ERRATA: MULTIDIMENSIONAL RATIONAL COVARIANCE EXTENSION WITH APPROXIMATE COVARIANCE MATCHING

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This note is as an errata to a few unfortunate errors in [A. Ringh, J. Karlsson, and A. Lindquist, *Multidimensional Rational Covariance Extension with Approximate Covariance Matching*, SIAM J. Control Optim., 56 (2018), pp. 913–944].

Structure of the weight matrix. A weight matrix W^{-1} in the norm $\|\cdot\|_{W^{-1}}$, introduced on page 915 in the paper, was only assumed to be Hermitian positive definite. However, since the space $\mathfrak{C}$, in which the covariances live, has a certain symmetry, the matrices W^{-1} need respect this symmetry. In particular, we therefore add the assumption that any weight matrix W^{-1} maps $\mathfrak{C}$ into $\mathfrak{C}$. The latter condition corresponds to W^{-1} being Hermitian centrosymmetric with respect to the index set Λ, i.e., $[W^{-1}]_{-k,-\ell} = \overline{[W^{-1}]_{k,\ell}}$ for all $k, \ell \in \Lambda$. All the results in the paper hold for this type of weight matrices.

Correction in the proof of theorem 6.1. In the paragraph just before theorem 6.1, on page 927, it is argued that $\gamma = 0$ cannot be optimal to the dual problem. However, this argument is invalid, since it uses strong duality between the two optimization problems, which cannot be assumed to hold.[1] Instead, the paragraph should be replaced by the following.

> For $\gamma = 0$, considering (6.2) we see that the supremum defining $\varphi(q, 0)$ is ∞ if $q \neq e$. However, for $\gamma = 0$ and $q = e$, then what remains in (6.2) is only $-\mathbb{D}(Pdm, d\mu)$, and the supremum is thus 0 and attained for $d\mu = Pdm$. This shows that (6.5) is optimal to (6.4) for all $\gamma \geq 0$, and inserting (6.5) into (6.4) and removing the constant term $-c_{\mathbf{0}}$ we obtain the modified dual functional
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> $$\mathbb{J}(q) = \langle c, q\rangle - \int_{\mathbb{T}^d} P \log Q \, dm + \|q - e\|_W. \tag{6.6}$$
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> Moreover, combining (6.3) and (6.5), we obtain
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> $$\|r - c\|_{W^{-1}} = 1\,, \tag{6.7}$$
>
> which also follows from strict convexity of $\mathbb{D}(Pdm, d\mu)$ and that $p \notin \mathfrak{S}_W$.

Note that this argument does not rule out $\gamma = 0$, i.e., that the optimal solution to the dual problem is $\hat{q} = e$. Instead, this has to be done in the proof of the theorem. While this part needs to be added to the proof, the rest of the proof follows as before, except for some details in the first two paragraphs of the proof that also need to be altered. This can be done as follows: In the first paragraph

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[1] The error in the argument is that the lower bound 0 on the primal problem, obtained from the dual problem by weak duality, is not necessarily attained unless we have strong duality.

- split the functional $\mathbb{J}$ as

$$\mathbb{J}(q) = h(q) + \tilde{h}(q) - \int_{\mathbb{T}^d} P \log Q \, dm,$$

where $h(q) := \langle c, q\rangle + \|q\|_W$ and $\tilde{h}(q) := \|q - e\|_W - \|q\|_W$.
- By the reverse triangle inequality, $|\tilde{h}(q)| \leq \|q - e - q\|_W = \|e\|_W$, and thus $\tilde{h}(q)$ is bounded for all $q \in \bar{\mathfrak{P}}_+$.
- Therefore, we need to have that the new term $h(q)$ tend to $+\infty$ as $\|q\|_\infty \to \infty$.
- This holds if we can ensure that the positive term $\|q\|_W$ dominates.

In the second paragraph

- the set K should be redefined as $K := \{q \in \bar{\mathfrak{P}}_+ \mid \|q\|_\infty = 1\}$.
- Since we have redefined $h(q)$, we need to change the estimate of it to

$$h(q) = \left(\left\langle c, \frac{q}{\|q\|_\infty}\right\rangle + \left\|\frac{q}{\|q\|_\infty}\right\|_W\right)\|q\|_\infty \geq \varepsilon\|q\|_\infty \geq \frac{\varepsilon}{|\Lambda|}\|Q\|_\infty.$$

- Finally, the inequality (6.14) should then be changed to

$$\rho \geq \mathbb{J}(q) \geq \frac{\varepsilon}{|\Lambda|}\|Q\|_\infty - \int_{\mathbb{T}^d} P \log \|Q\|_\infty dm - \|e\|_W \,. \tag{6.14}$$

Now, between paragraph two and three in the proof, insert the following paragraph.

> We now want to show that $\hat{q} \neq e$ since, in view of (6.5), this also means that $\hat{\gamma} > 0$, i.e., we have strict complementarity between the Lagrangian multiplier γ and the constraint $\|r - c\|_{W^{-1}}^2 \leq 1$. To this end, we first note that by the assumption $p \notin \mathfrak{S}_W$, we have that $\|c - p\|_{W^{-1}}^2 > 1$, and thus also that
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> $$\|c - p\|_{W^{-1}}^2 > \|c - p\|_{W^{-1}}. \tag{*}$$
>
> Now, consider the point $\tilde{q} = e + \varepsilon W^{-1}(c - p)$, i.e., a small perturbation around $q = e$. For $|\varepsilon|$ small enough, we also have that $\tilde{q} \in \mathfrak{P}_+$. Moreover, in this point the unmodified dual functional $\mathbb{J}(q) - c_{\mathbf{0}}$ takes the value
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> $$\mathbb{J}(\tilde{q}) - c_{\mathbf{0}} = \varepsilon\langle c, W^{-1}(c - p)\rangle - \int_{\mathbb{T}^d} P \log \tilde{Q} dm + |\varepsilon|\|(c - p)\|_{W^{-1}}.$$
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> Analyzing the middle term further, if we let the vector of basis functions $[e^{i(\boldsymbol{k},\boldsymbol{\theta})}]_{\boldsymbol{k}\in\Lambda}$ be ordered in the same way as elements in $\mathfrak{C}$, then any trigonometric polynomial Q can be written as $Q(e^{i\boldsymbol{\theta}}) = \langle [e^{i(\boldsymbol{k},\boldsymbol{\theta})}]_{\boldsymbol{k}\in\Lambda}, q\rangle$. Now, by a series expansion of the logarithm we get that
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> $$\begin{aligned}\int_{\mathbb{T}^d} P \log \tilde{Q} dm &= \int_{\mathbb{T}^d} P\left(\varepsilon\left\langle [e^{i(\boldsymbol{k},\boldsymbol{\theta})}]_{\boldsymbol{k}\in\Lambda}, W^{-1}(c - p)\right\rangle - \mathcal{O}(|\varepsilon|^2)\right) dm \\ &= \varepsilon\langle p, W^{-1}(c - p)\rangle - \mathcal{O}(|\varepsilon|^2),\end{aligned}$$
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> since P has finite total mass, and by moving the integration into each component of $[e^{i(\boldsymbol{k},\boldsymbol{\theta})}]_{\boldsymbol{k}\in\Lambda}$ and using that the complex exponentials

are orthogonal. This gives

$$\begin{aligned}\mathbb{J}(\tilde{q}) - c_{\mathbf{0}} &= \varepsilon\langle c, W^{-1}(c-p)\rangle - \varepsilon\langle p, W^{-1}(c-p)\rangle + \mathcal{O}(|\varepsilon|^2) + \|c-p\|_{W^{-1}} \\ &= \varepsilon\|c-p\|_{W^{-1}}^2 + |\varepsilon|\|c-p\|_{W^{-1}} + \mathcal{O}(|\varepsilon|^2) \\ &= \varepsilon\left(\|c-p\|_{W^{-1}}^2 + \operatorname{sign}(\varepsilon)\|c-p\|_{W^{-1}}\right) + \mathcal{O}(|\varepsilon|^2) \\ &\leq \delta\varepsilon + \mathcal{O}(|\varepsilon|^2)\end{aligned}$$

for some $\delta > 0$, where the last inequality follows from (*). Thus, for $\varepsilon < 0$ with $|\varepsilon|$ sufficiently small, we have that $\mathbb{J}(\tilde{q}) - c_{\mathbf{0}} < 0$. Since $\mathbb{J}(e) - c_{\mathbf{0}} = 0$, this shows that $q = e$ is not optimal to (6.8).

The rest of the proof now follows verbatim.