, it depends only on the joint spectrum (3). *Stability of H holds if and only if W_1 is minimum phase.*

PROOF. The formula follows from the partition (13) since both components are driven by the same full rank process $e(t)$. Formula (9) in Lemma 3, provides the alternative expression $H(z) = \Phi_{21}(z)\Phi_{11}(z)^{-1}$ which must obviously coincide with (14) since $\Phi_{2,1}(z) = W_2(z)W_1(z)^*$ and $\Phi_1(z) = W_1(z)W_1(z)^*$. It is then clear that $H(z)$ depends only on the joint spectrum (3) and must therefore be *unique* for a given partition of the vector process y. That stability of H holds if and only if W_1 is minimum phase follows since there cannot be cancellations in forming the quotient (14). It is shown in Appendix B that if $W(z)$ is minimum phase then $W_2(z)$ and $W_1(z)$ cannot have common unstable zeros which could cancel in forming the product (14). $\qquad\square$

Proposition 5 is in agreement with [24], where it was shown that $H(z)$, (F in [24]) is unique but in general not stable by a counterexample in Section V-A. (Also see its conference version [23].) This answered a question by Manfred Deistler in the negative. On the contrary we shall see that there are in general infinitely many transfer functions $F(z)$ generating y by means of the model (10).

3.1 Estimation of $H(z)$

Since the relation between y_2 and y_1 is completely deterministic we can identify $H(z)$ by imposing a deterministic transfer function model to the observed data. The model can be written as $A(z^{-1})y_2(t) - B(z^{-1})y_1(t) = 0$, $t = 1, \ldots, N$ (the minus sign is for convenience) where $A(z^{-1})$ and $B(z^{-1})$ are matrix polynomials in the delay variable z^{-1}, of dimension $p \times p$ and $p \times m$ such that

$$H(z) = A(z^{-1})^{-1}B(z^{-1}).$$

One can always choose $A(z^{-1})$ monic and parametrize the matrix polynomial $B(z^{-1})$ so that the transfer function corresponds to the difference equation

$$y_2(t) = -\sum_{k=1}^{q} A_k y_2(t-k) + \sum_{k=0}^{r} B_k y_1(t-k), \quad t = 1, \ldots, N, \qquad (15)$$

where we have written $A(z^{-1}) = I + \sum_{k=1}^{q} A_k z^{-k}$ and $B(z^{-1}) = \sum_{k=0}^{r} B_k z^{-k}$. The above equation involves delayed components of the observed trajectory data of y. The coefficients can then be estimated by solving a deterministic overdetermined linear system by least squares and a strongly consistent and unbiased result can be obtained whether the system is stable or not, assuming we know the true degrees of A and B. See the example in subsection 7.2.

Then, once W_1 is identified, the transfer function W_2 can be calculated using the relation

$$W_2(z) = H(z)W_1(z). \qquad (16)$$

This procedure however may fail if the true $W_1(z)$ in (12) is not minimum phase and the identification is done by a time-recursive least squares algorithm. In fact if $W_1(z)$ has unstable zeros then $H(z)$ is unstable and in this case the noise superimposed to the data may tend to excite the unstable modes of the system (15) and cause divergence. To bypass the constraint of minimum phase of the true system one should rely on algorithms processing the whole data batch in one shot.

3.2 Identification of W_1

Next, since y_1 (and W_1) is full rank, it seems that one could easily identify, say an ARMA innovation model for y_1 based only on observations of $y_1(t)$ on some large enough time interval. By this procedure we would ideally identify an innovation representation for y_1, say $y_1(t) = G_1(z)e_1(t)$ where however the minimum phase transfer function $G_1(z)$ does not necessarily coincide with the upper block of the joint innovation representation of y. This would be true only if the upper block $W_1(z)$ of the minimum phase $W(z)$ was also minimum phase, *which in general may not be true* (the same clearly holding also for the lower block). See Appendix B for a discussion of this point. In other words, the partitioned innovation representation of the full process y may not necessarily coincide with the separate innovation representations of the two components y_1 and y_2.

Therefore a PEM method applied to measurements of y_1 may not lead to a consistent estimate of the upper block $W_1(z)$ of the model (13) since there may be a nontrivial inner function $Q_1(z)$ such that

$$W_1(z) = G_1(z)Q_1(z) \qquad (17)$$

However it may easily be checked directly that estimating G_1 can nevertheless lead to a consistent estimate of the joint spectrum.

Proposition 6 *Assume that the transfer function $H(z)$ is estimated as described in the previous subsection, that is using the data (y_1, y_2) and asymptotically satisfying the relation (16). Then, even if the upper block $W_1(z)$ of the joint (minimum phase) transfer function $W(z)$ is not*

minimum phase, a consistent estimate of the minimum phase transfer function $G_1(z)$ does nevertheless produce a consistent estimate of the joint spectral density of the (joint) process y.

PROOF. The statement is obviously true for the auto spectral density $\Phi_{11}(z)$. Then just recall that the cross spectral density of y_2 and y_1 can be expressed as

$$\Phi_{21}(z) = H(z)\Phi_1(z) = H(z)G_1(z)G_1(z)^*.$$

Using the estimate $\hat{G}_1(z)$ in place of $W_1(z)$ in formula (16) to compute the estimate $\hat{W}_2(z)$, although $\hat{W}_2(z) := \hat{H}(z)\hat{G}_1(z)$ may be a non-consistent estimate of $W_2(z)$, it does result in a consistent estimate of the cross spectrum $\Phi_{21}(z)$. A similar argument can be used for Φ_{22}. $\qquad \square$

We may however be able to recover the missing inner factor Q_1 in the outer-inner factorization (17) from the expression $H(z) = W_2(z)Q_1^*(z)G_1(z)^{-1}$, that is from

$$H(z)G_1(z) = W_2(z)Q_1^*(z) = W_2(z)Q_1(z)^{-1}. \qquad (18)$$

One can get estimates of W_2 and Q_1 by performing a left-coprime factorization in the rational H^∞ space (see e.g. [14, sect. 5.4]), of the estimated product $\hat{H}(z)\hat{G}_1(z)$ imposing that Q_1 should be inner (see e.g. [6]). This guarantees uniqueness, see again [14, p. 368]. The conjugate inner function Q_1^* must contain exactly all the unstable poles of the left member.

Hence we may in principle be able to obtain a consistent estimate of the full minimum phase model W even when W_1 is not minimum phase. The calculations are easy when W_1 is scalar. In the matrix case, one would need to use coprime factorization algorithms in terms of state-space realization which we shall not dwell into.

4 Identifiability of the feedback model

The procedure described above does not take into account the structure of the first equation in model (10). We shall analyze now the identification of an "*internal*" description of y_1 involving the transfer functions F, K and H. Assume that the model (13) is in innovation form, with the process $e(t)$ the innovation of the joint process $y(t)$ and let

$$y_1 = F(z)y_2 + K(z)e, \qquad (19a)$$
$$y_2 = H(z)y_1. \qquad (19b)$$

be the corresponding feedback representation with $K(z)$ a square spectral factor such that $v(t) := K(z)e(t)$,

which we assume minimum phase for identifiability. From (6) we have

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = T \begin{bmatrix} K \\ 0 \end{bmatrix} = \begin{bmatrix} PK \\ QHK \end{bmatrix} = \begin{bmatrix} PK \\ HPK \end{bmatrix}, \qquad (20)$$

with both $P(z)K(z)$ and $H(z)P(z)K(z)$ submatrices of a minimum phase transfer function.

One may ask how one could recover the direct transfer function $F(z)$ from the identified $W_1(z)$ and $H(z)$. This would amount to solving for F the relation $W_1 = (I - FH)^{-1}K$ which, assuming H is given, contains *two unknowns*. Hence $F(z)$ and $K(z)$ are *not identifiable* as they do not correspond uniquely to the minimum phase representation (13) and hence do not correspond uniquely to the joint spectral density of $y(t)$. In other words, there are in general infinitely many pairs $(F(z), K(z))$ realizing in feedback form the innovation representation (13). This actually agrees with the well-known identifiability analysis of feedback systems which dates back to [12], see the example in Sect. VI.

4.1 On equivalent feedback structures

In our setting the causal transfer function $H(z)$ of the feedback channel is uniquely determined by the two components of the process y, once the partition is fixed and known, while there are in general a multitude of pairs (F, K) yielding the same transfer function $W_1(z)$. Note that each such pair should make W_1 stable. In particular, once H is given, each F should make the feedback configuration (10) internally stable. Characterizing the set of such F's can be regarded as the "dual" of a stabilization problem in control, which is also discussed in our companion paper [24] on modeling of low rank vector processes. Here we have a more limited scope than [24]; we only want to analyze the identifiability of the system by explicitly describing all pairs of transfer functions (F, K) which realize the same stable W_1. Consider first the transfer function from $v(t)$ to $y_1(t)$ (also called the sensitivity function):

$$P(z) = [I - F(z)H(z)]^{-1}.$$

Since the feedback system must be internally stable $P(z)$ needs to be analytic in the complement of the open unit disk, without unstable pole-zero cancellation between $F(z)$ and $H(z)$. Assuming for the moment that $H(z)$ is a proper stable rational function, there is a whole class of proper rational functions $F(z)$ which accomplish this job. In the scalar case they are all described by the formula [5, Chapter 5.1],

$$F(z) = \frac{S(z)}{1 + S(z)H(z)} \qquad (21)$$

where $S(z)$ is an arbitrary proper stable rational function. The corresponding sensitivity function is given by

$$P(z) = 1 + S(z)H(z)$$

linearly parametrized by an arbitrary such $S(z)$. All corresponding $K(z)$ are then gotten from the relation (19a), that is

$$K(z) = P(z)^{-1}W_1(z)$$

so that all such (F, K) yield the same transfer function $W_1(z)$.

When $W_1(z)$ is not minimum phase and $H(z) = W_2(z)W_1(z)^{-1}$ fails to be stable, closed-loop stability can still be characterized by using a coprime stable proper-rational factorization of $H(z)$ yielding a more general parametrization of all F's as described in [5, Sect. 5.4] (involving the so-called Youla parametrization).

In the matrix case, still assuming a stable H (actually F in the original setting), there is a parametrization formula similar to (21), see e.g. reference [14]. But for the unstable case one needs to use matrix coprime factorizations to obtain the stabilizing F. Reference [24, Sect. VI and Sect. VIII-D] discusses the general case for low rank processes, both applicable to stable and unstable H.

5 About stability of $F(z)$

Note that the transfer function (21) may in general not be stable. However, as seen from (21), there are infinitely many possible $F(z)$ and there may be one that is stable. In this section we shall ask the following question: If one restricts F to be stable with at least one unit delay, does there exists a unique feedback representation (19)? Since the identifiability analysis of the previous section involves also the transfer function $K(z)$, it is quite evident that the answer should be negative. The following example provides in fact a few different pairs (F, K), all with a strictly causal stable F, which realize the same transfer function $W(z)$.

Example 1 *Let a 2×1 transfer function $W(z)$ be partitioned by two scalar blocks of respective transfer functions*

$$W_1(z) = \frac{z^3}{(z - 0.5)(z + 0.5)(z - 0.2)}, \qquad (22a)$$

$$W_2(z) = \frac{z^3}{(z - 0.5)(z - 0.2)(z + 0.1)}. \qquad (22b)$$

the corresponding transfer function H being (from (14))

$$H(z) = \frac{z + 0.5}{z + 0.1}.$$

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We can provide three different pairs F, K realizing the system, all three with a stable strictly causal F. The first being

$$F_1 = \frac{-0.4}{z + 0.5}, \tag{23a}$$

$$K_1 = \frac{z^3}{(z - 0.5)(z - 0.2)(z + 0.1)}. \tag{23b}$$

the second,

$$F_2 = \frac{0.4}{z + 0.5}, \quad K_2 = \frac{z^3(z - 0.3)}{(z + 0.5)(z - 0.5)(z - 0.2)(z + 0.1)}. \tag{24a}$$

and finally

$$F_3 = \frac{(0.2z^2 + 0.25z - 0.5)(z + 0.1)}{(z + 0.5)z^3}, \tag{25a}$$

$$K_3 = 1. \tag{25b}$$

To check that all three pairs realize the minimum phase W_1 in the example, just calculate the noise transfer functions K_i from $K_i = (I - F_i H)W_1$, yielding all K_i to be minimum phase, and the corresponding $P_i = (I - F_i H)^{-1} = W_1 K_i^{-1}$ being stable.

Choosing $K(z)$ equal to a constant as in the third example turns out to provide a remarkable interpretation of $F(z)$. Indeed, in Proposition 7 below we shall show that under certain conditions, this will precisely provide the unique stable causal F.

Proposition 7 *Assume that W_2 is of full column rank[1] and minimum phase then there is a representation (19) where F is stable and strictly causal, that is $F(z) = z^{-1} F_1(z)$ with $F_1(z)$ analytic in $\{|z| \geq 1\}$ and $K(z)$ is a constant matrix K_+. In fact, this $F(z)$ coincides with the transfer function $F_+(z)$ of the causal Wiener filter*

$$F_+(z)y_2(t) = \mathbb{E}\{y_1(t) \mid \mathbf{H}_{t-1}^-(y_2)\} \tag{26}$$

and the prediction error $\tilde{y}_1(t) := y_1(t) - F_+(z)y_2(t)$ is equal to $K_+ e_1(t)$ where $e_1(t)$ is the innovation of the process y_1. The representation

$$y_1(t) = F_+(z)y_2(t) + K_+ e_1(t) \tag{27}$$

is the unique feedback representation of $y_1(t)$ in which $v(t)$ is uncorrelated with the strict past of y_2. If $W_1(z)$ is

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[1] Although $W_1(z)$ being square and invertible follows from the defining property of y_1 in our partition, $W_2(z)$ may generally be of column rank smaller than m. Although the result holds also in this circumstance, we shall not deal with this technical nuisance here as it would obscure the main ideas of the proof.

also minimum phase then $e_1 = e$ and one can substitute e_1 with the joint innovation process e.

PROOF. The statement is a consequence of the orthogonal decomposition

$$y_1(t) = G_1(z)e_1(t) = G_1(\infty)e_1(t) + [G_1(z)]_+ e_1(t - 1) \tag{28}$$

where e_1 is the innovation of y_1, $W_1 = G_1 Q_1$ is the outer-inner factorization of W_1 and $[\cdot]_+$ denotes the projection operator onto $z^{-1}H^2$ yielding the strictly causal part of a function. Then $e_1(t) = Q_1(z)e(t)$ and hence

$$\mathbf{H}_t^-(e_1) \subset \mathbf{H}_t^-(e).$$

However, by our assumptions $\mathbf{H}_t^-(e) = \mathbf{H}_t^-(y_2)$ since, being full column rank, W_2 has a left inverse whereby $e(t) = W_2^{-L}(z)y_2(t)$. Hence the last term of (28) belongs to $\mathbf{H}_{t-1}^-(y_2)$. But by the subspace inclusion above $e_1(t) \perp \mathbf{H}_{t-1}^-(e) = \mathbf{H}_{t-1}^-(y_2)$. Therefore, by the orthogonal projection lemma [4, p. 27] the last term is the Wiener predictor. Note that $K_+ = G_1(\infty)$ is non singular.
The proof of uniqueness is just based on the uniqueness of the orthogonal decomposition of $y_1(t)$ as a linear causal functional of the strict past of y_2 plus an orthogonal error part. The linear causal functional of the strict past of y_2 must then be the orthogonal projection onto $\mathbf{H}_{t-1}^-(y_2)$.
If $v(t) = K_+ e(t)$ with a constant K_+, since $e(t)$ is the innovation of the joint process, it must in particular be uncorrelated with the strict past $H_{t-1}^-(y_2)$ of y_2. Hence if $F(z)$ is strictly causal and analytic, $F(z)y_2(t)$ in (10) must coincide with the orthogonal projection $\mathbb{E}\{y_1(t) \mid y_2(s); s < t\}$ by the orthogonal projection lemma [4, p. 27]. Clearly $v(t)$ must then coincide with the prediction error $\tilde{y}_1(t)$. □

Remark 8 Note that for a general strictly causal F, the error process $v(t)$ is given by $v(t) = [W_1(z) - F(z)W_2(z)]e(t) := K(z)e(t)$ where $K(z)$ is square analytic but in general not constant.

In general, we may directly calculate F_+ and K_+ by the Wiener filtering formula [4, p. 105]

$$F_+ = [W_1 W_2^* G_2^{-*}]_+ G_2^{-L}. \tag{29}$$

where $G_2(z)$ is the minimum phase spectral factor of $\Phi_2(z) = W_2(z)W_2(z^{-1})^\top$, G_2^{-L} its (Moore-Penrose) left inverse and G_2^{-*} is the conjugate of G_2^{-L} which coincides with the right inverse of G_2^*. If W_2 is minimum phase the above simplifies to

$$F_+(z) = [W_1(z)]_+ W_2^{-L}(z). $$

In case $W_1(\infty)$ is normalized to the identity so is K_+, and from $I - F_+ H = W_1^{-1}$ one gets

$$F_+(z)W_2(z) = (I - W_1(z)^{-1})W_1(z) = W_1(z) - I \tag{30}$$

so, when you compute $F_+(z)y_2(t) = F_+(z)W_2(z)e(t)$ you generate a stochastic process which has spectral density

$$F_+(z)W_2(z)W_2(z)^* F_+(z)^* = (W_1(z) - I)(W_1(z) - I)^*$$

Hence this process is always well-defined. In general one should really not worry so much about stability of F but make sure that the process $F(z)y_2$ should have a well defined spectral density and a finite variance.

Remark 9 The model (27) is called an *equation error model* [15, p. 203] because of the white error term, which is unique and hence identifiable, so the Prediction Error method should be able to identify the transfer function directly from observed data. One should however recall that this model leads to a predictor which is a *nonlinear function of the parameters of the denominator* and is moreover *constrained by the stability condition on* $F_+(z)$. Because of these difficulties one may attempt a simple least squares estimation method by using matrix-fraction descriptions and transforming it into a special ARMAX model. The least squares method however can be consistent only if $F_+(z)$ is a FIR-type transfer function, that is the denominator of $F_+(z)$ is a constant (see again [15, Sect. 7.3]).
With this proviso, in spite of feedback, a suitably constrained PEM method may work anyway [22, p. 416], [15], see Appendix C.

6 Identification of a low rank model with an external input

Suppose we want to identify a multidimensional system with an external input $u(t)$, say

$$y(t) = F(z)u(t) + K(z)e(t) \tag{31}$$

where e is a white noise process whose dimension is *strictly smaller than the dimension of* y and the input u is completely uncorrelated with e (no feedback). In this case the model is called low-rank. This problem is discussed in [8] and [11].

When $\dim e = \dim y$ and $K(z)$ is square invertible, one could attack the problem by a standard PEM method. The method however runs into difficulties when the noise is of smaller dimension than y since then the predictor and the prediction error are not well-defined.

Referring to the general feedback model for the joint process we can always assume F causal and $K(\infty)$ full

rank and normalized in some way. Consider then the prediction error of $y(t)$ given the past history of u. We have

$$\tilde{y}(t) := y(t) - \mathbb{E}[y(t) \mid \mathbf{H}_t^-(u)] = K(z)e(t) \tag{32}$$

since, by causality of $F(z)$, the Wiener predictor is exactly $F(z)u(t)$. Hence $\tilde{y}$ is a low rank time series in the sense described in the previous sections (now with the current $K(z)$ playing the role of $W(z)$). In principle we could then use the procedure described above for time series as we could preliminarily estimate $F(z)$ by solving a deterministic regression of $y(t)$ on the past of u and hence get $\tilde{y}(t)$. If we choose linear least square methods, we will obtain a consistent estimation. Then a standard ARMA identification can be applied to estimate $K(z)$ in terms of the pre-processed data $\tilde{y}(t)$.

7 Simulation Examples

We shall mostly discuss scalar examples and use the standard notations

$$A(z^{-1}) = 1 + \sum_{k=1}^{q} a_k z^{-k},$$

$$B(z^{-1}) = z^{-1} B_1(z^{-1}) = z^{-1}(\sum_{k=0}^{r} b_k z^{-k}),$$

$$C(z^{-1}) = 1 + \sum_{k=1}^{q} c_k z^{-k}.$$

Note that A and C have the same degree q.

7.1 Example 2

As a first simulation example consider a two-dimensional process of rank 1 described by

$$y(t) = \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix} e(t)$$

where both $W_1(z)$ and $W_2(z)$ are minimum phase rational transfer functions and e is a scalar Gaussian white noise of zero mean and variance λ^2. By simulation we produce a sample of two-dimensional data. With these data we shall:

- Identify a model for y_1 and estimate $H(z)$ according to the first procedure. And do the same for the other component.
- Estimate $F_+(z)$ and $K_+(z)$ in (27) using the estimated value of $W_1(z)$ and $H(z)$.

We start by simulating a two-dimensional process $y(t)$ of rank 1 described by (13) where e is a scalar zero mean
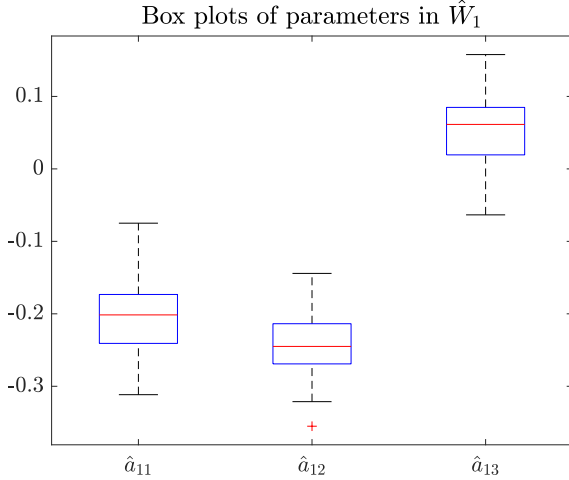
Fig. 2. Box plots of $\hat{a}_{1k}$ for $k = 1, 2, 3$ in example 2.

Fig. 3. Box plots of $\hat{a}_{2k}$ for $k = 1, 2, 3$ in example 2.

white noise of variance $\lambda^2 = 1$ and choose W_1 and W_2 as in (22). These functions are normalized at infinity and minimum phase rational transfer functions. Note that in this particular example both y_1 and y_2 are full rank so that our procedure would work for both.

We have generated 100 groups of two-dimensional time series with $N = 500$ data points $\{y_i(t); t = 1, \ldots, N, i = 1, 2\}$ and use the Monte-Carlo simulations in MATLAB. Box plots are used to show the results of Monte-Carlo simulations condensing some features of the estimates distribution.

Assume the orders of W_1 and W_2 are known. Since the two AR models of y_1 and y_2 are of order 3, we just implement two AR identification in MATLAB for models of the form

$$y_i(t) = -\sum_{k=1}^{3} a_{i,k} y_i(t-k) + e(t), \qquad t = 1, \ldots N,$$

The box plots of the estimated parameters in $\hat{W}_1$ and $\hat{W}_2$ are shown in Fig. 2 and Fig. 3. [2] In the two box plots, all median estimated values are close to the real ones, with the ranges of estimation values acceptable and only one outlier for $\hat{a}_{12}$. We use the average of 100 runs of Monte-Carlo simulation to estimate the asymptotic covariance of the estimated parameters which are of the order of magnitudes $10^{-2}/100 = 10^{-4}$, quite small compared with the magnitude of parameters. The box plots in Figure 2 and 3 show that our AR estimators work well.

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[2] In all box plots, the red horizontal line is the median of the data, the blue box contains half of the data points, the horizontal lines are at 25% and 75% level. The black tails (black horizontal lines) are at the minimum and maximum values, except for the outliers that are indicated by a red '+' sign.
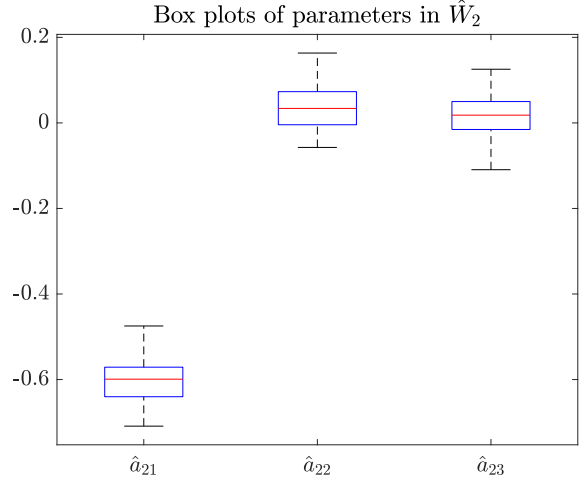
Next we do least squares estimation of the transfer function $H(z)$. Since it satisfies the identity

$$W_2(z) = H(z)W_1(z), \quad W_1(z) = \bar{H}(z)W_2(z),$$

we have the following theoretical formulas for H and $\bar{H}$:

$$H(z) = \frac{1 + 0.5z^{-1}}{1 + 0.1z^{-1}}, \quad \bar{H}(z) = \frac{1 + 0.1z^{-1}}{1 + 0.5z^{-1}}.$$

which is equivalent to the difference equation

$$(1 + 0.1z^{-1})y_2(t) = (1 + 0.5z^{-1})y_1(t),$$

This is just a theoretical model which we keep for comparison.

Assuming now that we don't know the true degrees of the model polynomials in (15); then we first carry on an order estimation to choose the appropriate q and r in the model

$$y_2(t) - y_1(t) = -\sum_{k=1}^{q} a_k y_2(t-k) + \sum_{k=1}^{r} b_j y_1(t-j),$$

and then use least square to get estimates of the parameters of the model

$$\hat{H}(z) = \frac{1 + \sum_{k=1}^{r} \hat{b}_k z^{-k}}{1 + \sum_{k=1}^{q} \hat{a}_k z^{-k}}.$$

From a BIC table values we see that when $(q, r) = (1, 1)$ the BIC index reaches a minimum. So we do least squares estimation of a first order model

$$y_2(t) - y_1(t) = -a_1 y_2(t-1) + b_1 y_1(t-1).$$

All the parameter estimates turn out to be equal to the true values of the parameters $a_1 = 0.1$, $b_1 = 0.5$, affected

by extremely small errors. In Monte-Carlo simulations, the calculated estimated variances are all smaller than 10^{-29}. We don't show box plots here. For estimating $\bar{H}(z)$, we obtain very similar results, which are therefore not presented. Here both H and $\bar{H}$ are stable functions. We shall check if our algorithm also works when H is not stable in the next example.

Next we shall use the previous estimates $\hat{W}_1$ and $\hat{H}$ to calculate estimates of F and K. We choose one estimate from the previous Monte-Carlo simulations, namely

$$\hat{W}_1 = \frac{1}{1 - 0.1627z^{-1} - 0.2256z^{-2} + 0.0505z^{-3}},$$
$$\hat{H} = \frac{1 + 0.5000z^{-1}}{1 + 0.1000z^{-1}}.$$

From Proposition 7, we know that there is one and only one pair of F_+ and K_+ with F_+ the strictly causal Wiener filter. In our case with W_1, W_2 are both normalized and minimum phase, and from (30) we have the estimate of F_+ described by

$$\hat{F}_+ = (1 - \hat{W}_1^{-1})\hat{H}^{-1}$$
$$= \frac{z^{-1}(0.1627 + 0.2256z^{-1} - 0.0505z^{-2})(1 + 0.1000z^{-1})}{1 + 0.5000z^{-1}}$$

and $\hat{K}_+$ the constant part of $\hat{W}_1$, i.e.,

$$\hat{K}_+ = \hat{W}_1(\infty) = 1.$$

The parameters of these functions are very close to the true values and hence appear to be consistent estimates of F_3, K_3 in (25).
In fact, we get $\hat{K}_+ = 1$ each time in different simulations. What's more, since we are identifying with true orders in the previous Monte-Carlo simulations, we have a $\hat{F}$ with true orders as in (25), i.e.,

$$F_+ = \frac{z^{-1}(0.2 + 0.27z^{-1} - 0.025z^{-2} - 0.005z^{-3})}{1 + 0.5z^{-1}}$$

The box plot of the estimated parameters in $\hat{F}_+$,

$$\hat{F}_+ = \frac{z^{-1}(\sum_{k=0}^{3} \hat{b}_k z^{-k})}{1 + \hat{a}_1 z^{-1}}.$$

are shown in Figure 4, showing that $\hat{F}_+$ obtained from $\hat{W}_1$ and the calculations in Section 5 is a good estimate of the true causal Wiener filter F_+.

7.2 Example 3

In this subsection, a simple simulation example will be given to show that our method can identify H well also
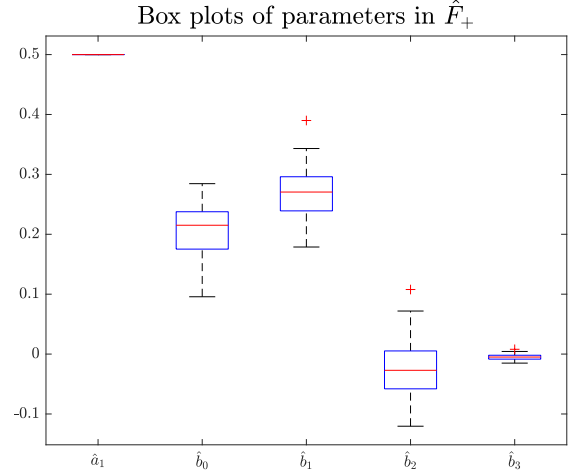

Fig. 4. Box plots of the parameters in $\hat{F}_+$ in example 2.

when it is unstable, and can recover the minimum phase factor when W_1 is not minimum phase as discussed in subsection 3.2.

Consider a two-dimensional process $y(t)$ described by (13), where e is a zero mean white scalar noise of variance $\lambda^2 = 1$, and W has the two blocks with transfer functions

$$W_1 = \frac{z+2}{z-0.2}, \quad W_2 = \frac{z-2}{z-0.2}.$$

It is easy to obtain an outer-inner factorization of W_1 as (17), where

$$G_1 = \frac{2z+1}{z-0.2} = \frac{2 + z^{-1}}{1 - 0.2z^{-1}}, \quad Q_1 = \frac{z+2}{2z+1}.$$

And we have the calculated value

$$H = \frac{1 - 2z^{-1}}{1 + 2z^{-1}},$$

which is not stable.
Here for simplicity, we do not use Monte-Carlo simulations and order estimations. We just generate one group of data as in Example 2, with e scalar zero mean and variance 1. Assume the orders of G_1 and H are known.

Though G_1 is not normalized at infinity, we may still implement an ARMA estimation first in MATLAB and obtain an estimated model

$$y_1(t) - 0.1442y_1(t-1) = \hat{e}(t) + 0.5666\hat{e}(t-1),$$

where the variance of the innovation $\hat{e}$ is $\hat{\lambda}^2 = 4.3127$. Then calculate the corresponding estimate of G_1

$$\hat{G}_1 = \frac{\lambda(1 + 0.5666z^{-1})}{1 - 0.1442z^{-1}} = \frac{2.077z + 1.177}{z - 0.1442},$$
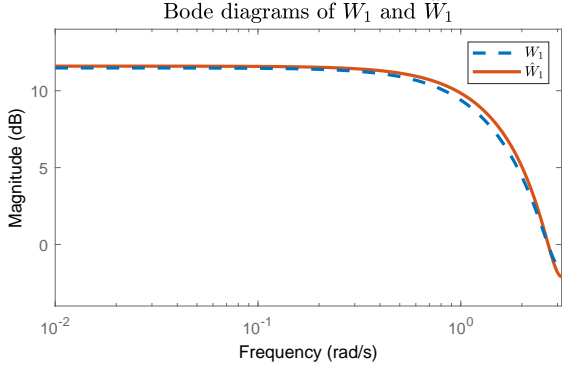
Fig. 5. Bode diagrams of W_1 and $\hat{W}_1(z)$ in example 3.

which is minimum phase.
Next we estimate H by least squares on the model

$$y_1(t) + a_1 y_1(t-1) = y_2(t) + b_1 y_2(t-1),$$

and obtain the estimate

$$\hat{H} = \frac{1 + \hat{b}_1 z^{-1}}{1 + \hat{a}_1 z^{-1}} = \frac{1 - 2.0000 z^{-1}}{1 + 2.0000 z^{-1}},$$

which is practically equal to the true H, with an estimation error variance of 1.0607×10^{-29}. Simulation results show that the least squares method works well in identifying unstable H's.

Since in this example G_1 and W_2 are scalar, we do not need coprime factorization for obtaining Q_1. In this case, Q_1^* is the conjugate inner factor of H of formula (18), i.e., Q_1 is the greatest inner factor of H^{-1}. From

$$\hat{H}^{-1} = \frac{z + 2.0000}{z - 2.0000} = \frac{2.0000z + 1}{z - 2.0000} \cdot \frac{z + 2.0000}{2.0000z + 1},$$

we have the estimate,

$$\hat{Q}_1 = \frac{z + 2.0000}{2.0000z + 1}.$$

Hence the estimate of W_1 is

$$\hat{W}_1 = \hat{G}_1 \hat{Q}_1 = \frac{2.077z^2 + 5.33z + 2.353}{2z^2 + 0.7117z - 0.1442},$$

whose magnitude Bode graph is compared with the true W_1 in Fig. 5. The Bode diagrams show that we can obtain a consistent estimate of W_1 even if it is not minimum phase. The correponding estimate of W_2 can be calculate from

$$\hat{W}_2 = \hat{H}\hat{W}_1 = \frac{1.038z^2 - 1.489z - 1.177}{z^2 + 0.3558z - 0.0721},$$

whose Bode diagram is close to that of the true W_2. We omit the graphs due to space limitations. It is easy to check $\hat{W} = [\hat{W}_1 \ \hat{W}_2]^\top$ is minimum phase.

7.3 Example 4

In this subsection we consider the identification of a two-dimensional process of rank 1 subjected to an external input u. We generate a scalar white noise u independent of e and identify a 2-dimensional process model (31) as described in the previous section 6.
In this example the true system is described by

$$F(z) = z^{-1} \begin{bmatrix} 0.3 + 0.7z^{-1} + 0.3z^{-2} \\ 0.15 + 0.9z^{-1} - 0.5z^{-2} \end{bmatrix},$$

$$K(z) = \begin{bmatrix} \frac{1 + 0.1z^{-1} + 0.4z^{-2}}{1 + 0.3z^{-1} + 0.4z^{-2}} \\ \frac{1 - 0.1z^{-1} + 0.4z^{-2}}{1 - 0.2z^{-1} + 0.1z^{-2}} \end{bmatrix}.$$

$$(33)$$

where we have used the same F as in [8] (called $G(q)$ there). Since the K_2 of [8] is not normalized to 1, we use a different one. Both components of our $K(z)$ here are normalized and minimum-phase so the overall model is an innovation model.

From the model (33) we generate 100 groups of two-dimensional time series of $N = 500$ data points $\{y_i(t); t = 1, \ldots, N, \ i = 1, 2\}$. Monte-Carlo simulations are run with u and e independent scalar white noises of variances 2 and 1. Of course here we also measure the input time series u. Suppose we do not know the orders of F_i for $i = i, 2$.

First, letting $F_i(z) = zA_i(z^{-1})^{-1}B_i(z^{-1})$ for $i = 1, 2$, where the polynomials

$$A_1(z^{-1}) = 1 + \sum_{k=1}^{q_1} a_{1,k} z^{-k}, \quad A_2(z^{-1}) = 1 + \sum_{k=1}^{q_2} a_{2,k} z^{-k}.$$

$$B_1(z^{-1}) = \sum_{k=0}^{r_1} b_{1,k} z^{-k}, \quad B_2(z^{-1}) = \sum_{k=0}^{r_2} b_{2,k} z^{-k}$$

correspond to the dynamic relations

$$A_i(z^{-1})y_i(t) = B_i(z^{-1})u(t-1) + \varepsilon_i(t), \quad t = 1, \ldots, N, \ i = 1, 2$$

where we have added a small white noise error term. We do a standard least squares regression on these models, written in the form,

$$\hat{y}_i(t) = -\sum_{k=1}^{q_i} a_{i,k} y_i(t-k) + \sum_{k=0}^{r_i} b_{i,k} u(t-1-k), \quad (i = 1, 2).$$

$$(34)$$

where the orders are to be estimated. Order estimation by minimum BIC leads to choose $(q_1, r_1) = (1, 3)$ and $(q_2, r_2) = (2, 4)$. Although we don't get the right model structures, with these orders we get the reasonable box plots shown in Fig. 6 and Fig. 7, with very few outliers.
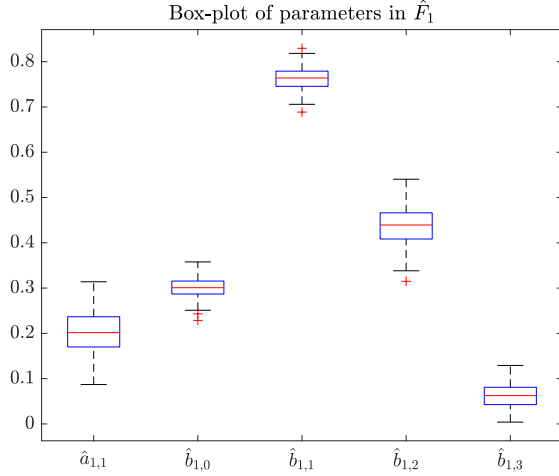
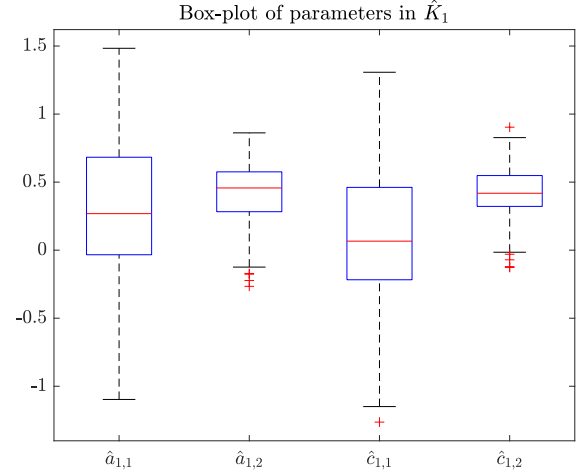Fig. 6. Box plots of parameters of $\hat{F}_1(z)$ in example 4.

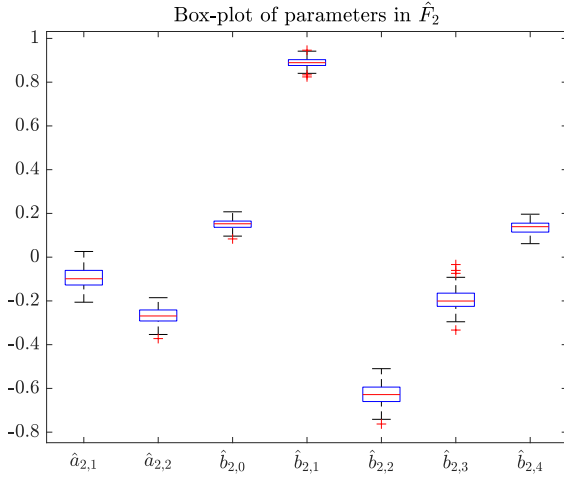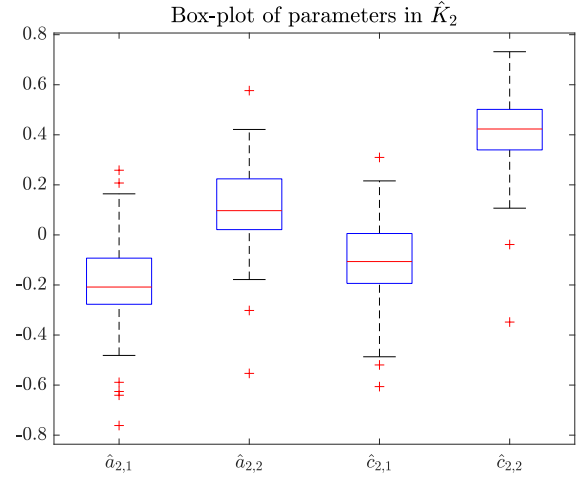Fig. 7. Box plots of parameters in $\hat{F}_2(z)$ in example 4.

Then we estimate $K_1(z)$ and $K_2(z)$ by an ARMA routine, this time with true degrees. This may be useful to test the influence of the wrong model structure used in estimating F_1 and F_2. We let $K_i = A_i^{-1}C_i$ so that

$$A_i(z^{-1})\tilde{y}_i(t) = C_i(z^{-1})e(t-1), \quad i = 1, 2$$

where

$$A_i(z^{-1}) = 1 + a_{i,1}z^{-1} + a_{i,2}z^{-2},$$
$$C_i(z^{-1}) = 1 + c_{i,1}z^{-1} + c_{i,2}z^{-2}.$$

Box plots of the Monte-Carlo simulations are shown in Fig. 8 and Fig. 9. The results are reasonable even if we didn't use the true model structures when estimating F. The plots show that these estimated parameters have however a rather larger variance.

Fig. 8. Box plots of parameters in $\hat{K}_1(z)$ in example 4.

Fig. 9. Box plots of parameters in $\hat{K}_2(z)$ in example 4.

8 Conclusions

In this paper we have shown that a rank-deficient process admits a special feedback representation with a deterministic feedback channel, which can be used to split the identification in two steps, one of which can be based on standard PEM algorithms while the other is based on a deterministic least squares fit. Identifiability of these feedback structures is not guaranteed and we show how to choose an identifiable representative. Simulations show that standard identification algorithms can be easily applied to identify the transfer functions of the model.

References

[1] Chiuso A. and Pillonetto G. A Bayesian approach to sparse dynamic network identification. *Automatica*, 48(8):1553–1565, 2012.

[2] Fuhrmann P. A. *Linear Systems and Operators in Hilbert spaces.* McGraw Hill, New York, 1981.

[3] Hidalgo C. A., Blumm N., Barabási A-L., and Christakis N. A. A dynamic network approach for the study of human phenotypes. *PLoS Computational Biology*, 5(4):e1000353, 2009.

[4] Lindquist A. and Picci G. *Linear Stochastic Systems: A Geometric Approach to Modeling, Estimation and Identification.* Springer, 2015.

[5] Doyle J. C., Francis B. A., and Tannenbaum A. R. *Feedback Control Theory.* McMillan, 1992.

[6] Oară C. and Varga A. Minimal degree coprime factorization of rational matrices. *SIAM Journal on Matrix Analysis and Applications*, 21(1):245–278, 1999.

[7] Peña D. and Box G. E. P. Identifying a simplifying structure in time series. *Journal of the American Statistical Association*, 82(399):836–843, 1987.

[8] Van den Hof P., Weerts H., and Dankers A. Prediction error identification with rank-reduced output noise. In *Proceedings of 2017 American Control Conference*, pages 382–387, Seattle, USA, 2017.

[9] Bottegal G. and Picci G. Modeling complex systems by generalized factor analysis. *IEEE Transactions on Automatic Control*, 60(3):759–774, 2015.

[10] Picci G., Cao W., and Lindquist A. Modeling and identification of low rank vector processes. In *Proceedings of the 2021 IFAC SYSID conference, also in arXiv :2012.05004v2[eess.SY]*, pages 631–636, Padova, Italy, 2021. Science Direct.

[11] Weerts H., Van den Hof P., and Dankers A. Identifiability of linear dynamic networks. *Automatica*, 89:247–258, 2018.

[12] Gustavsson I., Ljung L., and Söderström T. Identification of processes in closed loop—identifiability and accuracy aspects. *Automatica*, 13(1):59–75, 1977.

[13] Remple R. K and Tischler M. B. *Aircraft and Rotorcraft System Identification: Engineering Methods With Flight-Test Examples.* American Institute of Aeronautics and Astronautics (AIAA), Reston, VA, 2006.

[14] Zhou K., Doyle J., and Glover K. *Robust and Optimal Control.* Prentice Hall, 1995.

[15] Ljung L. *System identification: Theory for the User.* Prentice Hall, Englewood Cliffs, New Jersey, 2nd edition, 2002.

[16] Deistler M. Singular ARMA systems: A structural theory. *Numerical Algebra, Control and Optimization*, 9(3):383–391, 2019.

[17] Deistler M., Anderson B.D.O., Filler A, and Chen W. Generalized linear dynamic factor models: An approach via singular autoregressions. *European Journal of Control*, 16(3):211–224, 2010.

[18] Lichota P., Szulczyk J., Tischler M. B., and Berger T. Frequency responses identification from multi-axis maneuver with simultaneous multisine inputs. *Journal of Guidance, Control and Dynamics*, 42(11):2550–2556, 2019. doi: 10.2514/1.G004346.

[19] Basu S., Li X., and Mochailidis G. Low rank and structured modeling of high-dimensional vector autoregressions. *IEEE Transactions on Singnal Processing*, 67(5):1207–1222, 2019.

[20] Bazanella A. S., Gevers M., Hendrickx J. M., and Parraga A. Identifiability of dynamical networks: which nodes need be measured? In *Proceedings of 2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 5870–5875, 2017.

[21] Georgiou T. T. and Lindquist A. Dynamic relations in sampled processes. *Control Systems Letters*, 3(1):144–149, 2019.

[22] Söderström T. and Stoica P. *System Identification.* Prentice Hall International, New York, 1989.

[23] Cao W., Lindquist A., and Picci G. Spectral rank, feedback, causality and the indirect method for CARMA identification. In *Proceedings of 59th IEEE Conference on Decision and Control (CDC)*, pages 4299–4305, Jeju, Korea (South), 2020.

[24] Cao W., Lindquist A., and Picci G. Modeling of low rank time series. *submitted, arXiv:2109.11814v3*, 2021.

[25] Yuan Y., Stan G-B., Warnick S., and Goncalves J. Robust dynamical network structure reconstruction. *Automatica*, 47(6):1230–1235, 2011. Special Issue on Systems Biology.

A Proof of the existence of models (4) with uncorrelated noises

Consider a feedback model like (5) where the input noises (r, v) may be correlated and let

$$\hat{r}(t) := \mathbb{E}[r(t) \mid v(s); s \in \mathbb{Z}]$$

be the *acausal* Winer estimate of $r(t)$ given the whole history of the process v [4, p. 105]. Since the joint spectral density is rational there is a rational transfer function say $S(z)$ by which we can represent $\hat{r}$ as $\hat{r}(t) = S(z)v(t)$ (with the usual convention on the symbols). Hence

$$r(t) = S(z)v(t) + w(t)$$

where $w(t)$ is a stationary process uncorrelated with the whole history of v. Now, after substituting into the first equation, the second equation of (4) can be written

$$y_2(t) = [H(z) + S(z)]y_1(t) - S(z)F(z)y_2(t) + w(t)$$

from which

$$\begin{aligned} y_2(t) = {}& [I + S(z)F(z)]^{-1}[H(z) + S(z)]y_1(t) \\ & + [I + S(z)F(z)]^{-1}w(t) \end{aligned} \tag{A.1}$$

which, after setting $\tilde{r}(t) := [I + S(z)F(z)]^{-1}w(t)$ may be written $y_2(t) = \tilde{H}(z)y_1(t) + \tilde{r}(t)$, of the same form of the second equation in (4) but now with v and $\tilde{r}$ completely uncorrelated. □

B On minimum phase matrix functions

Let $W(z)$ be an $(m+p) \times m$ full column rank stable matrix possibly a spectral factor of our $(m + p) \times (m + p)$ spectral density matrix $\Phi(z)$ of rank m. Minimum phase functions are called *outer* in the mathematical literature. Although our functions are rational it will be convenient to refer to the general definitions in Hardy spaces of the

literature. For these we shall use the row-vector convention of the book [4]. The following is an intuitive definition which matches that for scalar functions [4, Theorem 4.6.11, p.137].

Definition 10 *A rational matrix function $W(z)$ is minimum-phase, i.e., outer , if and only if it has all its poles in the open unit disc and all its zeros in the closed unit disc.*

One should refer to the definition of (right) zeros [4, Definition 4.6.10, p.136] for full column rank matrix functions with rows in H_m^2. For example, α is a zero of a 2×1 matrix $W = [W_1, W_2]'$, if and only if it is a common zero of both W_1 and W_2. Equivalently there is a scalar inner function $q(z)$, a Blaschke product with a zero in α, such that $W(z) = \hat{W}(z)q(z)$ with $\hat{W}(\alpha) \neq 0$. More generally, we want to consider a partition of $W(z)$

$$W(z) = \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix} \qquad (\text{B.1})$$

where $W_1(z)$, $W_2(z)$ are $m \times m$, $p \times m$ analytic matrix functions with rows in H_m^2. Next we recall the classical definition of an outer matrix function in the matrix Hardy space $H_{(p+m),m}^2$. The matrix function $W(z) \in H_{(p+m),m}^2$ is outer, if the row-span

$$\overline{\mathrm{span}} \left\{ \phi(z)W(z) \, ; \, \phi \in H_{(p+m)}^\infty \right\}$$

is the whole space H_m^2. This is equivalent to saying that in the outer-inner factorization $W(z) = \hat{W}(z)Q(z)$, the inner (matrix) function Q must be a unitary constant, which we may identify with the the identity I_m.

Consider now the outer-inner factorizations

$$W_1(z) = \hat{W}_1(z)Q_1(z), \qquad (\text{B.2a})$$
$$W_2(z) = \hat{W}_2(z)Q_2(z), \qquad (\text{B.2b})$$

where $\hat{W}_1$, $\hat{W}_2$ are the outer (minimum-phase) factors and Q_1, Q_2 are inner (in fact matrix Blaschke products). The question we want to answer is: if W is outer, does it follow that any (or both) of the two components W_1, W_2 should also be outer? We shall see that the answer is in general negative.
Let us recall the definition of *greatest common right inner divisor* of two inner functions Q_1 and Q_2, see [2, p. 188 top] denoted $Q_1 \wedge_R Q_2$. This is the inner function representative of the closed vector sum $H_m^2 Q_1 \vee H_m^2 Q_2$.

Theorem 11 *Let a full column rank matrix function $W(z) \in H_{(p+m),m}^2$ be partitioned as in (B.1). The W is outer if and only Q_1 and Q_2 are right-coprime, equivalently, the greatest common right inner divisor of Q_1 and Q_2 is the identity, i.e. $Q_1 \wedge_R Q_2 = I_m$.*

PROOF. Follows from the identity see [2, p. 188 top].

$$H_m^2 Q_1 \vee H_m^2 Q_2 = H_m^2 (Q_1 \wedge Q_2)$$

Hence $W(z) \in H_{(p+m),m}^2$ can be outer even if none of the two submatrices W_1 and W_2 is. They just need to have no (unstable) zeros in common. On the other hand, when W_1 or W_2 have no unstable zeros, they are automatically outer.

C   Details of the identification of the predictor model

Since $F_+(z)$ has at least a unit delay we have

$$F_+(z) = z^{-1}F_1(z) = A(z^{-1})^{-1}[z^{-1}B_1(z^{-1})],$$

and we may write the transfer function of the one-step predictor as

$$\hat{y}_1(t \mid t - 1) = F_1(z)y_2(t - 1). \qquad (\text{C.1})$$

Without loss of generality we can normalize $A(z^{-1})$ to be monic so that $K_+ = I$ and the predictor model can then be written in ARMAX form as

$$A(z^{-1})y_1(t) = B_1(z^{-1})y_2(t - 1) + A(z^{-1})e(t),$$

which can be used to compute the prediction error

$$\varepsilon_1(t \mid t - 1) := y_1(t) - \hat{y}_1(t \mid t - 1). \qquad (\text{C.2})$$

Given data from time $i = N - T + 1$ to N, one minimizes

$$\hat{\lambda}^2 = \frac{1}{T} \sum_{i=N-T+1}^{N} \varepsilon_1(i \mid i - 1)^2$$

with respect to the model parameters, by an iterative descent method. To guarantee viability of the algorithm at each iteration one should check stability of the estimated polynomial $A(z^{-1})$ and if needed, substitute a spectrally equivalent version to impose stability of the estimated $F(z)$. To this end one needs to readjust on line the estimated parameters of $A(z^{-1})$.
Note that this procedure does not involve the dynamics of the "input" y_2 (i.e. no need to know $H(z)$). If needed, $H(z)$ can be identified independently as seen in the previous paragraph.
Assuming *parameter identifiability* of these representations, when the true model structure with true (augmented) degrees is in the model set considered in the minimization, the parameter estimates should converge to the true parameters with probability 1 when the data size tends to infinity.
We do not consider here the difficulties connected with

parameter identifiability of these representations in the
vector case, since this is a theme which has been amply
discussed in the literature.