# On the Partial Realization Problem

William B. Gragg* and Anders Lindquist†
*Department of Mathematics*
*University of Kentucky*
*Lexington, Kentucky 40506*

Submitted by P. A. Fuhrmann

ABSTRACT

In this paper we take a unified approach to the partial realization problem in which we seek to incorporate ideas from numerical linear algebra, most of which were originally developed in other contexts. We approach the partial realization problem from several different angles and explore the connections to such topics as factorization of Hankel matrices, block tridiagonalization, generalizations of the Lanczos process for biorthogonalization, the Euclidean algorithm and the principal-part continued fractions of Arne Magnus, the Padé table, and the Berlekamp-Massey algorithm. In this way we are able to clarify some previous results by Rissanen, Kalman, and others and place them in a broader context. This leads to several results and concepts which we think are new. Our analysis is restricted to the scalar case, but some definitions and formulations have been rigged to facilitate an extension to the matrix case.

## 1. INTRODUCTION

The following problem is of central importance in systems theory. Given a finite sequence $\gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ of real numbers, find a triplet $\Sigma := (A, B, C)$ of matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times 1}$, and $\mathbb{R}^{1\times n}$ respectively such that

$$CA^{i-1}B = \gamma_i \qquad \text{for} \quad i = 1,2,\ldots,N \tag{1.1}$$

and such that $n$ is as small as possible. This is the (scalar) *partial realization problem* [19, 21]. In a more general formulation of this problem, the elements of $\gamma$ are $m \times p$ matrices and $B$ and $C$ are $n \times p$ and $m \times n$ respectively.

*This work was supported by the National Science Foundation under grant MCS-81-02344.
†This work was supported in part by the Air Force Office of Scientific Research under grant AFOSR-78-3519 and in part by the National Science Foundation under grant ECS-7903731.




52 Vanderbilt Ave., New York, NY 10017 0024-3795/83/020277-43$3.00

A triplet $\Sigma$ satisfying (1.1) will be called a *realization* of $\gamma$, and the number $n$ is its *dimension*, denoted $\dim \Sigma$. Clearly there always exist such realizations. For example, consider the following *shift realization*. Let $A$ be the $N \times N$ downshift matrix $J$ with ones on the subdiagonal and zeros elsewhere, let $B$ be the axis unit vector in $\mathbb{R}^N$ with the one in the first position, and set $C := (\gamma_1, \gamma_2, \ldots, \gamma_N)$. In general, however this realization will not have the required minimal dimension. Let $\delta(\gamma)$ be the unique natural number with the property that there exists a realization of $\gamma$ with dimension $\delta(\gamma)$, but none with a smaller dimension. The number $\delta(\gamma)$ will be called the *McMillan degree* of $\gamma$ for reasons to be explained below. A realization of dimension $\delta(\gamma)$ is called *minimal*. The problem under consideration is then to find a minimal realization of $\gamma$. As we shall see below, this problem has more than one solution, and this nonuniqueness will be exploited to obtain certain additional properties.

All concepts defined so far make sense also if $\gamma$ is an infinite sequence $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ and we require that (1.1) hold for all natural numbers, except that there may exist no realization $\Sigma$ of finite dimension. If so, set $\delta(\gamma) = \infty$. This is perfectly consistent with the finite-sequence case, since there are always infinite-dimensional realizations. For example, the shift realization described above is well-defined also when $N = \infty$. A subsequence $\hat{\gamma} := \{\gamma_1, \gamma_2, \ldots, \gamma_M\}$ with $M < N$ will be called a *partial sequence* of $\gamma$, regardless of whether $N$ is finite or infinite.

Equivalently we may describe the partial realization problem in the following way, which better motivates its name. Let

$$\Gamma(z) = \sum_{i=1}^{\infty} \gamma_i z^{-i} \tag{1.2}$$

be a formal power series. Then, if (1.1) holds, we call $\Sigma := (A, B, C)$ a *partial realization of* $\Gamma$ *of order* $N$. To each such partial realization we associate the power series

$$W(z) = \sum_{i=1}^{\infty} CA^{i-1}Bz^{-i}. \tag{1.3}$$

If $\dim \Sigma < \infty$, this series converges in a neighborhood of infinity and

$$W(z) = C(zI - A)^{-1}B. \tag{1.4}$$

This is a rational function; let us write it $W = P/Q$, where $P$ and $Q$ are

relatively prime polynomials with $Q$ monic. Moreover $W$ is *strictly proper*, i.e. $W(\infty) = 0$. Hence the *degree* of $W$ is the same as the degree of $Q$. Now, the power series of the rational function $W$ matches that of $\Gamma$ at least up to powers of order $N$ in $z^{-1}$, and therefore $W$ is a *Padé fraction* for $\Gamma$ of at least order $N$ [12].

Next consider the degrees of all $W$ corresponding to partial realizations of $\Gamma(z)$ of a fixed order $N < \infty$. The smallest such degree is the McMillan degree $\delta(\gamma)$ of the finite sequence $\gamma := \{\gamma_1, \gamma_2, \dots, \gamma_N\}$. To see this, note that $\deg W \leqslant \dim \Sigma$ and that equality can be obtained by choosing $A$ to be a companion matrix of $Q$, $B$ as the first axis vector, and $C := (p_0, p_1, \dots, p_{n-1})$, where $P(z) = \sum p_i z^i$ [4, 21]. Consequently our definition of McMillan degree, which is standard in systems theory, is consistent with the use of the term "degree" by McMillan [27] in reference to rational (matrix) functions (although one would have to go to the matrix case to make this observation nontrivial). A partial realization with dimension $\delta(\gamma)$ is called *minimal*. A triplet $\Sigma$ such that (1.1) holds for all natural numbers is called a *complete realization* of $\Gamma$. Clearly $\Gamma$ has a finite-dimensional complete realization if and only if it converges to a rational function about $z = \infty$, and this happens precisely when the McMillan degree of the infinite sequence $\{\gamma_1, \gamma_2, \gamma_3, \dots\}$ is finite. Then $W = \Gamma$. We shall make no distinction between the sequence and power-series formulation, using them interchangably as best fits the situation at hand, and we shall analogously refer to a realization of a partial sequence of a finite or infinite sequence $\gamma$ as a partial realization of $\gamma$.

The importance of the partial realization problem in systems theory emanates from the fact that $\Sigma$ corresponds to a *linear system*

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{1.5}$$

$(t = 0, 1, 2, \dots)$, where $x(t) \in \mathbb{R}^n$ is called the *state*, $y(t) \in \mathbb{R}$ the *output*, and $u(t) \in \mathbb{R}$ the *input* at time $t$. Then the sequence $\gamma$ is a section of the *impulse response* of the system (1.5) in the sense that, if we choose $u(0) = 1$, $u(1) = u(2) = \cdots = u(N) = 0$, and $x(0) = 0$, we obtain the output $y(t) = \gamma_t$ for $t = 1, 2, \dots, N$. In other words, the partial realization problem is the *inverse problem* of determining a system (1.5) from its partial impulse response $\gamma$. A complete impulse response $\{\gamma_1, \gamma_2, \gamma_3, \dots\}$ is usually not available from data, and therefore we must content ourselves with a partial realization. Moreover, it is easy to see that, if $x(0) = 0$,

$$\tilde{y}(z) = W(z)\tilde{u}(z), \tag{1.6}$$

where $\tilde{y}(z) := \sum_{t=0}^{\infty} y_t z^{-t}$ and $\tilde{u}(z) = \sum_{t=0}^{\infty} u_t z^{-t}$ are the discrete Laplace transforms ("$z$-transforms") of $y$ and $z$ respectively. Hence $W$ is the *transfer function* of the system (1.5). The relation (1.6) is still true if we replace (1.5) by the continuous-time linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \tag{1.7}$$

($t \geqslant 0$), where the dot denotes differentiation, if now $\tilde{y}$ and $\tilde{u}$ are the regular (continuous) Laplace transforms of $y$ and $u$. Consequently, discrete-time and continuous-time linear systems lead to the same partial realization problem, and (1.5) and (1.7) will play no role in what follows.

The (complete) realization problem can be regarded as the fundamental inverse problem in systems theory. There are many algorithms for its solution in the literature, the most interesting of which are based on the Hankel matrix of $\gamma$ [14, 31, 37, 38]. These algorithms can also be applied to finite sequences $\gamma$, and the particular problems connected with this are discussed in [19, 21]. However, in [29] Rissanen adds an important new aspect to the partial realization problem, the *nesting property*. As the number of elements in the partial sequence $\gamma$ increases through the natural numbers, a (nonunique) sequence of minimal partial realizations $\{\Sigma_0, \Sigma_1, \Sigma_2, \ldots\}$ of increasing dimensions is obtained. Rissanen requires that these realizations be determined so that they are nested in the sense that the matrices of $\Sigma_m$ are submatrices of the corresponding matrices of $\Sigma_n$ whenever $m < n$, and he presents an algorithm that achieves this. (Actually Rissanen's algorithm may skip certain realizations in the sequence $\{\Sigma_0, \Sigma_1, \Sigma_2, \ldots\}$, as we shall see in Section 3.) This is an important property, since the addition of new numbers in the sequence $\gamma$ will not require complete recalculation of the realization but only that it be appropriately augmented. Hence none of the old calculations are wasted.

Our interest in the partial realization problem was triggered by a recent paper by Kalman [20], one of the basic ideas of which is to represent $W$ by a certain continued fraction. However, this is actually a special case of the *principal-part continued fraction* (*P*-fraction) of Arne Magnus [24, 25]. Thus the paper [20] led us to realize that certain problems, ostensibly unrelated to systems theory, with which the first author has been involved, are deeply connected to the more interesting aspects of the partial-realization problem. These include the block structure of the Padé table [12], block tridiagonalization of matrices, and generalizations of the Lanczos process of biorthogonalizations in the case where there are breakdowns [13, 24, 25, 28]. Some of these connections, more precisely the relations to the generalized Lanczos poly-

nomials [24, 25] and their connection to the Berlekamp-Massey algorithm [1, 26], had already been pointed out and elaborated upon in a thesis by Kung [22].

Our original objective was to consider the partial realization problem in the full generality of matrix sequences, but we encountered so many different aspects of the scalar problem that we shall be unable to discuss even all of them in the limited format of this article. Moreover, we do not have complete results on the vector case at this time. Therefore we shall only consider the scalar case. This strategy also has the advantage of not excessively burdening our presentation with detail: one needs to understand the scalar case completely before going to the matrix case, a step which is decidedly nontrivial. Various other aspects of the matrix case have been studied in [7, 22, 30].

The outline of the paper is as follows. In Section 2 we discuss factorization of Hankel matrices and block tridiagonalization, and in Section 3 we tie this up with the partial-realization problem, which is discussed in detail. The connections to the Euclidean algorithm and *P*-fractions are discussed in Section 4, and in Section 5 we present an algorithm of the Berlekamp-Massey type and consider some numerical questions.

In this paper we do not consider questions of numerical stability. This has been done, to some extent, by de Jong [16]. The problem of rank determination is inherently ill posed in the presence of noise, but can be modified to be well posed. Stable methods of linear algebra can be applied to the modified problem. There appear to be severe tradeoffs between efficiency and stability. These questions merit further study.

## 2. MATRIX FACTORIZATION AND BLOCK TRIDIAGONALIZATION

Let $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \dots\}$ be an infinite sequence of real numbers, and let $\{H_{ij};\ i, j = 1,2,3,\dots\}$ be the family of rectangular Hankel matrices

$$H_{ij} = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_j \\ \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{j+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_i & \gamma_{i+1} & \gamma_{i-2} & \cdots & \gamma_{i+j-1} \end{bmatrix}. \tag{2.1}$$

Define the *degree indices* $\{\nu(0), \nu(1), \nu(2), \dots\}$ of $\gamma$ in the following way. Set $\nu(0) := 0$, and, for $n = 0,1,2,\dots$, let $\nu(n+1)$ be the smallest integer greater than $\nu(n)$ for which

$$\hat{H}_{n+1} := H_{\nu(n)+1, \nu(n+1)} \tag{2.2}$$

has full rank. Set $d(n+1) := \nu(n+1) - \nu(n)$. At this point we disregard the fact that this procedure may not be numerically meaningful since it involves rank determination. We shall address this question in Section 5. In Section 3 we shall interpret the degree indices of $\gamma$ as the McMillan degrees of the partial sequences of $\gamma$; see this section for an example.

Let $H$ be the infinite Hankel matrix

$$H := \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \gamma_4 & \cdots \\ \gamma_3 & \gamma_4 & \gamma_5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.3}$$

In the sequel we shall refer to $H$ as the *Hankel matrix of the sequence* $\gamma$. If $H$ has finite rank, the process defining the degree indices stops after finitely many steps, at $n = \hat{n}$. Therefore we shall refer to the sequence of degree indices as $\{\nu(n);\ n \in \mathfrak{N}\}$, where $\mathfrak{N}$ is either $\{0,1,2,\ldots\}$ or $\{0,1,2,\ldots,\hat{n}\}$, whichever case applies. In the latter case, for notational convenience, we set $\overline{\mathfrak{N}} := \{0,1,2,\ldots,\hat{n}+1\}$ and $\nu(\hat{n}+1) := \infty$; then $d(\hat{n}+1) = \infty$. If $\mathfrak{N}$ is infinite, $\overline{\mathfrak{N}} := \mathfrak{N}$. Moreover let $\mathfrak{N}^+$ ($\overline{\mathfrak{N}}^+$) denote $\mathfrak{N}$ ($\overline{\mathfrak{N}}$) with the zero removed. Define $J$ to be the infinite downshift matrix with ones in positions $(k+1, k)$, $k = 1,2,3,\ldots$, and zeros elsewhere. Then it is easy to see that, for $i = 0,1,2,\ldots$,

$$HJ^i = (J')^i H = \sigma^i(H), \tag{2.4}$$

where the prime denotes transpose and $\sigma^i(H)$ is the Hankel matrix of the shifted sequence $\{\gamma_{i+1}, \gamma_{i+2}, \gamma_{i+3}, \ldots\}$. For each $n \in \mathfrak{N}^+$, let $H_n$ be the $\nu(n) \times \nu(n)$ leading principal submatrix of $H$, i.e.

$$H_n := H_{\nu(n),\nu(n)}. \tag{2.5}$$

Let us agree to call a square matrix *right* (*left*) *triangular* if all elements below (above) the diagonal are zero, and *upper* (*lower*) *triangular* if all elements below (above) the antidiagonal are zero. If in addition all elements on the diagonal (antidiagonal) are ones, we add the attribute *unit*.

THEOREM 1. *The matrices* $\{H_n;\ n \in \mathfrak{N}^+\}$ *are nonsingular. All other leading principal submatrices of* $H$ *are singular. For* $n \in \mathfrak{N}^+$, *let* $r_n := -H_n^{-1}h_n$, *where* $h_n := (\gamma_{\nu(n)+1}, \ldots, \gamma_{2\nu(n)})'$, *and define* $q_n := (r_n', 1, 0, 0, \ldots)' \in \mathbb{R}^\infty$; *set* $q_0 := (1,0,0,\ldots)'$. *Let* $R \in \mathbb{R}^{\infty\times\infty}$ *be the unit right triangular*

*matrix*

$$R := \left(q_0, Jq_0, \ldots, J^{d(1)-1}q_0, q_1, Jq_1, \ldots, J^{d(2)-1}q_1, q_2, Jq_2, \ldots\right). \quad (2.6)$$

*Moreover, let* $\{\pi_1, \pi_2, \pi_3, \ldots\}$ *be the sequence defined, for each* $n \in \mathfrak{N}$, *by*

$$\pi_{2\nu(n)+i} = q_n' H J^{i-1} q_n, \qquad i = 1, 2, \ldots, 2d(n+1), \quad (2.7)$$

*set* $\lambda_n := \pi_{2\nu(n)+d(n+1)}$, *and, for* $n \in \mathfrak{N}^+$, *define* $\Pi_n$ *to be the* $d(n) \times d(n)$ *Hankel matrix of the subsequence* $\{\pi_{2\nu(n-1)+1}, \pi_{2\nu(n-1)+2}, \ldots, \pi_{2\nu(n)-1}\}$. *Then, for* $n \in \mathfrak{N}^+$,

$$R_n' H_n R_n = D_n, \quad (2.8)$$

*where* $R_n$ *is the* $\nu(n) \times \nu(n)$ *leading principal submatrix of* $R$, *and* $D_n$ *is the block diagonal matrix*

$$D_n = \operatorname{diag}(\Pi_1, \Pi_2, \ldots, \Pi_n). \quad (2.9)$$

*Each* $\Pi_k$ *is nonsingular and lower triangular,* $\lambda_{k-1} \neq 0$ *being the element on the antidiagonal of* $\Pi_k$. *If* $\mathfrak{N}$ *is finite, all elements of* $D := R'HR$ *outside* $D_{\hat{n}}$ *are zero.*

The results concerning the factorization (2.8) were announced in Gragg [13], which paper constituted an impetus for our study. The fact that the sizes of the nonsingular matrices $\{H_n;\ n \in \mathfrak{N}\}$ are precisely the degree indices (as defined above) is suggested by systems theory; see in particular Theorem 11 in Silverman [32]. The proof presented here is constructed so as to emphasize this connection to realization theory.

*Proof.* The proof is by induction. First note that $\hat{H}_1 = (0, \ldots, 0, \lambda_0)$, where $\lambda_0 := \gamma_{d(1)}$ is the first nonzero element in the sequence $\gamma$. Hence $H_1$ is lower triangular and Hankel with nonzero elements on the antidiagonal. Therefore $H_1$ is nonsingular, and all leading principal submatrices of $H_1$ are singular. Moreover $D_1 = H_1$, which establishes the factorization (2.8) for $n = 1$, since $R_1 = I$. Now assume that $H_n$ is nonsingular and that $R_n' H_n R_n = D_n$. We want to show that $H_{n+1}$ is nonsingular, that all leading submatrices of $H_{n+1}$ of orders $\nu(n)+1, \nu(n)+2, \ldots, \nu(n+1)-1$ are singular, and that $R_{n+1}' H_{n+1} R_{n+1} = D_{n+1}$. To this end, consider the family of unit right trian-

gular matrices

$$R_n(k) := \begin{bmatrix} R_n & r_n(1) & r_n(2) & \cdots & r_n(k) \\ & 1 & \rho_{21} & \cdots & \rho_{k1} \\ & & 1 & \cdots & \rho_{k2} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \tag{2.10}$$

Set $d := d(n+1)$. We want to determine $r_n(1), r_n(2), \ldots, r_n(d) \in \mathbb{R}^{\nu(n)}$ and $\rho_{21}, \rho_{31}, \cdots, \rho_{d,d-1} \in \mathbb{R}$ so that

$$R_n(1)'\hat{H}_{n+1}R_n(d) = \begin{bmatrix} D_n & & & & \\ & * & * & \cdots & * \end{bmatrix}, \tag{2.11}$$

where numbers which are not affected by the present analysis are denoted by an asterisk. (Blank spaces are zero.) Setting

$$\hat{H}_{n+1} = \begin{bmatrix} H_n & h_n(1) & h_n(2) & \cdots & h_n(d) \\ h_n(1)' & \eta_1 & \eta_2 & \cdots & \eta_d \end{bmatrix}, \tag{2.12}$$

it is seen that (2.11) holds if and only if

$$H_n r_n(i) + \rho_{i1}h_n(1) + \cdots + \rho_{i,i-1}h_n(i-1) + h_n(i) = 0 \tag{2.13}$$

for $i = 1,2,\ldots,d$. Hence we must have $r_n(1) = -H_n^{-1}h_n(1)$, which is precisely the relation $r_n = -H_n^{-1}h_n$ mentioned in the theorem. However, there is a certain arbitrariness in the choice of the other parameters. The dimension of the solution space of the system (2.13) equals the number of $\rho$-parameters; the vectors $r_n(2), \ldots, r_n(d)$ are uniquely determined in terms of these. (Of course, this whole situation happens only if $d > 1$.) Although next we shall choose a particular solution which exploits the Hankel structure, we shall have reason to refer to this lack of uniqueness in Section 3. Now let $q_n(i) \in \mathbb{R}^\infty$, $i = 1,2,\ldots,d$, be the last $d$ columns of $R_n(d)$, each augmented with infinitely many zeros, i.e.

$$q_n(i) := [r_n(i)', \rho_{i1}, \rho_{i2}, \ldots, \rho_{i,i-1}, 1, 0, 0, \ldots]'; \tag{2.14}$$

hence, in particular, $q_n(1) = q_n$ as defined in the theorem. Then (2.13) can be

written

$$e_k' H q_n(i) = 0 \qquad \text{for} \quad k = 1,2,\ldots,\nu(n) \tag{2.15}$$

where $e_k \in \mathbb{R}^\infty$ is the axis unit vector with a one in position $k$ and zeros elsewhere. However, due to the rank condition defining $\hat{H}_{n+1}$, we actually have

$$R_n(1)'\hat{H}_{n+1}R_n(d) = \begin{bmatrix} D_n & & & & \\ & 0 & \cdots & 0 & \lambda_n \end{bmatrix}, \tag{2.16}$$

where $\lambda_n \neq 0$. Hence, since $R_n(d)^{-1}$ is unit right triangular,

$$(r_n', 1)\hat{H}_{n+1} = (0,\ldots,0,\lambda_n)$$

(noting that $r_n(1) = r_n$), or equivalently

$$e_k' H q_n = \begin{cases} 0 & \text{for} \quad k = 1,2,\ldots,\nu(n+1)-1, \\ \lambda_n & \text{for} \quad k = \nu(n+1), \end{cases} \tag{2.17}$$

which is stronger than the corresponding equation ($i = 1$) in (2.15). So far we have not used the fact that $H$ is a Hankel matrix. In view of (2.4) and the fact that $e_{k+i} = J^i e_k$, (2.17) yields

$$e_k' H J^{i-1} q_n = 0 \qquad \text{for} \quad k = 1,2,\ldots,\nu(n+1) - i.$$

Comparing this with (2.15), we see that we may choose

$$q_n(i) = J^{i-1} q_n, \qquad i = 1,2,\ldots,d(n+1). \tag{2.18}$$

This choice is easy to implement since, once $r_n$ has been uniquely determined, the last $d-1$ columns of $R_n(d)$ are obtained by merely shifting the previous column down one step. Then $R_n(d) = R_{n+1}$ as defined in the theorem. Next we extend $\hat{H}_{n+1}$ to the square matrix $H_{n+1}$. The relation (2.16) can be written

$$[R_n(1)', 0] H_{n+1} R_{n+1} = \begin{bmatrix} D_n & & & & \\ & 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

and therefore, by symmetry,

$$R'_{n+1}H_{n+1}R_{n+1} = \begin{bmatrix} D_n & \\ & \Pi \end{bmatrix}, \tag{2.19}$$

where the components of the $d(n+1)\times d(n+1)$ matrix $\Pi$ are

$$\begin{aligned} \Pi_{ij} &= q_n'(J')^{i-1}HJ^{j-1}q_n \\ &= q_n'HJ^{i+j-2}q_n, \end{aligned} \tag{2.20}$$

in which (2.4) has been used. Consequently $\Pi$ is the Hankel matrix $\Pi_{n+1}$. It is lower triangular with $\lambda_n \neq 0$ on the antidiagonal. Hence $H_{n+1}$ is nonsingular and all leading principal submatrices of orders $\nu(n)+1,\dots,\nu(n+1)-1$ are singular. If $\nu(n)$ is the largest degree index, there is no $\lambda_n \neq 0$, so the above procedure yields instead an infinite $\Pi = 0$. Therefore the singularity of the leading principal submatrices of orders $\nu(n)+1, \nu(n)+2,\dots$ is established. ∎

REMARK 1. One of the motivations for the generalizations proposed in [13; p. 222] was the study of signatures of Hankel matrices; see [15] for many connections. In view of (2.8), $H_n$ and $D_n$ are congruent. Hence they have the same signature (Sylvester's law of inertia). But $D_n$ is the direct sum of $\Pi_1, \Pi_2,\dots,\Pi_n$ and consequently the signature of $D_n$ is the sum of the signatures of $\Pi_1, \Pi_2,\dots,\Pi_n$. Therefore, since the signature of a nonsingular lower triangular matrix is the sign of its middle element if the latter exists and zero otherwise [8], the signature of $H_n$ is $\Sigma_i \operatorname{sign} \lambda_{i-1}$, where the sum is taken over those $i = 1,2,\dots,n$ for which $d(i)$ is odd. These observations are equivalent to those made in [20, p. 21], derived via the Euclidean algorithm. To see this we need to establish certain equivalences between our matrix factorization and certain continued fractions. This will be done in Section 4. We shall discuss several procedures for obtaining the factorization (2.8) below. Any of these algorithms can be used for the signature problem as well. Since the problem of determining the stability of a matrix can be formulated in similar terms [9], the same comment goes for this problem. However, a complete treatment of the "polynomial inertia problem" which is based only on Euclid's algorithm and the argument principle, totally avoiding Cauchy indices and Sturm sequences, can be found in Talbot [35].

We associate with the sequence $\gamma$ a linear functional on the vector space of real polynomials defined by

$$\gamma^*(z^{i-1}) = \gamma_i, \qquad i = 1,2,3,\dots. \tag{2.21}$$

Then, if $F(z) := \Sigma f_i z^i$ and $G(z) := \Sigma g_i z^i$ are any two polynomials,

$$\gamma^*(FG) = f'Hg, \tag{2.22}$$

where $f, g \in \mathbb{R}^\infty$ are defined as $f := (f_0, f_1, f_2, \dots)'$ and $g := (g_0, g_1, g_2, \dots)'$. To be able to make the transformation between vectors in $\mathbb{R}^\infty$ and polynomials we introduce the infinite vector

$$v(z) := (1, z, z^2, z^3, \dots)' \tag{2.23}$$

of $z$-powers. Then $F(z) = v(z)'f$ and $G(z) = v(z)'g$.

THEOREM 2. *Let $\gamma$ be an infinite sequence (of real numbers) with degree indices $\{\nu(n);\ n \in \mathfrak{N}\}$. For each $n \in \mathfrak{N}$, define the monic polynomial $Q_n(z) := v(z)'q_n$ of degree $\nu(n)$, where $q_n$ is defined as in Theorem 1. Then*

$$\gamma^*(z^{i-1}Q_n) = \begin{cases} 0 & \text{for } i = 1,2,\dots,\nu(n+1)-1, \\ \lambda_n \neq 0 & \text{for } i = \nu(n+1), \end{cases} \tag{2.24}$$

*where $\lambda_n$ is given in Theorem 1, or equivalently,*

$$\gamma^*(z^{i-1}Q_mQ_n) = \begin{cases} 0 & \text{for } m \neq n \text{ and } i = 1,2,\dots,d(m+1) \\ & \quad + d(n+1) - 1, \qquad (2.25a) \\ 0 & \text{for } m = n \text{ and } i = 1,2,\dots,d(n+1)-1, \qquad (2.25b) \\ \lambda_n & \text{for } m = n \text{ and } i = d(n+1). \qquad (2.25c) \end{cases}$$

*Moreover, $\{Q_n;\ n \in \mathfrak{N}\}$ satisfy the three-term recursion*

$$Q_{n+1}(z) = \alpha_{n+1}(z)Q_n(z) - \beta_nQ_{n-1}(z) \tag{2.26}$$

*with initial conditions $Q_0(z) = 1$ and $Q_{-1}(z) = 0$, where, for each $n \in \mathfrak{N}^+$,*

$$\alpha_n(z) = z^{d(n)} - \alpha_{n1}z^{d(n)-1} - \cdots - \alpha_{n,d(n)} \tag{2.27}$$

*is a monic polynomial of degree $d(n) := \nu(n) - \nu(n-1)$ such that $a_n := (\alpha_{n,d(n)}, \dots, \alpha_{n1})' \in \mathbb{R}^{d(n)}$ is the unique solution of the system of linear equations*

$$\Pi_n a_n = p_n \tag{2.28}$$

*with* $p_n := (\pi_{2\nu(n-1)+d(n)+1}, \dots, \pi_{2\nu(n)})' \in \mathbb{R}^{d(n)}$, *and*

$$\beta_{n-1} = \lambda_{n-1}/\lambda_{n-2} \tag{2.29}$$

*are nonzero real numbers. Here* $\lambda_{-1} := 1$, *and the matrices* $\{\Pi_n;\ n \in \mathfrak{N}^+\}$ *and the sequence* $\{\pi_1, \pi_2, \pi_3, \dots\}$ *are defined as in Theorem* 1 *or, equivalently, via*

$$\pi_{2\nu(n)+i} = \gamma^*\big(z^{i-1}Q_n^2\big), \qquad i = 1,2,\dots,2d(n+1), \tag{2.30}$$

*for each* $n \in \mathfrak{N}$.

The conditions (2.24) and (2.25) are actually corollaries of Theorem 1, reported here for convenience. In fact, (2.24) is the same as (2.17), and (2.25) is equivalent to the factorization (2.8). From the linearity of the functional $\gamma^*$, it is easy to see that (2.24) and (2.25) are equivalent. If the sequence $\gamma$ has degree indices $\{0,1,2,3,\dots\}$, i.e. $d(n) = 1$ for all $n \geqslant 0$, then $\{Q_n;\ n = 0,1,2,\dots\}$ are the *Lanczos polynomials* and the corresponding three-term recursion provides a well-known technique for tridiagonalization of matrices and computation of eigenvalues [23]. The generalized three-term recursion (2.26) is due to Magnus [24, 25]. A derivation via a matrix factorization can be found in Kung [22].

*Proof.* It remains to show that $\{Q_n;\ n \in \mathfrak{N}\}$ satisfy (2.26) with $\{\alpha_n, \beta_{n-1};\ n \in \mathfrak{N}^+\}$ defined by (2.27)–(2.29). To this end, first note that, since

$$\{Q_0, zQ_0, \dots, z^{d(1)-1}Q_0, Q_1, zQ_1, \dots, z^{d(2)-1}Q_1, Q_2, zQ_2, \dots,$$
$$z^{d(n-1)-1}Q_{n-2}, Q_{n-1}, zQ_{n-1}, \dots, z^{d(n)}Q_{n-1}\}$$

is a sequence of monic polynomials of degrees $0,1,2,\dots,\nu(n)$, there are polynomials $\{\phi_i;\ i = 1,2,\dots,n-1\}$ such that $\deg \phi_i < d(i)$ and a monic polynomial $\alpha_n$ of degree $d(n)$ such that

$$Q_n(z) = \alpha_n(z)Q_{n-1}(z) + \phi_{n-1}(z)Q_{n-2}(z) + \cdots + \phi_1(z)Q_0(z). \tag{2.31}$$

Then it follows from (2.25) that

$$\gamma^*\big(z^{i-1}\phi_{k+1}Q_kQ_m\big) = 0 \qquad \text{for} \quad i = 1,2,\dots,d(m+1) \tag{2.32}$$

except when $k = m = n - 2$. To see this, first note that (2.25a) implies that (2.32) holds for $k \neq m$, because $i + \deg \phi_{k+1} \leqslant i + d(k+1) - 1 \leqslant d(m+1) + d(k+1) - 1$. Next multiply (2.31) by $z^{i-1}Q_k$ and apply $\gamma^*$ to obtain

$$\gamma^*\left(z^{i-1}Q_nQ_k\right) = \gamma^*\left(z^{i-1}\alpha_nQ_{n-1}Q_k\right) + \gamma^*\left(z^{i-1}\phi_{k+1}Q_k^2\right) \tag{2.33}$$

for $k = 0,1,2,\ldots,n-2$ and $i = 1,2,\ldots,d(k+1)$. In view of (2.25a), the left member of (2.33) is zero. Moreover, $i + \deg(\alpha_nQ_k) = i + d(n) + \nu(k) \leqslant \nu(n)$, with equality when $k = n-2$ and $i = d(n-1)$. Hence, by (2.24), the first term in the right member of (2.33) is zero except in the case that $k = n-2$ and $i = d(n-1)$, in which case it equals $\lambda_{n-1}$. This establishes (2.32) for $k = m \neq n-2$ and

$$\gamma^*\left(z^{i-1}\phi_{n-1}Q_{n-2}^2\right) = \begin{cases} 0 & \text{for } i = 1,2,\ldots,d(n-1)-1, \\ \lambda_n & \text{for } i = d(n-1). \end{cases} \tag{2.34}$$

Now let $f_i \in \mathbb{R}^{d(i)}$ be the vector of coefficients of $\phi_i$. Then, (2.32) with $k = m = j-1$ is equivalent to the system of linear equations $\Pi_j f_j = 0$. But $\Pi_j$ is nonsingular (Theorem 1), and therefore $f_j = 0$, i.e. $\phi_j = 0$, for $j = 1,2,\ldots,n-2$. Likewise (2.34) is equivalent to $\Pi_{n-1}f_{n-1} = (0,0,\ldots,\lambda_{n-1})$, which yields $\phi_{n-1} = \lambda_{n-1}/\lambda_{n-2}$. Hence $Q_n = \alpha_nQ_{n-1} + \beta_{n-1}Q_{n-2}$, where $\beta_{n-1}$ is given by (2.29). Multiply this by $z^{i-1}Q_{n-1}$ and proceed as above to obtain

$$\gamma^*\left(z^{i-1}\alpha_nQ_{n-1}^2\right) = 0, \qquad i = 1,2,\ldots,d(n). \tag{2.35}$$

But this is precisely (2.28). ■

We shall need a matrix interpretation of the three-term recursion (2.26). To each monic polynomial $P(z) = z^m + p_1z^{m-1} + \cdots + p_m$ of degree $m$ we associate the $m \times m$ matrix

$$F(P) := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -p_m \\ 1 & 0 & 0 & \cdots & 0 & -p_{m-1} \\ 0 & 1 & 0 & \cdots & 0 & -p_{m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -p_1 \end{bmatrix}, \tag{2.36}$$

which we call the *companion matrix* of $P$. [This name is often used for various other forms also, but from now on we shall reserve it for (2.36).]

THEOREM 3. *Given a sequence* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *of monic polynomials and a sequence* $\{\beta_1, \beta_2, \ldots, \beta_{n-1}\}$ *of real numbers, let* $\{Q_0, Q_1, \ldots, Q_n\}$ *be the monic polynomials defined by the three-term recursion* (2.26). *For* $k = 1, 2, \ldots, n$ *set* $d(k) := \deg \alpha_k$ *and* $\nu(k) := \deg Q_k$. *Let* $R_n$ *be the* $\nu(n) \times \nu(n)$ *unit right triangular matrix defined by*

$$\begin{aligned}&\left[1, z, z^2, \ldots, z^{\nu(n)-1}\right] R_n \\ &= \left(Q_0, zQ_0, \ldots, z^{d(1)-1}Q_0, Q_1, zQ_1, \ldots, z^{d(2)-1}Q_1, Q_2, \ldots, z^{d(n)-1}Q_{n-1}\right).\end{aligned} \tag{2.37}$$

*Then*

$$F_n R_n = R_n A_n, \tag{2.38}$$

*where* $F_n$ *is the companion matrix* $F(Q_n)$ *and* $A_n$ *is the block tridiagonal* $\nu(n) \times \nu(n)$ *matrix*

$$A_n = \begin{bmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & A_{23} & & \\ & A_{32} & A_{33} & & \\ & & & & A_{n-1,n} \\ & & & A_{n,n-1} & A_{nn} \end{bmatrix}. \tag{2.39}$$

*Here* $A_{kk} := F(\alpha_k)$ *is the* $d(k) \times d(k)$ *companion matrix of* $\alpha_k$, *and* $A_{k,k+1}$ *and* $A_{k+1,k}$ *are matrices (of appropriate dimensions), each of which has zeros in all positions except the northeast corner, where (as for* $A_{k,k+1}$*) there is a* $\beta_k$ *and (as for* $A_{k+1,k}$*) a one.*[1] *Conversely, if* $\{Q_0, Q_1, \ldots, Q_n\}$ *are polynomials satisfying* (2.37)–(2.39), *they also satisfy the three-term recursion* (2.26). *Moreover, if* $B_n := (1, 0, 0, \ldots, 0)' \in \mathbb{R}^{\nu(n)}$,

$$R_n^{-1} = \left(B_n, A_n B_n, A_n^2 B_n, \ldots, A_n^{\nu(n)-1} B_n\right). \tag{2.40}$$

*Proof.* The relation $Q_k = \alpha_k Q_{k-1} - \beta_{k-1} Q_{k-2}$ is equivalent to

$$\begin{aligned}\left[zQ_{k-1}, z^2 Q_{k-1}, \ldots, z^{d(k)} Q_{k-1}\right] &= \beta_{k-1}\left[0, \ldots, 0, Q_{k-2}\right] \\ &\quad + \left[zQ_{k-1}, z^2 Q_{k-1}, \ldots, \left(z^{d(k)} - \alpha_k\right) Q_{k-1}\right] \\ &\quad + \left[0, \ldots, 0, Q_k\right].\end{aligned} \tag{2.41}$$

[1] See Figure 2 for an example.

Let $R(k) \in \mathbb{R}^{\nu(n)\times d(k)}$ be defined by

$$\left[1, z, z^2, \ldots, z^{\nu(n)-1}\right] R(k) = \left[Q_{k-1}, zQ_{k-1}, \ldots, z^{d(k)-1}Q_{k-1}\right].$$

Then, for $k = 0,1,2,\ldots,n-1$, (2.41) can be written

$$J_n R(k) = R(k-1)A_{k-1,k} + R(k)A_{kk} + R(k+1)A_{k+1,k}, \quad (2.42a)$$

where $J_n$ is the $\nu(n)\times\nu(n)$ leading principal submatrix of the downshift $J$. For $k = n$ we have

$$J_n R(n) = R(n-1)A_{n-1,n} + R(n)A_{nn} + (0,\ldots,0,r_n) \quad (2.42b)$$

where $r_n \in \mathbb{R}^{\nu(n)}$ is defined by $Q_n(z) = z^{\nu(n)} + (1, z, \ldots, z^{\nu(n)-1})r_n$. Now $R_n = [R(1), R(2), \ldots, R(n)]$, and therefore, since $R(0) = 0$, (2.42) can be written

$$J_n R_n = R_n A_n + (0,\ldots,0,r_n). \quad (2.43)$$

But, since $R_n$ is unit right triangular, the last term in (2.43) can be replaced by $(0,\ldots,0,r_n)R_n$, and therefore (2.43) is the same as (2.38). It just remains to prove (2.40). Since $R_n$ is unit right triangular, $R_n B_n = B_n$, which together with (2.38) yields

$$R_n\left(B_n, A_n B_n, A_n^2 B_n, \ldots, A_n^{\nu(n)-1}B_n\right) = \left(B_n, F_n B_n, F_n^2 B_n, \ldots, F_n^{\nu(n)-1}B_n\right) = I.$$

This gives us (2.40). ∎

In the special case that all $d(k) = 1$, $A_n$ is tridiagonal. This is the regular case when the Lanczos algorithm applies. The relation (2.40) has certain implications in realization theory, as we shall see next.

## 3. COMPLETE FAMILIES OF MINIMAL PARTIAL REALIZATIONS

Our interest in the degree indices of a sequence $\gamma$ emanates from the fact that they are precisely the McMillan degrees of all partial sequences of $\gamma$.

THEOREM 4. *Let $\gamma$ be an infinite sequence (of real numbers) with degree indices $\{\nu(n);\ n \in \mathfrak{N}\}$. Then the McMillan degree of $\gamma$ is given by*

$$\delta(\gamma) = \sup\{\nu(n);\ n \in \mathfrak{N}\}. \quad (3.1)$$

*Moreover, for each* $N = 1,2,3,\ldots$, *the partial sequence* $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ *has McMillan degree* $\nu(n)$, *where* $n$ *is determined by*

$$\nu(n-1)+\nu(n) \leqslant N < \nu(n)+\nu(n+1). \tag{3.2}$$

*Here* $\nu(-1) := 0$, *and when* $\mathfrak{N}$ *is finite the convention of Section 2 applies.*

Consequently each partial sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ satisfying (3.2) has a minimal realization $\Sigma_n = (A_n, B_n, C_n)$ of dimension $\nu(n)$. Hence we can construct a family $\Sigma = \{\Sigma_n;\ n \in \mathfrak{N}\}$ of minimal partial realizations of $\gamma$, one for each degree index, so that the matchings $\gamma_i = C_n A_n^{i-1} B_n$ hold for increasing portions of $\gamma$. (Here $\Sigma_0 := (0,0,0)$ will only be needed if $\gamma_1 = 0$, as can be seen from (3.2); nevertheless we shall always include it for completeness.) Such a family will be called a *complete family of minimal partial realizations* of $\gamma$. There are many of them. We shall be interested in families which are *nested* in the sense that, whenever $0 < m < n$, $A_m$, $B_m$, and $C_m$ appear in the upper left corners of $A_n$, $B_n$, and $C_n$ respectively. This implies that data computed for $\Sigma_m$ can also be used for $\Sigma_n$.

Before continuing we shall illustrate Theorem 4 by an example, to which we shall return repeatedly in the sequel. Let $\gamma$ be the infinite sequence

$$\gamma = \{1,1,1,2,3,4,5,6,7,8,\ldots\}. \tag{3.3}$$

Then the full-rank matrices $\hat{H}_n$ are the ones framed in Figure 1. Consequently the degree indices are $\{0,1,3,4\}$ and $\delta(\gamma) = 4$. For $N = 1,2,3,\ldots$, the McMillan degrees of the subsequences $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ are respectively $1,1,1,3,3,3,4,4,4,4,\ldots$. There is a change of degree as we reach the lower right corner of each matrix $\hat{H}_n$; these elements have been circled in Figure 1.

The degree indices of a *finite* sequence $\gamma$ are then simply defined as the indices $\{\nu(0), \nu(1), \ldots, \nu(n)\}$ which the above procedure has produced by the time we run out of numbers $\gamma_i$; and, by Theorem 4, $\delta(\gamma) = \nu(n)$. Hence there is really no need to consider Hankel matrices with undetermined elements as in Kalman [20]. If we do not have enough data to form a full-rank matrix $\hat{H}_n$, there is no index $\nu(n)$.

Theorem 4 follows quite easily from Theorem 11 in Silverman [32], but here we shall provide a proof which is based on our Theorem 1. For this purpose, and for latter use, we shall need the following lemma.

LEMMA 1. *Let* $\gamma$ *be an infinite sequence with degree indices* $\{\nu(n);\ n \in \mathfrak{N}\}$, *and let* $\sigma^i(H_n)$ *denote the* $\nu(n) \times \nu(n)$ *Hankel matrix of the subse-*

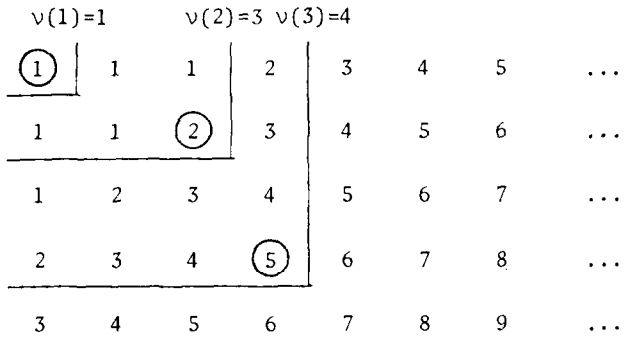



FIG. 1.

*quence* $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{i+2\nu(n)-1}\}$. *Then, for each* $n \in \mathfrak{N}^+$,

$$\sigma^i(H_n) = H_n F_n^i, \qquad i = 0,1,2,\ldots,d(n+1), \tag{3.4}$$

*where* $F_n := F(Q_n)$ *is the companion matrix of* $Q_n$ *(which is defined as in Theorem* 2*). Moreover, if* $e_1 := (1,0,\ldots,0)' \in \mathbb{R}^{\nu(n)}$,

$$\gamma_i = e_1' H_n F_n^{i-1} e_1, \qquad i = 1,2,\ldots,\nu(n)+\nu(n+1)-1. \tag{3.5}$$

*Proof.* Let $r_n \in \mathbb{R}^{\nu(n)}$ be given by $Q_n = z^{\nu(n)} + (1, z, z^2, \ldots, z^{\nu(n)-1})r_n$. Then (2.24) yields

$$\gamma_{\nu(n)+i} + r_{n,\nu(n)-1}\gamma_{\nu(n)+i-1} + \cdots + r_{n0}\gamma_i = 0$$

for $i = 1,2,3,\ldots,\nu(n+1)-1$. Therefore, since $F_n = J_n - r_n e_{\nu(n)}'$, where $e_{\nu(n)} = (0,0,\ldots,0,1)' \in \mathbb{R}^{\nu(n)}$, a simple calculation shows that $\sigma^i(H_n)F_n = \sigma^{i+1}(H_n)$ for $i = 0,1,2,\ldots,d(n+1)-1$. This proves the first part. As $i$ becomes greater than $d(n+1)$, $\sigma^i(H_n)$ and $H_n F_n^i$ will differ in an increasing number of positions, beginning in the southeast corners. The northwest corners will agree until $i = \nu(n)+\nu(n+1)-1$. ∎

Note that, if $\gamma$ has only finitely many degree indices $\{\nu(0), \nu(1), \ldots, \nu(\hat{n})\}$, then (3.4) holds for all nonnegative integers $i$ when $n = \hat{n}$, since, by the convention introduced in Section 2, $d(\hat{n}+1) = \infty$.

*Proof of Theorem 4.* Let $(A, B, C)$ be an arbitrary realization of the sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$. Set $m := \dim(A, B, C)$ and $\bar{\gamma}_i := CA^{i-1}B$ for $i =$

$1,2,3,\ldots$, and let $\bar{H}$ be the Hankel matrix of the infinite sequence $\bar{\gamma} = \{\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \ldots\}$. Then

$$\bar{H} = \left[C, CA, CA^2, \ldots\right] \begin{bmatrix} B \\ AB \\ A^2B \\ \vdots \end{bmatrix}, \tag{3.6}$$

and therefore rank $\bar{H} \leqslant m$. But $\bar{\gamma}_i = \gamma_i$ for $i = 1,2,3,\ldots,N$, where $N$ satisfies (3.2). Hence $\nu(n)$ is a degree index of $\bar{\gamma}$ also, and consequently rank $\bar{H} \geqslant \nu(n)$ (Theorem 1). This implies that $m \geqslant \nu(n)$, i.e., there is no realization of $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ with smaller dimension than $\nu(n)$. However, by choosing $(A, B, C)$ to be $(F_n, e_1, e_1'H_n)$ we achieve this lower bound (Lemma 1), and consequently the partial sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ has McMillan degree $\nu(n)$. If $\gamma$ has a largest degree index $\nu(\hat{n})$, then $(F_{\hat{n}}, e_1, e_1'H_{\hat{n}})$ realizes the complete sequence $\gamma$; hence $\delta(\gamma) = \nu(\hat{n})$. If $\gamma$ has infinitely many degree indices, it has no finite-dimensional complete realization, and therefore $\delta(\gamma) = \infty$. ■

Consequently we obtain a complete family of minimal partial realizations of $\gamma$ by forming the triplet $(F_n, e_1, e_1'H_n)$ for each degree index $\nu(n) > 0$. However, this family is not nested, so we shall consider another.

THEOREM 5. *Let $\gamma$ be an infinite sequence with degree indices $\{\nu(n);\ n \in \mathfrak{N}\}$, and let the polynomials $\{\alpha_n;\ n \in \mathfrak{N}^+\}$ and real numbers $\{\beta_{n-1};\ n \in \mathfrak{N}^+\}$ be determined from $\gamma$ as in Theorem 2. Set $\Sigma_0 := (0,0,0)$, and, for each $n \in \mathfrak{N}^+$, let $\Sigma_n$ be the triplet $(A_n, B_n, C_n)$, where $A_n$ and $B_n$ are defined as in Theorem 3 and $C_n := \beta_0 e_{\nu(1)}'$; $e_{\nu(1)} \in \mathbb{R}^{\nu(n)}$ is the unit vector with a one in position $\nu(1)$ and zeros elsewhere. Then $\{\Sigma_n;\ n \in \mathfrak{N}\}$ is a nested complete family of minimal partial realizations of $\gamma$. If $\mathfrak{N}$ is finite, the last $\Sigma_n$ is a (complete) realization of $\gamma$. For each $n \in \mathfrak{N}^+$, $\Sigma_n$ is uniquely determined by the parameters $\{\gamma_1, \gamma_2, \ldots, \gamma_{2\nu(n)}\}$.*

*Proof.* Let $n \in \mathfrak{N}^+$ be arbitrary. It follows from (2.38) that $F_n^i R_n = R_n A_n^i$ for $i = 0,1,2,\ldots$. Moreover, since $R_n$ is unit right triangular, $R_n e_1 = e_1$. Therefore, $e_1'H_n F_n^i e_1 = e_1'R_n'H_n F_n^i R_n e_1 = e_1'R_n'H_n R_n A_n^i e_1$. But, by Theorem 1, $R_n'H_n R_n = D_n$, and therefore Lemma 1 implies that

$$\gamma_i = e_1'D_n A_n^i e_1 \qquad \text{for} \quad 0 \leqslant i < \nu(n) + \nu(n+1).$$

However, the first row of $D_n$ is zero except for the northeast corner of $\Pi_1$, which contains $\beta_0$; hence $e_1'D_n = \beta_0 e_{d(1)}' =: C_n$. Moreover, $B_n := e_1$. Therefore

for each $N$ satisfying (3.2), $\Sigma_n := (A_n, B_n, C_n)$ is a realization of $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ of dimension $\nu(n)$. Hence it is minimal (Theorem 4). The family $\{\Sigma_n; n \in \mathfrak{N}\}$ defined in this way is clearly nested. The last statement of the theorem is a direct consequence of Theorem 2. ■

The family of partial realizations defined in Theorem 5 is, with trivial modifications, the one proposed by Kalman in [20]. What we have added, among other things, is the connection to block tridiagonalization, and other results will follow. The connection to Magnus's $P$-fractions will be discussed in the next section. It is suggested in [20] that the realizations $\{\Sigma_n; n \in \mathfrak{N}\}$ are "canonical," but it is not mentioned in what sense. We shall make this point clear.

The realization $\Sigma_n = (A_n, B_n, C_n)$ is uniquely determined by $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_0, \beta_1, \ldots, \beta_{n-1}\}$, the sizes of the blocks being determined by the degrees of the $\alpha$-polynomials. To pinpoint the structural properties of $\Sigma_n$, let us first consider an example in which the degree indices are $\{0,2,5,7,8\}$. Then $\{\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$ is a complete family of minimal partial realizations, and $\Sigma_4$ has the form described in Figure 2. Note that $A_n$ is the sum of the downshift $J_n$ and a right triangular matrix. Such a matrix is called a *unit right Hessenberg* matrix. The partial realizations $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$ are the subsystems of $\Sigma_4$ obtained by merely deleting an appropriate number of blocks in the matrices of Figure 2. More precisely, $\Sigma_n$ is determined by $\Sigma_{n-1}$, $\beta_{n-1}$, and

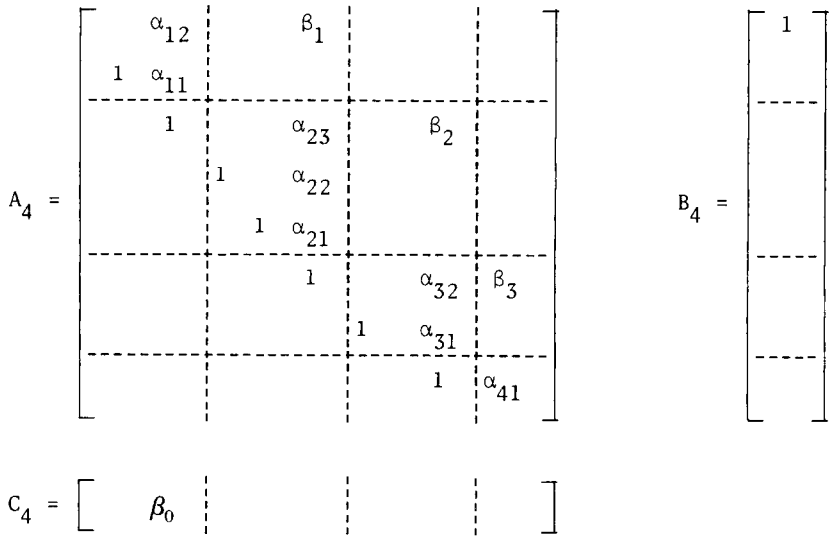



FIG. 2.

$\alpha_n$. This is the nesting property. For a numerical example, let us consider (3.3). Theorem 2 yields $\alpha_1(z) = z - 1$, $\alpha_2(z) = z^2$, $\alpha_3(z) = z - 1$, $\beta_0 = 1$, $\beta_1 = 1$, and $\beta_2 = -1$. Hence the complete family $\{\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3\}$ of minimal partial realizations is $\Sigma_0 = (0,0,0)$, $\Sigma_1 = (1,1,1)$,

$$\Sigma_2 = \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [1,0,0] \right\}, \tag{3.7a}$$

and

$$\Sigma_3 = \left\{ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [1,0,0,0] \right\}. \tag{3.7b}$$

Given a sequence $\gamma$, it is convenient to collect the parameters defined from it by Theorem 2 in a sequence $\rho := \{\rho_1, \rho_2, \rho_2, \dots\}$ constructed as follows. For each $n \in \overline{\mathfrak{N}}^+$, set

$$\rho_{2\nu(n-1)+i} = \begin{cases} 0 & \text{for } 0 \leqslant i < d(n), \\ \beta_{n-1} & \text{for } i = d(n), \\ \alpha_{n,i-d(n)} & \text{for } d(n) < i \leqslant 2d(n). \end{cases} \tag{3.8}$$

Then, if $\delta(\gamma) < \infty$, $\rho_{2\delta(\gamma)+i} = 0$ for $i = 1,2,3,\dots$. We shall call $\rho$ the *parameter sequence* of $\gamma$. For the example illustrated by Figure 2 we have

$$\rho = \{0, \beta_0, \alpha_{11}, \alpha_{12}, |0,0, \beta_1, \alpha_{21}, \alpha_{22}, \alpha_{23}, |0, \beta_2, \alpha_{31}, \alpha_{32}, |\beta_3, \alpha_{41}, |0,0,0,\dots\}, \tag{3.9}$$

where we have divided the sequence $\rho$ into sections to indicate how many data are required for each of the partial realizations defined by Theorem 5: the first $\sigma(n) := 2\nu(n)$ elements of $\rho$ are needed for $\Sigma_n$. We shall call $\{\sigma(n); n \in \mathfrak{N}\}$ the *section indices* of $\rho$. In (3.9) they are $\{0,4,10,14,16\}$. Note that, since all $\beta_n \neq 0$, it is easy to decode the sequence $\rho$ to obtain the $\alpha$- and $\beta$-parameters as well as the degree indices of $\gamma$.

Now let $\rho = \{\rho_1, \rho_2, \rho_3, \dots\}$ be an *arbitrary* sequence of real numbers. Then we define the *section indices* $\{\sigma(m); m \in \mathfrak{M}\}$ of $\rho$ in the following way.

Set $\sigma(0) := 0$. For $m = 0, 1, 2, \ldots$, find the first nonzero $\rho_i$ such that $i > \sigma(m)$, and set $d(m+1) := i - \sigma(m)$. Then $\sigma(m+1) := \sigma(m) + 2d(m+1)$. Clearly this definition, which also applies to finite sequences $\rho$, is consistent with the one given above. The index set $\mathfrak{M}$ is defined *by this process* in complete analogy with $\mathfrak{N}$ above, and we shall also use the notations $\mathfrak{M}^+$, $\overline{\mathfrak{M}}$, etc. with the same meaning as before. Then, for $m \in \mathfrak{M}^+$, set

$$\begin{aligned} \beta_{m-1} &= \rho_{\sigma(m-1)+d(m)}, \\ \alpha_{mi} &= \rho_{\sigma(m-1)+d(m)+i} \qquad \text{for} \quad 0 < i \leqslant d(m), \end{aligned} \tag{3.10}$$

and, for $m \in \mathfrak{M}$, set $\nu(m) := \frac{1}{2}\sigma(m)$. Hence $\rho$ defines a family $\{\Sigma_m;\ m \in \mathfrak{M}\}$ of triplets $\Sigma_m := (A_m, B_m, C_m)$ of the type described above. To emphasize its dimension and its dependence on $\rho$ we shall write $\Sigma(\nu(m), \rho)$ instead of $\Sigma_m$. Certainly each $\Sigma(\nu(m), \rho)$ is a realization of some sequence $\gamma$. In fact, it defines a sequence $\gamma$ via (1.1). However, we have not established that it is the realization of $\gamma$ defined by Theorem 5, i.e that $\rho$ is the parameter sequence of $\gamma$. For example, let $\gamma$ be a sequence with finite McMillan degree and parameter sequence $\rho$. Then, there could conceivably be a $\hat{\rho} \neq \rho$ such that both $\Sigma(\delta(\gamma), \hat{\rho})$ and $\Sigma(\delta(\gamma), \rho)$ are minimal realizations of $\gamma$. Next we shall demonstrate that this cannot be the case.

THEOREM 6. *Let $f$ be a function on the space of all infinite sequences of real numbers $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ into itself defined in the following way. For each $\gamma$, let $\rho = f(\gamma)$ be the parameter sequence of $\gamma$, i.e. the sequence determined by the algorithm consisting of Equations* (2.26)–(2.30) *plus* (3.8). *Then the map $f$ is a bijection. Moreover, if $\gamma$ and $\hat{\gamma}$ are two sequences and $\rho = f(\gamma)$ and $\hat{\rho} = f(\hat{\gamma})$, then $\gamma_i = \hat{\gamma}_i$ for $i = 1, 2, \ldots, N$ if and only if $\rho_i = \hat{\rho}_i$ for $i = 1, 2, \ldots, N$.*

*Proof.* The algorithm yields the sequence $\pi := \{\pi_1, \pi_2, \pi_3, \ldots\}$ as an intermediate result. Note that $\pi$ and $\rho$ have identical section indices. Hence $f$ is the composition of two functions $g$ and $h$, defined on the same space, such that $\pi = g(\gamma)$ and $\rho = h(\pi)$, and $h$ preserves the section indices. Then the theorem follows if we can show that $g$ and $h$ have the properties claimed for $f$. Let us begin with $g$. For any partial sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ the algorithm (2.26)–(2.30) produces a sequence $\{\pi_1, \pi_2, \ldots, \pi_N\}$ of precisely the same length. We need to show that this restricted map has an inverse. To this end, let $\{\pi_1, \pi_2, \ldots, \pi_N\}$ be arbitrary with section indices $\{\sigma(0), \sigma(1), \ldots, \sigma(n)\}$. Set $\nu(k) := \frac{1}{2}\sigma(k)$ and $d(k) := \nu(k) - \nu(k-1)$, for $k = 1, 2, \ldots, n$ $[\nu(-1) = 0]$. Set $\gamma_i := \pi_i$ for $i = 1, 2, \ldots, \sigma(1)$. Now, for $k = 1, 2, \ldots, n-1$, proceed as follows.

First use (2.26)–(2.29) to determine $Q_k$ from $\{\pi_1, \pi_2, \dots, \pi_{\sigma(k)}\}$. Then use (2.30) to determine $\{\gamma_{\sigma(k)+1}, \dots, \gamma_{\sigma(k+1)}\}$ from $\{\pi_{\sigma(k)+1}, \dots, \pi_{\sigma(k+1)}\}$ and previously computed $\gamma$-parameters. If $k = n-1$ and $N < \sigma(n)$, the triangular structure of (2.30) implies that we obtain precisely $\{\gamma_{\sigma(n-1)+1}, \dots, \gamma_N\}$ before we run out of $\pi$-parameters. If $N > \sigma(n)$, the last $N - \sigma(n)$ $\pi$-parameters are zero, and then we can solve (2.30) directly for $\{\gamma_{\sigma(n)+1}, \dots, \gamma_N\}$. This is clearly the required inverse. A similar (but much simpler) argument applied to (2.28) and (2.29) and using the triangular structure of (2.28) establishes the required one-one correspondence between the partial $\pi$- and $\rho$-sequences. ∎

This theorem has two interesting corollaries dealing with canonical forms and realizations of partial sequences respectively. Let us begin with the first topic. Consider the family $\mathfrak{R}_N$ of all $\Sigma := (A, B, C)$ such that $\Sigma$ is a minimal realization of some sequence $\gamma$ of McMillan degree $N$; hence $\dim \Sigma = N$. Let $\mathfrak{S}_N$ be the group of transformations on $\mathbb{R}^N$. We shall say that two elements in $\mathfrak{R}_N$ are *equivalent* if one can be obtained from the other by a transformation

$$(A, B, C) \xrightarrow{T} (TAT^{-1}, TB, CT^{-1}) \tag{3.11}$$

where $T \in \mathfrak{S}_N$. It is well known [4, 21] that two realizations in $\mathfrak{R}_N$ are equivalent if and only if they realize the same infinite sequence $\gamma$. A family $\{\Sigma(\theta);\ \theta \in \Theta\}$ of realizations in $\mathfrak{R}_N$ is a set of *canonical forms* for $\mathfrak{R}_N$ if each $\Sigma \in \mathfrak{R}_N$ is equivalent to one and only one $\Sigma(\theta)$; cf. [2; p. 244]. Define $\mathfrak{P}_N$ to be the class of infinite sequences $\rho := \{\rho_1, \rho_2, \rho_3, \dots\}$ having $2N$ among its section indices. (Nothing is being assumed about which one in order it is.)

COROLLARY 1. *The family* $\mathfrak{S}_N := \{\Sigma(N, \rho);\ \rho \in \mathfrak{P}_N\}$ *is a set of canonical forms for* $\mathfrak{R}_N$.

*Proof.* Let $\gamma$ be an arbitrary infinite sequence such that $\delta(\gamma) = N$, and let $\rho := f(\gamma)$. Then $\Sigma(N, \rho)$ belongs to the equivalence class of realizations of $\gamma$ (Theorem 5). Now, no other $\Sigma(N, \hat{\rho})$ with $\hat{\rho} \neq \rho$ will belong to this equivalence class, because $\hat{\gamma} := f^{-1}(\hat{\rho})$ is different from $\gamma$ (Theorem 6), and $\Sigma(N, \hat{\rho})$ is a realization of $\hat{\gamma}$ (Theorem 5). ∎

Consequently, if $\gamma$ is an infinite sequence with degree indices $\{\nu(0), \nu(1), \dots, \nu(n)\}$, $\gamma$ has one and only one minimal realization in $\mathfrak{S}_{\nu(n)}$, namely $\Sigma_n = (A_n, B_n, C_n)$ as determined by Theorem 5. Moreover $\{\beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \beta_{n-1}, \alpha_n\}$ is a complete set of invariants. For finite sequences this question is not quite as simple. This leads us to the next corollary of Theorem 6.

Many results for finite sequences $\gamma$ can be obtained trivially from the corresponding results for infinite sequences by merely embedding $\gamma$ in an infinite sequence with the same McMillan degree. However, in doing so we must realize that this *degree preserving extension* may not be unique. The reason for this is that the degree index $\nu(n)$ is determined by $\hat{H}_n$, while $H_n + \gamma_{2\nu(n)}$ is needed to determine $\Sigma_n$ (Theorem 5). Hence if $\gamma$ contains fewer than $2\nu(n)$ elements but enough elements to form $\hat{H}_n$, $\gamma$ will be represented by many equivalence classes in $\mathfrak{R}_N$. This uniqueness question has been studied by Kalman [19, 20], and, in the context of differential geometry, by Brockett [3]. However, the parameterization of the family of minimal realizations of a partial sequence $\gamma$ is very simply described in terms of the canonical forms $\Sigma(N, \rho)$, something which was apparently overlooked by Kalman [20].

COROLLARY 2. *Let* $\gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ *be a finite sequence (of real numbers) with degree indices* $\{\nu(0), \nu(1), \ldots, \nu(n)\}$. *For each* $k = 0, 1, 2, \ldots, n$ *there is a* $\Sigma_k \in \mathcal{S}_{\nu(k)}$, *so that* $\{\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ *is a nested complete family of minimal partial realizations of* $\gamma$. *The partial realizations* $\{\Sigma_0, \Sigma_1, \ldots, \Sigma_{n-1}\}$ *are uniquely determined by* $\gamma$ *(as in Theorem 5), whereas* $\Sigma_n$ *is unique if and only if* $N \geqslant 2\nu(n)$. *If* $m := 2\nu(n) - N > 0$, *there are* $m$ *undetermined parameters in* $\Sigma_n$, *namely* $\alpha_{n,d(n)}, \alpha_{n,d(n)-1}, \ldots, \alpha_{n,d(n)-m}$, *the possible realizations forming an* $m$*-dimensional affine manifold. There are at most* $d(n)$ *undetermined parameters,* $\beta_{n-1}$ *being completely determined.*

*Proof.* The sequence $\gamma$ can be embedded in an infinite sequence $\bar{\gamma}$ with the same degree indices, and consequently the existence of a family $\{\Sigma_0, \Sigma_1, \ldots, \Sigma_n\}$ with the required properties is insured by Theorem 5. To each degree preserving extension $\bar{\gamma}$ there corresponds a unique parameter sequence $\rho = f^{-1}(\bar{\gamma})$ with section indices $\{\sigma(0), \sigma(1), \ldots, \sigma(n)\}$, where $\sigma(k) := 2\nu(k)$, $k = 0, 1, \ldots, n$. In the family of such $\rho$, $\{\rho_1, \rho_2, \ldots, \rho_N\}$ are fixed and $\rho_{\sigma(n)+i} = 0$ for $i = 1, 2, 3, \ldots$; any other $\rho_i$ is arbitrary (Theorem 6). Since $N > \sigma(n-1)$ (Theorem 4), $\Sigma_0, \Sigma_1, \ldots,$ and $\Sigma_{n-1}$ are uniquely determined. If $N \geqslant \sigma(n)$, there are no arbitrary $\rho$-parameters, so $\Sigma_n$ is unique. If $N < \sigma(n)$, there are $m := \sigma(n) - N$ arbitrary $\rho$-parameters, namely $\rho_{N+1}, \rho_{N+2}, \ldots, \rho_{\sigma(n)}$. Since $N \geqslant \nu(n) + \nu(n-1) = \sigma(n) - d(n)$ (Theorem 4), there are at most $d(n)$ such undetermined parameters. If $\Sigma_n$ and $\hat{\Sigma}_n$ are two realizations of $\gamma$, then so is $\lambda\Sigma_n + (1-\lambda)\hat{\Sigma}_n$ for all $\lambda \in \mathbb{R}$. ∎

Let us illustrate this with the example in Figure 2. Look at the parameter sequence (3.9). If $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, $\Sigma_1$ is completely determined. We obtain $\gamma_5$ and $\gamma_6$ "for free" as the corresponding positions in the sequence $\rho$ contain zeros. If $\gamma_7$ is added, we need to determine $\Sigma_2$ in which there are three undetermined parameters, namely $\alpha_{21}$, $\alpha_{22}$, and $\alpha_{23}$. Adding $\gamma_8$ will fix $\alpha_{21}$,

adding $\gamma_9$ will fix $\alpha_{22}$, etc. In the numerical example (3.3), the parameter sequence is

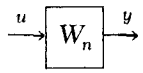
$$\rho = \{1,1,0,1,0,0,\ -1,1,0,0,0,\ldots\}. \tag{3.12}$$

The partial sequence $\hat{\gamma} = \{1,1,1,2\}$ has the canonical realizations

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & \xi \\ 0 & 1 & \eta \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C_2 = [1,0,0],$$

where $\xi$ and $\eta$ are arbitrary real numbers. If we augment $\hat{\gamma}$ to be $\{1,1,1,2,3\}$, $\eta$ is fixed at $\eta = 0$. Also adding $\gamma_6 = 4$ will fix $\xi$ too, at $\xi = 0$. In Section 5 we shall present an algorithm which computes the elements of the parameter sequence $\rho$ in the correct order, sequentially producing $\rho_k$ from $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ for $k = 1,2,3,\ldots$ rather than in complete sections.

It is instructive to illustrate the structure of the canonical family $\{\Sigma_n;\ n \in \mathfrak{N}\}$ of minimal partial realizations by a block diagram. As explained in Section 1, the partial realization $\Sigma_n = (A_n, B_n, C_n)$ can be interpreted as a dynamical system, symbolically depicted




(3.13)

where $W_n$ is the transfer function

$$W_n(z) = C_n(zI - A_n)^{-1}B_n \tag{3.14}$$

and $u(z)$ and $y(z)$ are the (discrete or continuous) Laplace transforms of the input and output sequences respectively. In the sequel we shall refer to $u$ and $y$ simply as the input and the output respectively, and for notational convenience, we have dropped the tilde ( $\tilde{}$ ) used in Section 1 to distinguish the input and the output from the corresponding time-domain quantities. From (1.6) we have

$$y(z) = W_n(z)u(z), \tag{3.15}$$

so the diagram (3.13) should be interpreted multiplicatively. Now setting $x(z) := (zI - A_n)^{-1}B_n u$, (3.14) and (3.15) yield

$$\begin{aligned} zx &= A_n x + B_n u, \\ y &= C_n x, \end{aligned} \tag{3.16}$$

which is merely a Laplace-transformed version of (1.5) or of (1.7). To exhibit the block structure of the system (3.16), we partition $x(z) \in \mathbb{R}^{\nu(n)}$ into a column $x = (x_1', x_2', \ldots, x_n')'$ of $n$-subvectors $x_1(z) \in \mathbb{R}^{d(1)}$, $x_2(z) \in \mathbb{R}^{d(2)}, \ldots,$ $x_n(z) \in \mathbb{R}^{d(n)}$. Moreover, for each $k = 1,2,\ldots,n$, define $b_k \in \mathbb{R}^{d(k)}$ as the (column) unit vector $b_k = (1,0,\ldots,0)'$ and $c_k \in \mathbb{R}^{1 \times d(k)}$ as the row unit vector $c_k = (0,\ldots,0,1)$. Then

$$zx_k = A_{k,k-1}x_{k-1} + A_{kk}x_k + A_{k,k+1}x_{k+1} + b_1 u \delta_{k1}, \qquad k = 1,2,\ldots,n,$$

$$y = \beta_0 c_1 x_1$$

if we define $x_0$ and $x_{n+1}$ to be zero, and the submatrices $A_{ij}$ are defined as in Theorem 3. But $A_{k,k-1} = b_k c_{k-1}$ and $A_{k,k+1} = \beta_k b_k c_{k+1}$, and therefore, setting $y_1 := \beta_0^{-1} y$,

$$\begin{aligned} zx_k &= A_{kk}x_k + b_k u_k, \\ y_k &= c_k x_k \end{aligned} \tag{3.17a}$$

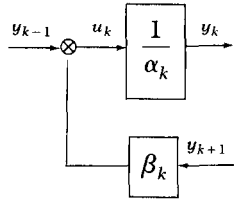
for $k = 1,2,\ldots,n$, where $\{u_1(z), u_2(z),\ldots,u_n(z)\}$ are defined by

$$u_k = y_{k-1} + \beta_k y_{k+1} \tag{3.17b}$$

for $k = 1,2,\ldots,n$, with $y_0 = u$ and $y_{n+1} = 0$. Note that the subsystem (3.17a) has the same structure as (3.16) except that $A_{kk}$ is the companion matrix of the polynomial $\alpha_k$. It is a routine matter to see that the transfer function of (3.17a) is

$$c_k(zI - A_{kk})^{-1} b_k = 1/\alpha_k, \tag{3.18}$$

and consequently (3.17) has the block representation




(3.19)

where $\otimes$ denotes addition. The block diagram of the partial realization $\Sigma_n$ is

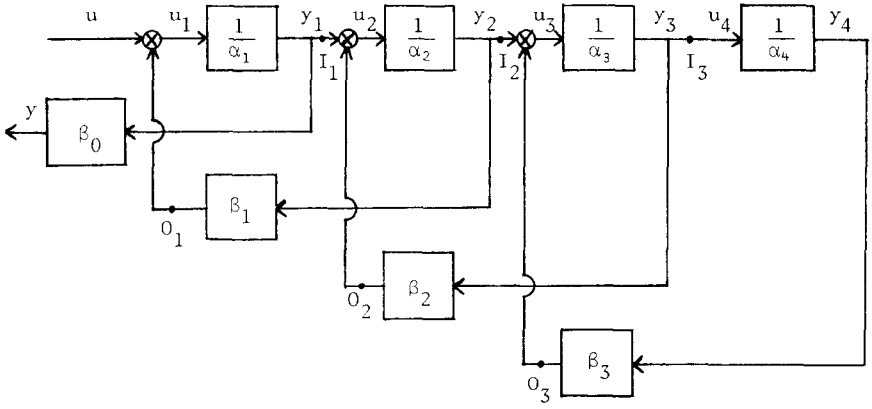



FIG. 3.

then obtained by coupling together the $n$ subsystems (3.19), remembering that $y_{n+1} = 0$, $y_0 = u$, and $y = \beta_0 y_1$. The case $n = 4$ is shown in Figure 3. (The nodes $O_k$ and $I_k$ are for later reference in Section 4.) Hence the partial realization $\Sigma_k$ is obtained from $\Sigma_{k-1}$ by merely adding the subsystem (3.19) in the appropriate manner. This is again a manifestation of the nesting property.

The diagram of Figure 3 is equivalent to the one obtained by Kalman in [20] via the Euclidean algorithm. In the next section we shall further investigate this connection and demonstrate that the procedure used by Kalman [20] is actually a special case of the (infinite) process of Magnus [24, 25], leading to a type of continued fraction known as a $P$-fraction.

Note that, as new subsystems are added to the diagram in Figure 3, the stability of the overall system may be affected. In fact, in [5] we showed that preservation of stability is not even a generic property, as one might have hoped [20; p. 23].

However, before leaving the present topic, we shall briefly discuss the connections to Rissanen's algorithm [29]. Let us return to the proof of Theorem 1 for reference. Assume that $d(n+1) > 1$, so that the matrix $\bar{H}_n := H_{\nu(n)+1,\,\nu(n)+1}$ has rank $\nu(n)$, and consider the factorization $R(1)'\bar{H}_n R(1)$, i.e.

$$\begin{bmatrix} R' & \\ r' & 1 \end{bmatrix} \bar{H}_n \begin{bmatrix} R & r \\ & 1 \end{bmatrix} = \begin{bmatrix} D & \\ & 0 \end{bmatrix},$$

where, for simplicity, we have dropped the indices. Now, remember that, as explained in the proof of Theorem 1, there is a certain arbitrariness in

defining the unit right triangular matrix $R$ (but not $r$). In fact, the $R$ used by Rissanen is different from our $R_n$, and this is yet another reason for using different symbols. Then

$$\bar{H}_n = \begin{bmatrix} (R')^{-1} & \\ (R^{-1}r)' & 1 \end{bmatrix} \begin{bmatrix} R'H_n & -R'H_n r \\ 0 & 0 \end{bmatrix}. \tag{3.20}$$

To obtain the second factor we have used the fact that $D = R'H_nR$. Following Rissanen, let us call the two factors in (3.20) $P$ and $Q$ respectively, so that $\bar{H}_n = PQ$. This is the factorization used by Rissanen in [29] to obtain a partial realization of dimension $\nu(n)$. (From [29] it may seem that some partial realizations are constructed from factorizations of rectangular Hankel matrices, but this generality is illusory. In fact, our Theorem 1 insures that any row added to $\hat{H}_n$ will increase the rank until $H_n$ is obtained. Hence the first alternative in step 5 of [29; p. 428] will never occur, forcing the algorithm always to pass through step 2 before it goes to step 3.)

Next delete the first row and the last column of $\bar{H}_n$ in (3.20), leaving us with $\sigma(H_n)$ in the left member. This corresponds to removing the first row of $P$ and the last column of $Q$. Since the last row of $Q$ is zero, we can also remove it, together with the last column of $P$. This yields

$$\sigma(H_n) = P_* R'H_n \tag{3.21}$$

where $P_*$ is the submatrix of $P$ obtained from this reduction. Now, choosing a particular $R$ (which is different from our $R_n$), the $A$-matrix in Rissanen's realization $(\bar{A}, \bar{B}, \bar{C})$ is

$$\bar{A} = R'P_*. \tag{3.22}$$

We want to relate this matrix to our $A_n$. To this end, note that $A_n = R_n^{-1}F_nR_n$ (Theorem 3) and $F_n = H_n^{-1}\sigma(H_n)$ (Lemma 1), and therefore (3.21) yields $A_n = (H_nR_n)^{-1}P_*R'H_nR_n$. Then, from (3.22) and $R_n'H_nR_n = D_n$, we obtain

$$\bar{A} = TA_nT^{-1},$$

where $T := (R_n^{-1}R)'D_n$. The complete triplet $(\bar{A}, \bar{B}, \bar{C})$ proposed by Rissanen is then obtained by applying the transformation (3.11) to $(A_n, B_n, C_n)$. If we modify Rissanen's factorization so that $R = R_n$, $T$ takes the particularly simple form $T = D_n$, i.e., we have a blockwise change of scale.

Rissanen's algorithm requires that $d(n+1) > 1$, and it will not produce a partial realization of McMillan degree $\nu(n)$ if $d(n+1) = 1$. In our numerical

example (3.3), partial realizations of dimensions $\nu(1) = 1$ and $\nu(3) = 4$ will be generated, but none of dimension $\nu(2) = 3$, and in the generic case when the degree indices are the nonpositive integers, no realization will result. Hence Rissanen's algorithm does not produce complete families of minimal partial realizations, and that does not seem to be its stated purpose. In fact, if $\delta(\gamma) = \nu(n) < \infty$, then $d(n+1) = \infty$, so a (complete) minimal realization of $\gamma$ will eventually appear.

## 4. PRINCIPAL-PART CONTINUED FRACTIONS

Let $\gamma$ be an infinite sequence (of real numbers) with degree indices $\{\nu(n); n \in \mathfrak{N}\}$, and let

$$\Gamma(z) = \sum_{i=1}^{\infty} \gamma_i z^{-i} \tag{4.1}$$

be the corresponding formal power series. Consider the parameter sequence $\rho$ of $\gamma$. We know that there is a one-one correspondence between $\rho$ and $\gamma$ and that the section indices of $\rho$ are $\{\sigma(n); n \in \mathfrak{N}\}$, where $\sigma(n) := 2\nu(n)$ (Theorem 6). Now for each $n \in \mathfrak{N}$, define a new sequence

$$\rho^{(n)} = \{\rho_{\sigma(n)+1}, \rho_{\sigma(n)+2}, \rho_{\sigma(n)+3}, \cdots\} \tag{4.2}$$

from $\rho$ by deleting the first $\sigma(n)$ elements. By Theorem 6, $\rho^{(n)}$ is the parameter sequence of some other sequence

$$\gamma^{(n)} = \{\gamma_{n1}, \gamma_{n2}, \gamma_{n3}, \cdots\} \tag{4.3}$$

with degree indices $\{\nu(k) - \nu(n); k \in \mathfrak{N}, k \geqslant n\}$. Let

$$\Gamma_n(z) = \sum_{i=1}^{\infty} \gamma_{ni} z^{-i} \tag{4.4}$$

be the corresponding formal power series.

For motivation let us consider the block diagram in Figure 3. Assume (for the moment only) that $\delta(\gamma) < \infty$ and that $\gamma$ has degree indices $\{\nu(0), \nu(1), \ldots, \nu(n)\}$. Then $\Gamma(z) = W_n(z)$, the transfer function of the complete minimal realizations of $\gamma$. Deleting the $\sigma(k)$ first elements of $\rho$ corresponds to removing the blocks corresponding to $\{\beta_0, \alpha_1, \beta_1, \alpha_2, \ldots, \beta_{k-1}, \alpha_k\}$, taking the input and the output to be $y_k$ and $\beta_k y_{k+1}$ respectively. Clearly $\Gamma_k$ is the

transfer function of this reduced system and therefore

$$\Gamma_k(z) = \frac{\beta_k y_{k+1}(z)}{y_k(z)}, \tag{4.5}$$

which also holds for $k = 0$ if we set $\Gamma_0(z) := \Gamma(z)$. For example, in Figure 3, $\Gamma_k$ is the transfer function of the subsystem with input $I_k$ and output $O_k$ for $k = 1,2,3$ (provided, of course, that $\Sigma_4$ is a complete realization of $\gamma$). Now, since $u_k(z) = \alpha_k(z)y_k(z)$, (3.17b) yields

$$y_{k-1} = \alpha_k y_k - \beta_k y_{k+1}, \tag{4.6}$$

and consequently, by (4.5),

$$\frac{\beta_{k-1}}{\Gamma_{k-1}(z)} = \alpha_k(z) - \Gamma_k(z) \tag{4.7}$$

for $k = 1,2,\ldots,n$, where $\Gamma_0 = \Gamma$ and $\Gamma_n = 0$. From this we see that $\{\beta_0, \alpha_1, \beta_1, \alpha_2, \ldots, \beta_{n-1}, \alpha_n\}$ can be determined from $\Gamma_0 = \Gamma$ by inverting $\Gamma_{k-1}$, successively for $k = 1,2,\ldots,n$, and splitting it into a polynomial part ($\alpha_k$) and a strictly proper part ($\Gamma_k$). The role of $\beta_{k-1}$ is to normalize so that $\alpha_k$ is monic. The process terminates at $k = n$, since $\Gamma_n = 0$. This is precisely the Euclidean algorithm applied to the pair of relatively prime polynomials $(Q_n, P_n)$ where $W_n = P_n/Q_n$ is the rational function $\Gamma$. The connection between the Euclidean algorithm and the canonical realizations of Theorem 5 was established by Kalman in [20]; in fact, this is the way they were derived in [20].

However, from the way we have written (4.7), it should be clear that $\Gamma$ need not be rational, but that we may consider an infinite process. Therefore let us now remove the assumption of rationality on $\Gamma$ and consider the following *generalized Euclidean algorithm for formal Laurent series*, which was presented by Magnus in [24, 25]:

$\Gamma_0(z) = \Gamma(z)$
for $n = 1,2,3,\ldots$ while $\Gamma_{n-1} \neq 0$
  reciprocate and split
  $\beta_{n-1}/\Gamma_{n-1}(z) = \alpha_n(z) - \Gamma_n(z)$
  with $\alpha_n$ a monic polynomial
  $\alpha_n(z) = z^{d(n)} - \alpha_{n1}z^{d(n)-1} - \cdots - \alpha_{n,d(n)}$,
  $\beta_{n-1}$ a normalizing constant, and
  $\Gamma_n(z) = \sum_{i=1}^{\infty}\gamma_{ni}z^{-i}$.

We are here working in the field $\mathcal{F}$ of real formal Laurent series with finitely many positive powers. (We could, of course, at no extra cost consider complex power series, but there is no need for this generality in our present setting.) The algorithm terminates if and only if $\Gamma(z)$ is the Laurent series about $z = \infty$ of a rational function. As described here, the algorithm uses in general an infinite sequence of reciprocations of formal Laurent series, each of which is itself an infinite process. However, as suggested by the results above, there are sequential versions of this process in which the coefficients $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ enter one by one in order and the output consists of $\{\beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\}$ in this order.

The polynomial $\alpha_n$ is the *principal part* at $z = \infty$ of the formal Laurent series $\beta_{n-1}/\Gamma_{n-1}(z)$. We have

$$\Gamma_{n-1}(z) = \frac{\beta_{n-1}}{\alpha_n(z) - \Gamma_n(z)}. \tag{4.8}$$

Hence we may associate with $\Gamma$ the continued fraction

$$G(z) = \cfrac{\beta_0}{\alpha_1(z) - \cfrac{\beta_1}{\alpha_2(z) - \cfrac{\beta_2}{\alpha_3(z) - \cdots}}} \tag{4.9}$$

which terminates if and only if $\Gamma$ is rational. Since we are interested in the algebraic properties of this object and not its convergence properties, we denote it by $G(z)$. This is the *principal-part continued fraction* (*P-fraction*) introduced by Magnus [24, 25]. By multiplying both numerator and denominator in the $n$th fraction by $1/\alpha_n$ for $n = 1,2,3,\ldots$ we obtain an alternative form of $G(z)$, from which someone with a little systems know-how can construct a block diagram as in Figure 3; cf. Kalman [20].

We associate with $G(z)$ the linear fractional transformations $t_0(w) = w$ and

$$t_n(w) = \frac{\beta_{n-1}}{\alpha_n - w}, \qquad n = 1,2,3,\ldots, \tag{4.10}$$

in terms of which we define

$$T_n = t_0 \circ t_1 \circ t_2 \circ \cdots \circ t_n \tag{4.11}$$

for $n = 0, 1, 2, \ldots$, where $\circ$ denotes composition and the dependence on the variable $z$ is surpressed to simplify the notation. Then $\Gamma = T_n(\Gamma_n)$. Since $\beta_n \neq 0$, these linear fractional transformations are invertible maps of the extended complex plane onto itself. The following theorem summarizes some of the results of [24, 25] in our framework.

THEOREM 7. *Let $\Gamma(z)$ be a formal power series of the type* (4.1), *and let $\{\beta_{n-1}, \alpha_n;\ n \in \mathfrak{N}^+\}$ be defined by the generalized Euclidean algorithm. For each $n \in \mathfrak{N}^+$, set $d(n) := \deg \alpha_n$, $\nu(n) := \sum_{k=1}^{n} d(k)$, and $\mu(n) := \sum_{k=2}^{n} d(k)$; set $\nu(0) := 0$. Then, for $n \in \mathfrak{N}$,*

$$T_n(w) = \frac{P_n - wP_{n-1}}{Q_n - wQ_{n-1}}, \tag{4.12}$$

*where $\{P_n;\ n \in \mathfrak{N}\}$ and $\{Q_n;\ n \in \mathfrak{N}\}$ are polynomials generated by the three-term recursions*

$$P_{n+1}(z) = \alpha_{n+1}(z)P_n(z) - \beta_n P_{n-1}(z), \qquad P_0 = 0, \quad P_{-1} = -1, \tag{4.13a}$$

$$Q_{n+1}(z) = \alpha_{n+1}(z)Q_n(z) - \beta_n Q_{n-1}(z), \qquad Q_0 = 1, \quad Q_{-1} = 0. \tag{4.13b}$$

*For each $n \in \mathfrak{N}^+$ we have* $\deg P_n = \mu(n)$, $\deg Q_n = \nu(n)$, *and*

$$\gcd(P_n, Q_n) = \gcd(P_n, P_{n+1}) = \gcd(Q_n, Q_{n+1}) = 1, \tag{4.14}$$

*where* gcd *denotes greatest common divisor. Moreover, the Laurent series about $z = \infty$ of the rational function*

$$W_n(z) = P_n(z)/Q_n(z) \tag{4.15}$$

*coincides with $\Gamma(z)$ up to powers in $z^{-1}$ of order $\nu(n) + \nu(n+1)$, i.e.*

$$W_n(z) = \sum_{i=1}^{\nu(n)+\nu(n+i)-1} \gamma_i z^{-i} + O(z^{-\nu(n)-\nu(n+1)}). \tag{4.16}$$

*If the algorithm terminates in step $n$, $\Gamma(z) = W_n(z)$.*

For the proof we need two lemmas, the first of which is well known and easy to prove.

LEMMA 2. *Let* $g_i(w)=(a_iw+b_i)/(c_iw+d_i)$, $i=1,2$, *be two linear fractional transformations with coefficients* $a_i, b_i, c_i, d_i$ *(in the field* $\mathfrak{F}$*) satisfying* $a_id_i-b_ic_i\neq 0$. *Then* $(g_1\circ g_2)(w)=(aw+b)/(cw+d)$, *where*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

LEMMA 3. *For* $n\in\mathfrak{N}$, *set*

$$E_n(z):=\Gamma(z)Q_n(z)-P_n(z), \tag{4.17}$$

*where* $P_n$ *and* $Q_n$ *are defined in terms of* $\Gamma$ *as in Theorem 7. Then*

$$E_n(z)=\sum_{k=1}^{\infty}\varepsilon_{nk}z^{-\nu(n)-k} \tag{4.18a}$$

*with* $\{\varepsilon_{nk}\}$ *satisfying*

$$\varepsilon_{nk}=\begin{cases} 0 & \text{for } k=1,2,\dots,d(n+1)-1, \\ \beta_0\beta_1,\dots,\beta_n & \text{for } k=d(n+1), \end{cases} \tag{4.18b}$$

*where all parameters and indices are defined as in Theorem 7. If the generalized Euclidean algorithm terminates in step* $n$, *then* $E_n=0$.

*Proof.* By the homogenity of the three-term recursions (4.13),

$$E_{n+1}(z)=\alpha_{n+1}(z)E_n(z)-\beta_nE_{n-1}(z), \qquad E_0(z)=\Gamma_0(z), \quad E_{-1}(z)=1.$$

Therefore, since $\alpha_{n+1}\Gamma_n-\beta_n=\Gamma_n\Gamma_{n+1}$, we obtain by induction

$$E_n=\Gamma_0\Gamma_1\Gamma_2\cdots\Gamma_n. \tag{4.19}$$

But, from (4.8), we have

$$\Gamma_n(z)=\beta_nz^{-d(n+1)}\left[1+O(z^{-1})\right], \tag{4.20}$$

which together with (4.19) yields (4.18). If the Euclidean algorithm stops in step $n$, then $\Gamma_n = 0$. ∎

*Proof of Theorem 7.* The function $T_n$ is clearly a linear fractional transformation of the form

$$T_n(w) = \frac{\hat{P}_n w + P_n}{\hat{Q}_n w + Q_n},$$

where $P_n$, $Q_n$, $\hat{P}_n$, and $\hat{Q}_n$ are polynomials in $z$ (Lemma 2). Now applying Lemma 2 to the composition $T_{n+1} = T_n \circ t_{n+1}$ for $n = 0, 1, 2, \ldots$, we obtain

$$\begin{bmatrix} \hat{P}_{n+1} & P_{n+1} \\ \hat{Q}_{n+1} & Q_{n+1} \end{bmatrix} = \begin{bmatrix} \hat{P}_n & P_n \\ \hat{Q}_n & Q_n \end{bmatrix} \begin{bmatrix} 0 & \beta_n \\ -1 & \alpha_{n+1} \end{bmatrix}, \qquad \begin{bmatrix} \hat{P}_0 & P_0 \\ \hat{Q}_0 & Q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But this holds if and only if $\hat{P}_n = -P_{n-1}$, $\hat{Q}_n = -Q_{n-1}$, and the recursions (4.13) are satisfied. Moreover, (4.13) yields

$$Q_{n+1}P_n - P_{n+1}Q_n = \beta_n(Q_n P_{n-1} - P_n Q_{n-1}),$$

from which we have $Q_{n+1}P_n - P_{n+1}Q_n = \beta_0\beta_1 \cdots \beta_n \neq 0$. Hence (4.14) follows. Finally, since $\Gamma - W_n = E_n/Q_n$, (4.16) follows from Lemma 3. If the algorithm terminates in step $n$, then $E_n = 0$, and consequently $\Gamma = W_n$. ∎

The connection to realization is then established by the following corollary.

COROLLARY 3. *Let $\gamma$ be an infinite sequence of real numbers, and let $\Gamma(z)$ be the formal power series* (4.1). *Let $\rho := \{\rho_1, \rho_2, \rho_3, \ldots\}$ be the sequence defined by* (3.8), *where $\{\beta_{n-1}, \alpha_n, d(n);\ n \in \mathfrak{N}^+\}$ is the output of the generalized Euclidean algorithm, $\nu(0) := 0$, $\nu(n) := \sum_{k=1}^{n} d(n)$ for $n \in \mathfrak{N}^+$, and, if the algorithm terminates in step $\hat{n}$, $d(\hat{n}+1) := \infty$. Then $\rho$ is the parameter sequence of $\gamma$. Moreover, for each $n \in \mathfrak{N}$, the rational function $W_n$ defined by* (4.15) *is the transfer function of $\Sigma_n$, where $\{\Sigma_n;\ n \in \mathfrak{N}\}$ is any complete family of minimal partial realizations of $\gamma$.*

*Proof.* Set $\hat{\gamma} := f^{-1}(\rho)$, where $f$ is the function of Theorem 6, and let $\{\hat{\Sigma}_n;\ n \in \mathfrak{N}\}$ be a complete family of minimal partial realizations of $\hat{\gamma}$. We want to show that $\hat{\gamma} = \gamma$. Let $n \in \mathfrak{N}$ be arbitrary. It follows from Theorem 4

that the transfer function $\hat{W}_n$ of $\hat{\Sigma}_n$ has the property

$$\hat{W}_n(z) = \sum_{i=1}^{\nu(n)+\nu(n+1)-1} \hat{\gamma}_i z^{-i} + O(z^{-\nu(n)-\nu(n+1)}), \tag{4.21}$$

where we set $\nu(\hat{n}+1) := \infty$ if the algorithm terminates in step $\hat{n}$. Now apply the Euclidean algorithm to the rational function $\hat{\Gamma} = \hat{W}_n$. Then it follows from the analysis (for the rational case) in the beginning of this section that $\hat{W}_n = T_n(0)$, and consequently $\hat{W}_n = W_n$ (Theorem 7). Hence, in view of (4.16) and (4.21), $\hat{\gamma}_i = \gamma_i$ for $i = 1,2,\ldots,\nu(n)+\nu(n+1)-1$. Therefore, since $n \in \mathfrak{N}$ is arbitrary, we must have $\hat{\gamma} = \gamma$. ∎

In our numerical example (3.3), we obtain $W_0 = 0$, $W_1(z) = 1/(z-1)$, $W_2(z) = z^2/(z^3 - z^2 - 1)$, and $W_3(z) = (z^3 - z^2 + 1)/(z^4 - 2z^3 + z^2)$. Since $\delta(\gamma) = 4 < \infty$, $\Gamma = W_3$. In view of (4.14), two consecutive minimal partial realizations have no poles (or zeros) in common.

REMARK 2. A more direct way of proving Corollary 3 is to first establish a link between the generalized Euclidean algorithm and the algorithm of Theorem 2 by first showing that

$$\gamma_{nk} = \pi_{\sigma(n)+k}, \qquad k = 1,2,\ldots,2d(n+1), \tag{4.22}$$

for each $n \in \mathfrak{N}$. Then (2.28) and (2.29) are obtained from (4.7) by identifying the coefficients of $z^{-i}$ in

$$\beta_{n-1} = \alpha_n \Gamma_{n-1} - \Gamma_n \Gamma_{n-1} \tag{4.23}$$

for $i = 0,1,2,\ldots,d(n)$. Note that, by (4.20), the last term of (4.23) does not contribute to these coefficients.

REMARK 3. From Corollary 3 we can now see that the occurrence of nongeneric jumps $[d(n) > 0]$ in the ranks of the leading principal submatrices of the Hankel matrix $H$ is actually a manifestation of the block structure of the corresponding Padé table [12, 24]. To see this, introduce the reciprocal polynomials

$$P_n^*(z) := z^{\nu(n)-1} P_n(z^{-1}), \tag{4.24a}$$

$$Q_n^*(z) := z^{\nu(n)} Q_n(z^{-1}), \tag{4.24b}$$

and set $\Gamma^*(z) := z^{-1}\Gamma(z^{-1})$. We then have the setting used in [12], $\Gamma^*$ being a Taylor series. Since $\deg P_n^* \leqslant \nu(n)-1$, $\deg Q_n^* \leqslant \nu(n)$, and $\gcd(P_n^*, Q_n^*) = 1 = Q_n^*(0)$, it follows from

$$E_n^*(z) := \Gamma^*(z)Q_n^*(z) - P_n^*(z) = O(z^{2\nu(n)})$$

that $W_n^* := P_n^*/Q_n^*$ is (the normalized reduced representation of) the *Padé fraction* $r_{\nu(n)-1,\nu(n)}$ for the formal power series $\Gamma^*(z)$. Hence we obtain the sequence $\{W_n;\ n \in \mathfrak{N}^+\}$ of transfer functions by moving along the superdiagonal in the Padé table (which is the matrix $\{r_{ij}\}$ of rational functions), at each distinct entry making the transformation $W_n(z) := z^{-1}r_{\nu(n)-1,\nu(n)}(z^{-1})$. In doing so, the jump $d(n+1)$ can be read off as the number of times the fraction $r_{\nu(n)-1,\nu(n)}$ is repeated. If $\Gamma$ is rational, the last $d(n+1)$ is infinite, corresponding to an infinite block in the Padé table. (See Table 3 on p. 12 in [12], where the numerical example (3.3) is illustrated.) The connection between the Euclidean algorithm and the Padé table was established in [24].

From Theorem 7 we can now establish, independently of the results in Section 2, the generalized orthogonality property (2.24).

COROLLARY 4. *Let $\gamma$ be an infinite sequence of real numbers, let $\Gamma(z)$ be the corresponding formal power series, and let the polynomials $\{Q_n;\ n \in \mathfrak{N}\}$ be as defined in Theorem 7. Then*

$$\gamma^*(z^{i-1}Q_n) = \begin{cases} 0 & \text{for} \quad 1 \leqslant i < d(n+1), \\ \beta_0\beta_1 \cdots \beta_n \neq 0 & \text{for} \quad i = d(n+1). \end{cases} \tag{4.25}$$

*Proof.* A routine calculation shows that (4.17) can be written

$$E_n(z) = \sum_{i=0}^{\infty} \gamma^*(z^{i-1}Q_n)z^{-i}. \tag{4.26}$$

Then (4.25) follows from (4.18). ∎

It should now be clear that we could have taken the generalized Euclidean algorithm as a starting point for our study. In fact, from (4.25) we immediately obtain (2.25), which is equivalent to the matrix factorizations $R_n' H_n R_n = D_n$. The other matrix results of Theorem 2 and 3 are actually derived via the polynomial formulation and could be adopted with no change.

COROLLARY 5 (Magnus's algorithm). *Let* $\{\beta_{n-1}, \alpha_n;\ n \in \mathfrak{N}^+\}$ *and* $\{Q_n;\ n \in \mathfrak{N}\}$ *be as defined in Theorem 7. Then, for each n for which it is defined,* $\{\beta_n, \alpha_{n+1}(z)\}$ *is the unique solution of*

$$\begin{bmatrix} & & & \varepsilon_{n,d(n+1)} \\ & & \varepsilon_{n,d(n+1)} & \varepsilon_{n,d(n+1)+1} \\ & \iddots & & \vdots \\ \varepsilon_{n,d(n+1)} & \cdots & \varepsilon_{n,2d(n+1)-2} & \varepsilon_{n,2d(n+1)-1} \end{bmatrix} \begin{bmatrix} \alpha_{n+1,d(n+1)} \\ \alpha_{n+1,d(n+1)-1} \\ \vdots \\ \alpha_{n+1,1} \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_{n,d(n+1)+1} \\ \vdots \\ \varepsilon_{n,2d(n+1)} \end{bmatrix} - \beta_n \begin{bmatrix} \varepsilon_{n-1,d(n)+1} \\ \vdots \\ \varepsilon_{n-1,d(n)+d(n+1)} \end{bmatrix} \tag{4.27a}$$

*and*

$$\beta_n = \varepsilon_{n,d(n+1)} / \varepsilon_{n-1,d(n)}, \tag{4.27b}$$

*where*

$$\varepsilon_{nk} = \gamma^*\left(z^{\nu(n)+k-1} Q_n\right). \tag{4.28}$$

*Proof.* Multiply the recursion (4.13b) by $z^{\nu(n)+k-1}$ and apply the functional $\gamma^*$ to obtain

$$\gamma^*\left(z^{\nu(n)+k-1} Q_{n+1}\right) = \varepsilon_{n,d(n+1)+k} - \sum_{i=0}^{d(n+1)-1} \alpha_{n+1,d(n+1)-i} \varepsilon_{n,k+i} - \beta_n \varepsilon_{n-1,d(n)+k}.$$

But, by (4.25), the left member is zero for $0 \leqslant k \leqslant d(n+1)$. Then (4.27a) is obtained by taking $k = 1, 2, \ldots, d(n+1)$ and (4.27b) by taking $k = 0$. The triangular structure of (4.27a) is due to (4.25). ∎

This algorithm belongs to a class of algorithms usually attributed to Berlekamp [1] and Massey [26]. This version, however, had been presented in

1962 by Magnus [24, 25]. For the special case when the polynomials $\{\alpha_n;$ $n \in \mathfrak{N}^+\}$ are linear, algorithms of this type had been introduced by Chebyshev [6] and Stieltjes [34]. See Gautschi [10], who gives a nice description of these algorithms, and Wall [36], who provides some different interpretations.

Magnus's algorithm corresponds to the factorizations

$$R_n' H_n = M_n, \tag{4.29}$$

where of course $M_n := D_n R_n^{-1}$. (In this context, cf. Kung [22], who derives the algorithm via a similar matrix factorization.) It is easy to see that element $(\nu(k)+i, j)$ of $M_n$ equals $\gamma^*(z^{i+j-2}Q_k)$ for $i = 1,2,\ldots,d(k+1)$, and therefore, for $k = 0,1,2,\ldots,n-1$, rows $\nu(k)+1$ through $\nu(k+1)$ of $M_n$ form a Hankel matrix for the sequence consisting of $\nu(k)$ zeros followed by $\varepsilon_{k1}, \varepsilon_{k2}, \varepsilon_{k3}, \ldots$. Moreover, $M_n$ is the observability matrix

$$M_n = \begin{bmatrix} C_n \\ C_n A_n \\ C_n A_n^2 \\ \vdots \\ C_n A_n^{\nu(n)-1} \end{bmatrix} \tag{4.30}$$

of the canonical realization $\Sigma_n$ of Theorem 5. This follows from (2.40).

## 5. AN ALGORITHM FOR COMPUTING THE PARAMETER SEQUENCE

We have presented two algorithms above for determining the parameter sequence $\rho$ (Theorem 2 and Corollary 5), each corresponding to a different factorization of the Hankel matrix. However, both of them produce the $\rho$-parameters in whole sections, the output of step $n$ being $\{\rho_{\sigma(n-1)+1}, \cdots, \rho_{\sigma(n)}\}$. This makes them less suitable for realization of partial sequences. What we want is an algorithm which, given the input $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$, produces the output $\{\rho_1, \rho_2, \ldots, \rho_N\}$ for any natural number $N$. In [39] Zierler presented a modified version of the Berlekamp algorithm [1], which, translated to our present setting, updates the $Q_n$-polynomials via a sequence of monic polynomials of degrees $1,2,3,\ldots$. However, since Zierler was not interested in the parameter sequence, we shall have to modify his procedure for our purposes.

For each $n \in \mathfrak{N}^+$, define

$$Q_n^{(i)}(z) := \left[z^{d(n)} - \alpha_{n1} z^{d(n)-1} - \cdots - \alpha_{ni} z^{d(n)-i}\right] Q_{n-1}(z) + \beta_{n-1} Q_{n-2}(z) \tag{5.1}$$

for $i = 0, 1, 2, \ldots, d(n)$. Then $Q_n^{(d(n))} = Q_n$.

LEMMA 4. *Let* $\{\lambda_{n-1};\ n \in \mathfrak{N}^+\}$ *be defined by* (2.24). *Then, for each* $n \in \mathfrak{N}$,

$$\alpha_{ni} = \frac{\gamma^*\left(z^{\nu(n-1)+i-1} Q_n^{(i-1)}\right)}{\lambda_{n-1}} \tag{5.2}$$

*for* $i = 1, 2, \ldots, d(n)$.

*Proof.* Apply $\gamma^*$ to

$$z^{\nu(n-1)+i-1}\left[Q_n^{(i-1)}(z) - Q_n(z)\right]$$
$$= \alpha_{ni} z^{\nu(n)-1} Q_{n-1}(z) + \alpha_{n,i+1} z^{\nu(n)-2} Q_{n-1}(z)$$
$$+ \cdots + \alpha_{n,d(n)} z^{\nu(n-1)+i-1} Q_{n-1}(z)$$

and use the condition (2.24) to obtain (5.2). ∎

To compute (5.2) we define monic polynomials $\{\phi_0, \phi_1, \phi_2, \ldots\}$ with $\deg \phi_k = k$ in the following way. For each $n \in \mathfrak{N}$, set

$$\phi_{2\nu(n)+i}(z) := z^{\nu(n)+i} Q_n(z) \qquad \text{for} \quad i = 0, 1, \ldots, d(n+1) - 1, \tag{5.3a}$$

$$\phi_{2\nu(n)+d(n+1)+i}(z) := z^{\nu(n)+i} Q_{n+1}^{(i)}(z) \qquad \text{for} \quad i = 0, 1, \ldots, d(n+1). \tag{5.3b}$$

[Note that there is overlap between (5.3a) and (5.3b), so that $\phi_{2\nu(n+1)}$ is given by both (5.3a) and (5.3b).] We shall also need a sequence $\{\theta_1, \theta_2, \theta_3, \ldots\}$ of real numbers which keeps track of the factors $1/\lambda_{n-1}$ in (5.2) and (2.29).

Theorem 4 suggests that, for each $n \in \mathfrak{N}$, we set

$$\theta_{\nu(n-1)+\nu(n)+i} := \frac{1}{\lambda_{n-1}} \qquad \text{for} \quad i = 0,1,\ldots,d(n)+d(n+1)-1, \quad (5.4)$$

where $\nu(-1) = d(0) = 0$ and $\lambda_{-1} = 1$. Hence $\theta_k$ is constant in each of the intervals (3.2). Now recall that, for each finite or infinite sequence $\gamma$, the parameter sequence $\rho$ is a sequence of the same length, the appropriate elements of which are defined by (3.8).

THEOREM 8. *Let $\gamma$ be an (infinite or finite) sequence with degree indices $\{\nu(n);\ n \in \mathfrak{N}\}$. Then its parameter sequence $\rho$ is given by*

$$\rho_{k+1} = \gamma^*(\phi_k)\theta_k, \quad (5.5)$$

*where $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is a sequence of monic polynomials generated by the recursion*

$$\phi_{k+1}(z) = z\phi_k(z) - \rho_{k+1}\psi_k(z), \qquad \phi_0(z) = 1 \quad (5.6)$$

*with $\psi_k$ defined as*

$$\psi_k(z) = \begin{cases} 0 & \text{for} \quad k = 0, \\ z\phi_{k-1}(z) & \text{for} \quad k = \nu(n)+\nu(n+1), \quad n \in \mathfrak{N}, \\ \psi_{k-1}(z) & \text{otherwise}, \end{cases} \quad (5.7)$$

*and where $\{\theta_0, \theta_1, \theta_2, \ldots\}$ is the sequence of real numbers*

$$\theta_k = \begin{cases} 1 & \text{for} \quad k = 0, \\ \theta_{k-1}/\rho_k & \text{for} \quad k = \nu(n)+\nu(n+1), \quad n \in \mathfrak{N}, \\ \theta_{k-1} & \text{otherwise}. \end{cases} \quad (5.8)$$

Moreover, the polynomials $\{Q_n;\ n \in \mathfrak{N}\}$ are given by

$$Q_n(z) = z^{-\nu(n)}\phi_{2\nu(n)}(z). \quad (5.9)$$

*Proof.* Let $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be given by (5.3) and $\{\theta_0, \theta_1, \theta_2, \ldots\}$ by (5.4). Then (5.5) is a consequence of (2.24), (2.29), and (5.2), and (5.9) follows from (5.3a). The sequence $\{\theta_k\}$ is constant except for a jump at $k = \nu(n) + \nu(n+1)$ for each $n \in \mathfrak{N}$. Then $\theta_k = 1/\lambda_n$ and $\theta_{k-1} = 1/\lambda_{n-1}$. But, by (2.29), $\rho_k := \beta_n = \lambda_n/\lambda_{n-1} = \theta_{k-1}/\theta_k$. This establishes (5.8). Hence it only remains to show that $\{\phi_k\}$ satisfies the recursion (5.6) where $\{\psi_k\}$ is given by (5.7). To this end, let $n \in \mathfrak{N}$ be arbitrary, and, for $i = 0, 1, 2, \ldots, d(n+1) - 1$, set $k = 2\nu(n) + i$. Then (5.6) holds trivially for $i = 0, 1, \ldots, d(n+1) - 2$, since $\rho_{k+1} = 0$ then. For $i = d(n+1) - 1$, we have

$$\begin{aligned}\phi_{k+1}(z) &= z^{\nu(n+1)}Q_n(z) - \beta_n z^{\nu(n)}Q_{n-1}(z) \\ &= z\phi_k(z) - \beta_n z\phi_{\nu(n-1)+\nu(n)-1}(z),\end{aligned}$$

which is precisely (5.6). Next, for $j = 0, 1, 2, \ldots, d(n+1) - 1$, set $k = \nu(n) + \nu(n+1) + j$. Then, from (5.3b) we have

$$\begin{aligned}\phi_{k+1}(z) - z\phi_k(z) &= z^{\nu(n)+j+1}\left[Q_{n+1}^{(j+1)}(z) - Q_{n+1}^{(j)}(z)\right] \\ &= \alpha_{n+1,\,j+1}z^{\nu(n+1)}Q_n(z) \\ &= \rho_{k+1}z\phi_{\nu(n)+\nu(n+1)-1}(z),\end{aligned}$$

which proves (5.6) for $k = 2\nu(n) + d(n+1), \ldots, 2\nu(n+1) - 1$. ∎

Hence we obtain the following algorithm which, given a sequence $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$, produces its parameter sequence $\rho = \{\rho_1, \rho_2, \ldots, \rho_N\}$.

$\phi_0 = 1$, $\psi_0 = 0$, $\theta_0 = 1$, $N_0 = 0$
for $k = 1, 2, 3, \ldots$ until $\gamma_k \notin \gamma$
  $\rho_k = \gamma^*(\phi_{k-1})\theta_{k-1}$
  $N_k = N_{k-1} + 1$
  if $N_k > 0$ and $\rho_k \neq 0$ then
    $N_k \leftarrow -N_k$
    $\theta_k = \theta_{k-1}/\rho_k$
    $\psi_k(z) = z\phi_{k-1}(z)$
  else
    $\theta_k = \theta_{k-1}$
    $\psi_k(z) = \psi_{k-1}(z)$
  $\phi_k(z) = z\phi_{k-1}(z) - \rho_k\psi_{k-1}(z)$

A few comments are in order. The integer $N_k$ is a marker which is zero whenever a complete section $\{\rho_1, \rho_2, \ldots, \rho_{\sigma(n)}\}$ has been produced. The "if"

statement is true whenever $k = \nu(n) + \nu(n+1)$ for some $n \in \mathcal{N}$. Then $N_k = d(n+1)$. Hence the sequence $\{N_k\}$ can be used to obtain the degree indices as an output. However, strictly speaking this is unnecessary, because the degree indices can be read directly from the parameter sequence $\rho$ via its section indices $\{\sigma(n);\ n \in \mathcal{N}\}$ as explained in Section 3; then $\nu(n) := \frac{1}{2}\sigma(n)$. If, upon termination of the algorithm, $N_k < 0$, this means that the last partial realization, i.e. the (complete) realization of $\gamma$, is not unique. There are precisely $|N_k|$ arbitrary parameters, namely the $\rho$-parameters needed to fill the section. If $N_k \geqslant 0$, then, no jumps in the rank of $H_{\nu(n)+1,\nu(n)+j}$ have occurred, and $N_k$ extra $\gamma$-parameters have been fitted beyond those needed to determine the last realization.

The determination of the degree indices of a sequence $\gamma$ is based on deciding whether some $\rho$-parameters are nonzero, i.e. whether the "if" statement in the algorithm should be implemented. Consequently a small variation in the data $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ may produce quite different degree indices. Since the sequence $\gamma$ may be corrupted with measurement errors, this is an important consideration. Unlike most situations in which the Lanczos algorithm is applied, we want the jumps to be large, because this will postpone the need for a partial realization of higher dimension. Therefore, if we are allowed to adjust the elements of the sequence $\gamma$ to achieve this, we would probably do so.

Let us associate with $\gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ a sequence $\varepsilon := \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$ of error limits such that replacing $\gamma_k$ by any element in the interval $[\gamma_k - \varepsilon_k, \gamma_k + \varepsilon_k]$ is within the limits of accuracy of $\gamma_k$. The sequence $\varepsilon$ will depend on the manner in which $\gamma$ has been obtained, and it is of course possible to consider unsymmetric intervals instead. Now, since $\phi_k$ is monic,

$$\eta_k := \gamma^*(\phi_{k-1}) = \gamma_k + \text{linear combination of } \{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}.$$

Therefore, if $|\eta_k| \leqslant \varepsilon_k$, we may exchange $\gamma_k$ for $\gamma_k - \eta_k$. Then $\gamma^*(\phi_{k-1}) = 0$, and hence so is $\rho_k$. This change of $\gamma_k$ will not complicate any other calculations, since $\gamma_k$ enters the algorithm for the first time in determining $\rho_k$. Consequently we may want to alter the algorithm as follows. Replace the line

$$\rho_k = \gamma^*(\phi_{k-1})\theta_{k-1}$$

by

$$\begin{aligned}
&\eta_k = \gamma^*(\phi_{k-1}) \\
&\text{if } N_k \geqslant 0 \text{ and } |\eta_k| \leqslant \varepsilon_k \text{ then} \\
&\quad \gamma_k \leftarrow \gamma_k - \eta_k \\
&\quad \eta_k \leftarrow 0 \\
&\rho_k = \eta_k \theta_{k-1}
\end{aligned}$$

Moreover, replace $\theta_k = \theta_{k-1}/\rho_k$ by $\theta_k = 1/\eta_k$. Note that we never need to adjust the $\gamma$-parameters when $N_k < 0$.

## REFERENCES


1 E. R. Berlekamp, *Algebraic Coding Theory*, McGraw-Hill, New York, 1968.
2 G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, 3rd ed., McMillan, 1965.
3 R. W. Brockett, The geometry of the partial realization problem, in *Proceedings of the 1978 IEEE Conference on Decision and Control*, pp. 1048–1052.
4 R. W. Brockett, *Finite Dimensional Linear Systems*, Wiley, 1970.
5 C. I. Byrnes and A. Lindquist, The stability and instability of partial realizations, *Systems and Control Letters* 2:99–105 (1982).
6 P. L. Chebyshev, Sur l'interpolation par la méthode de moindres carrés, *Mém. Acad. Imp. Sci. St. Pétersbourg* 1:1–24 (1859).
7 B. W. Dickinson, M. Morf, and T. Kailath, A minimal realization algorithm for matrix sequences, *IEEE Trans. Automat. Contr.*, AC-19:31–38 (1974).
8 F. G. Frobenius, Über das Trägheitsgesetz der quadratischen Formen, *J. Reine Angew. Math.* 114:187–230 (1895).
9 F. R. Gantmacher, *The Theory of Matrices*, Chelsea, 1959.
10 W. Gautschi, On generating orthogonal polynomials, *SIAM J. Sci. Stat. Comput.*, to appear.
11 E. G. Gilbert, Controllability and observability in multivariable control systems, *SIAM J. Control* 1:128–151 (1963).
12 W. B. Gragg, The Padé table and its relation to certain algorithms of numerical analysis, *SIAM Rev.* 14:1–62 (1972).
13 W. B. Gragg, Matrix interpretations and applications of the continued fraction algorithm, *Rocky Mountain J. Math.* 4:213–225 (1974).
14 B. L. Ho and R. E. Kalman, Effective construction of linear state-variable models from input/output functions, *Regelungstechnik* 14:545–548 (1966).
15 A. S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw-Hill, New York, 1970.
16 L. S. deJong, Numerical aspects of recursive realization algorithms, *SIAM J. Control Optimiz.* 16:646–659 (1978).
17 R. E. Kalman, Mathematical description of linear dynamical systems, *SIAM J. Control* 1:152–192 (1963).
18 R. E. Kalman, Irreducible realizations and the degree of a rational matrix, *SIAM J. Control* 13:520–544 (1965).
19 R. E. Kalman, On minimal partial realizations of a linear input/output map, in *Aspects of Network and System Theory* (R. E. Kalman and N. De Claris, Eds.), Holt, Rinehart and Winston, 1971, pp. 385–408.
20 R. E. Kalman, On partial realizations, transfer functions and canonical forms, *Acta Polytech. Scand.* MA31:9–32 (1979).
21 R. E. Kalman, P. Falb, and M. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969.

22 S.-Y. Kung, Multivariable and multidimensional systems: Analysis and design, Ph.D. thesis, Dept. of Electrical Engineering, Stanford Univ., June 1977.
23 C. Lanczos, An iteration method for the solution of the eigenvalue problem of linear differential and integral operators, *J. Res. Nat. Bur. Standards* 45:255–282 (1950).
24 A. Magnus, Certain continued fractions associated with the Padé table, *Math. Z.* 78:361–374 (1962).
25 A. Magnus, Expansion of power series into *P*-fractions, *Math. Z.* 80:209–216 (1962).
26 J. L. Massey, Shift register synthesis and BCH decoding, *IEEE Trans. Inform. Theory* IT-15:122–127 (1969).
27 B. McMillan, Introduction to formal realizability theory, *Bell System Tech. J.* 31:217–279, 591–600 (1952).
28 B. N. Parlett and D. Taylor, A look ahead Lanczos algorithm for unsymmetric matrices, *Math. Comp.*, to appear.
29 J. Rissanen, Recursive identification of linear systems, *SIAM J. Control* 9:420–430 (1971).
30 J. Rissanen, Realizations of matrix sequences, IBM Research Lab Report RJ 1032, 1972.
31 L. Silverman, Representation and realization of time-variable linear systems, Tech. Rep. 94, Dept. of Electrical Engineering, Columbia Univ., New York, 1966.
32 L. Silverman, Realization of linear dynamical systems, *IEEE Trans. Automat. Control* AC-16:554–567 (1971).
33 T. J. Stieltjes, Quelques recherches sur la théorie des quadratures dites méchaniques, *Ann. Sci. École Norm. Sup.* 1:406–426 (1884).
34 T. J. Stieltjes, Sur la réduction en fraction continue d'une série précédent suivant les pouissances descendants d'une variable, *Ann. Fac. Sci. Toulouse* 3:1–17 (1889).
35 A. Talbot, The number of zeros of a polynomial in a half plane, *Proc. Cambridge Philos. Soc.* 52:132–147 (1960).
36 H. S. Wall, Note on the expansion of a power series into a continued fraction, *Bull. Amer. Math. Soc.* 51:97–105 (1945).
37 D. C. Youla, The synthesis of linear dynamical systems from prescribed weighting patterns, *SIAM J. Appl. Math.* 14:527–549 (1966).
38 D. C. Youla and P. Tissi, *n*-port synthesis via reactive extraction Part I, in *IEEE International Convention Record*, Part 7, Vol. 14, 1966.
39 N. Zierler, Linear recurring sequences and error correcting codes, in *Error Correcting Codes* (H. B. Mann, Ed.), Wiley, New York, 1968, pp. 47–59.

*Received 24 January 1982; revised 2 July 1982*