On Optimal Stochastic Control with Smoothed Information

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ABSTRACT

This paper presents a generalization of the Separation Theorem of stochastic control. The generalization consists in assuming observations ahead of running time. We consider the following problem of optimal pursuit: Given noisy incomplete observations of a linear stochastic system, control another linear stochastic system so that a quadratic functional of the difference (in some generalized sense) between the two processes is minimized. Provided that we have access to observations only up to the time of control, the Separation Theorem states that the solution to this problem is given by a linear combination of the Kalman filtering estimates. Now suppose instead that we have observations ahead of running time. (Consider for instance an airplane with terrain following radar.) Then the basic result of this paper is that a weighted integral of the minimum variance smoothing estimate should be included in the optimal control. This result is obtained by use of Fejér kernels and enlargement of the state space. Finally we provide proof of optimality.

1. INTRODUCTION

The problem of controlling a (continuous time) linear stochastic system, given noisy incomplete information of the system's state, for the purpose of minimizing a quadratic functional has been solved by Potter [8] and others in the case of linear feedback (the Separation Theorem). Wonham [9] has given a rigorous proof of the Separation Theorem,† and very recently he has shown [10] that it is valid in the more general case with a nonquadratic cost functional and when the feedback is not a priori assumed to be linear. However in this paper we shall consider quadratic criteria only.

The Separation Theorem states that the optimal control of the process can be determined by solving one deterministic control problem (the one we should

† Another rigorous proof has been given by Zachrisson [14].

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have if the noise were not present and the state were exactly known) and one problem of filtering. The second one, of course, is the well-known minimum-variance filtering problem of Kalman and Bucy [4]. (This problem has also been solved by Zachrisson [12] in a very simple manner.)

Since lately the problem of smoothing (interpolation) has attracted a certain interest and has been solved by Bryson and Frazier [2], Meditch [7], Zachrisson [11], and others† (these papers differ in method rather than result), the question arises whether the following stochastic problem of control can be solved, using the smoothing estimate:

Control the linear stochastic system described by the differential equations (x is an m_1-vector, y an m_2-vector and u an m_3-control vector):

$$\begin{cases} \dfrac{dx}{dt} = A(t)\,x(t) + B(t)\,u(t) + v_1(t) & \text{(a)} \\[2ex] \dfrac{dy}{dt} = C(t)\,y(t) + v_2(t) & \text{(b)} \end{cases} \tag{1.1}$$

(where $v_1(t)$ and $v_2(t)$ are Gaussian white noise vectors with the appropriate dimensions) when the object is to minimize:

$$E\left\{\int_0^T \{[x(t) - D_1(t)\,y(t-h)]^*\, Q_1(t)[x(t) - D_1(t)\,y(t-h)] \right.$$

$$\left. + u(t)^*\, Q_2(t)\,u(t)\}\,dt + [x(T) - D_3\,y(T-h)]^*\, Q_3[x(T) - D_3\,y(T-h)]\right\} \tag{1.2}$$

given the distributions of $x(0)$ and $y(t_0)$ and also the observations ($z(t)$ is an m_4-vector and $w(t)$ is a Gaussian white noise vector):

$$z(t) = H(t)\,y(t) + w(t) \qquad (0 \leqslant t \leqslant T) \tag{1.3}$$

where $-T < t_0 < -h < 0$, $T > 2h$ and * denotes transposition. Thus the problem is to find a control $u(t)$ that is a functional on $\{z(s); s \in [0,t]\}$ for every t on $[0,T]$, and this control function should minimize (1.2).

The reason for choosing $t_0 \in (-T,-h)$ as starting time for $y(t)$ will be evident in Section 2 (Eq. 2.1). Since in Section 7 we will find that there are different expressions for the control law on $[0,h), [h,T-h]$ and $(T-h,T]$ respectively, we introduce the restriction $T > 2h$ (which incidentally should be met in most applications).

The matrices $A(t)$, $B(t)$, $C(t)$, $D_1(t)$, $H(t)$, $Q_1(t)$, $Q_2(t)$ and $Q_2(t)^{-1}$ are Lebesgue measurable and bounded on the interval $[t_0,T]$. (All matrices should

† Very recently Kailath [13] has given a unified (and very elegant) approach to filtering and interpolation (which incidentally also gives a solution to nonlinear estimation problems).

have appropriate dimensions so that the expressions above are defined.) Furthermore we assume that $Q_1(t)$ and Q_3 are positive semidefinite, $Q_2(t)$ is positive definite, and that all three are symmetric.

$x(0)$, $y(t_0)$ and the Gaussian white noise vectors have zero mean† and covariance matrices:

$$E\{v_1(t)\,v_1(\tau)^*\} = Q_x(t)\,\delta(t-\tau), \qquad E\{v_2(t)\,v_2(\tau)^*\} = Q_y(t)\,\delta(t-\tau),$$
$$E\{v_1(t)\,v_2(\tau)^*\} = Q_{xy}(t)\,\delta(t-\tau), \qquad E\{w(t)\,w(\tau)^*\} = Q_z(t)\,\delta(t-\tau),$$
$$E\{x(0)\,x(0)^*\} = S_x, \qquad E\{y(t_0)\,y(t_0)^*\} = S_y, \qquad E\{x(0)\,y(t_0)^*\} = S_{xy}.$$

$w(t)$ is independent of $v_1(t)$ and $v_2(t)$; $v_1(t)$, $v_2(t)$ and $w(t)$ are independent of $x(0)$ and $y(t_0)$. $Q_x(t)$, $Q_y(t)$, $Q_{xy}(t)$, $Q_z(t)$ and $Q_z(t)^{-1}$ are matrices with the appropriate dimensions that are measurable and bounded on $[t_0, T]$.

Now the white noise vectors (these are stochastic vector processes in a *generalized* sense) can be considered to be the *formal* derivatives of standard Wiener processes (of course the Wiener process is nowhere differentiable in any usual sense). To give the systems (1.1) and (1.3) a more respectable appearance we could use the Ito form, where in (1.3) we observe the integral of z rather than z itself. However, we will find it convenient not to do this, although we will mean the same thing. Thus stochastic integrals of the type $\int f(s)v(s)\,ds$, where $v(s)$ is white noise and $f(s)$ is a function belonging to L_2, are defined in quadratic mean ([3] p. 426, [9] p. 95). Furthermore, we have an underlying probability space $(\Omega, \mathscr{B}, P)$, where Ω is the sample space (for convenience the sample variable $\omega \in \Omega$ will be suppressed), $\mathscr{B}$ is a σ-algebra taking care of $x(0)$, $y(t_0)$ and the Wiener processes (these should be separable), and P is the (complete) probability measure.

Of course we may regard the more general problem of controlling (1.1) given not only (1.3) but also noisy observations of $x(s)$ $(s \in [0,t])$. Since this part of the problem is a filtering problem, our equations would become somewhat more complicated without giving any further new results, and therefore the problem at stake might get obscured. For this reason we do not choose to extend our problem in this way.

If we regard the special case when $v_1(t) = 0$ and $x(0)$ is deterministic (we obtain the optimal control for this case if we put $Q_x = Q_{xy} = S_x = S_{xy} = 0$ in the equations derived in the sequel), $x(t)$ is still a stochastic process, but the stochastic element is introduced through $u(\tau)$ $\tau \in [0,t]$, which is a function of the observations received so far. Therefore, given the observations, at time t we will have perfect knowledge of $x(t)$.

† The restriction $Ex(0) = Ey(t_0) = 0$ is not crucial. When treating the general case, just replace $\hat{p}(0|0) = 0$ by $\hat{p}(0|0) = Ep(0)$ in (5.1) and change the following calculations accordingly. S_x, S_y, and S_{xy} now signify convariance matrices.

Another possible modification of problem formulation (1.1)–(1.3) is that $x(t)$ is perfectly known (although $v_1(t) \neq 0$ and $x(0)$ is stochastic). This problem is not a trivial modification of the one posed, and it will not be treated here.

It should be noted that in our problem $x(t)$ and $y(t)$ are correlated (for $v_1(t)$ and $v_2(t)$, and $x(0)$ and $y(t_0)$ are), although in many applications this is not so. (In these cases $Q_{xy} = S_{xy} = 0$.)

Problem formulation (1.1)–(1.3), modified in one of the ways mentioned above, might adequately describe the following situation: We want to fly an airplane at a constant vertical distance (ρ) above an undulating ground (that is along the dotted line in Fig. 1.1). The x-process (which can be controlled) describes the motion of the airplane and the y-process the ground (or rather the dotted line). The airplane has a terrain following radar making noisy

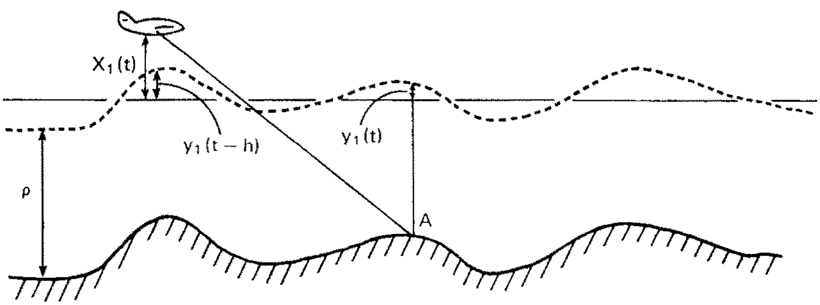

Fig. 1.1.

incomplete measurements of $y(t)$, or rather $y(t) + \rho$ which of course amounts to the same calculations. As indicated in the figure, at time t we will have measurements up to point A while actually $y(t-h)$ is of interest. It is then natural to minimize a quadratic mean of the difference $x_1(t) - y_1(t-h)$ as for example (1.2); (x_1 and y_1 are the first components of vectors x and y and their meaning is given in the figure).

2. TRANSFORMATION OF THE PROBLEM THROUGH EXTENSION OF THE STATE SPACE

Since $y(t)$ appears in (1.2) with delayed time argument, we cannot apply the Separation Theorem to our problem. For this reason we will transform the problem by using Dirac's δ-function:

$$y(t-h) = \int_{t_0}^{t} \delta(t-h-\tau)\, y(\tau)\, d\tau \qquad (-T < t_0 < -h < 0 \leqslant t \leqslant T).$$

As an approximation of $y(t-h)$ we will use $y^{(n)}(t-h,t)$ where:

$$y^{(n)}(\tau,t)=\int_{t0}^{t}\delta_n(\tau-s)y(s)\,ds \tag{2.1}$$

and where $\delta_n(t)$ is the Fejér kernel:

$$\delta_n(t)=\frac{1}{4T}\frac{1}{n+1}\frac{\sin^2\left(\frac{n+1}{2}\frac{\pi}{2T}t\right)}{\sin^2\left(\frac{1}{2}\frac{\pi}{2T}t\right)};\qquad -2T\leqslant t\leqslant 2T. \tag{2.2}$$

(The reason for choosing the period in this way is that

$$y^{(n)}(t-h,t)=\int_{-h}^{t-h-t_0}\delta_n(s)y(t-h-s)\,ds\quad\text{and}\quad t-h-t_0\leqslant 2T-h.)$$

It is a well known property of the Fejér kernel that (for example reference [1]):

$$\delta_n(t)=\sum_{k=-n}^{n}\gamma_k e^{ki\lambda t} \tag{2.3}$$

where:

$$\begin{cases}\gamma_k=\dfrac{1}{4T}\left(1-\dfrac{|k|}{n+1}\right)\\[2ex] \lambda=\dfrac{\pi}{2T}.\end{cases}$$

Let $q_k(t)$ be an m_2-vector function and $\alpha_k(t)$ a scalar function given by:

$$q_k(t)=\int_{t0}^{t}e^{-ki\lambda\tau}y(\tau)\,d\tau \tag{2.4}$$

$$\alpha_k(t)=\gamma_k e^{ki\lambda(t-h)} \tag{2.5}$$

$$\therefore\quad y^{(n)}(t-h,t)=\int_{t0}^{t}\sum_{-n}^{n}\gamma_k e^{ki\lambda(t-h-\tau)}y(\tau)\,d\tau=\sum_{-n}^{n}\alpha_k(t)q_k(t). \tag{2.6}$$

Then for $q_k(t)$ we have the following differential equation:

$$\begin{cases}\dfrac{dq_k(t)}{dt}=e^{-ki\lambda t}y(t)\\[2ex] q_k(0)=\displaystyle\int_{t_0}^{0}e^{-ki\lambda\tau}y(\tau)\,d\tau.\end{cases} \tag{2.7}$$

Let $e^{-ki\lambda t}$ be denoted by $\beta_k(t)$, p be an $m_1+2(n+1)m_2$-vector and F an $(m_1+2(n+1)m_2)\times(m_1+2(n+1)m_2)$ matrix given by:

$$p=\begin{pmatrix} x \\ y \\ q_{-n} \\ \vdots \\ q_n \end{pmatrix} \qquad F=\begin{pmatrix} A & 0 & 0 & \dots & 0 \\ 0 & C & 0 & \dots & 0 \\ 0 & \beta_{-n}I & 0 & \dots & 0 \\ . & . & . & . & . \\ 0 & \beta_n I & 0 & \dots & 0 \end{pmatrix}$$

$$\beta_k(t)=e^{-ki\lambda t}.$$

Furthermore:

$$B'(t)=\begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \qquad H'(t)=(0, H, 0\dots 0); \qquad v(t)=\begin{pmatrix} v_1(t) \\ v_2(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now, if we replace $y(t-h)$ with $y^{(n)}(t-h,t)$ in the problem of Section 1, we will have the following problem:

Control the system of differential equations:

$$\frac{dp}{dt}=F(t)p(t)+B'(t)u(t)+v(t) \tag{2.8}$$

given the observations:

$$z(t)=H'(t)p(t)+w(t) \qquad (0\leqslant t\leqslant T) \tag{2.9}$$

when the object is to minimize:

$$E\left\{\int_0^T [(x-\sum \alpha_k D_1 q_k)^* Q_1(x-\sum \alpha_k D_1 q_k)+u^* Q_2 u]\,d\tau \right.$$

$$\left. +[x(T)-\sum \alpha_k(T) D_3 q_k(T)]^* Q_3[x(T)-\sum \alpha_k(T) D_3 q_k(T)]\right\}$$

$$=E\left\{\int_0^T (p^* Q_1' p+u^* Q_2 u)\,d\tau+p(T)^* Q_3' p(T)\right\} \tag{2.10}$$

(where now * denotes Hermitian transposition).

As it is quite clear that the Separation Theorem is valid for a complex state vector of this type, and that the equations for the deterministic problem as well as that of the filtering problem are unaltered (this is clear from the proofs by Zachrisson [12]), we now have a problem of standard form. It remains to be shown that the solution of (2.8)–(2.10) is equivalent to that of (1.2)–(1.3) as $n \to \infty$. This will be done in Section 8.

3. QUESTIONS CONCERNING THE CONVERGENCE OF THE APPROXIMATE SYSTEM

We will need the following lemma, which is a slight modification of Fejér's Theorem [1]:

LEMMA 3.1. *If the function $f(t)$ is measurable and bounded on $[a,b]$, where a and b are real numbers such that $0 < b - a \leqslant 2T$, and the function $\phi(t)$ is defined by:*

$$\phi(t) = \begin{cases} f(t) & \text{for} \quad a < t < b \\ \frac{1}{2}f(t) & \text{for} \quad t = a \quad \text{and} \quad t = b \\ 0 & \text{for} \quad b - 2T \leqslant t < a \quad \text{and} \quad b < t \leqslant a + 2T, \end{cases} \tag{3.1}$$

then

$$\int_a^b \delta_n(\tau - t) f(\tau)\, d\tau \to \phi(t)$$

almost everywhere on the interval

$$I = [b - 2T, a + 2T] \quad \text{as} \quad n \to \infty.$$

If $f(t)$ is continuous on $[a,b]$

$$\int_a^b \delta_n(\tau - t) f(\tau)\, d\tau \to \phi(t)$$

everywhere on I.

Proof. Extend $\phi(t)$ outside $[b - 2T, a + 2T]$ so that $\phi(t)$ is a periodic function with period $4T$. Now, $b - t \leqslant b - b + 2T = 2T$ and $a - t \geqslant a - a - 2T = -2T$, which should explain the choice of I. (In addition, it is clear that $b - 2T \leqslant a < b \leqslant a + 2T$ since $0 < b - a \leqslant 2T$.) Therefore,

$$\int_a^b \delta_n(\tau - t) f(\tau)\, d\tau = \int_a^b \delta_n(\tau - t) \phi(\tau)\, d\tau = \int_{a-t}^{b-t} \delta_n(s) \phi(t + s)\, ds$$

$$= \int_{-2T}^{2T} \delta_n(s) \phi(t + s)\, ds.$$

Evidently $\phi(t + s)$ is zero on the intervals added to the integral in the last step (except possibly for the end points), for $\phi(t)$ has period $4T$, $t + s \leqslant t + 2T \leqslant a + 4T$ and $t + s \geqslant t - 2T \geqslant b - 4T$.

Now according to reference [1] p. 113,

$$\int_{-2T}^{2T} \delta_n(s)\phi(t+s)\,ds \to \phi(t) \qquad \text{as} \quad n \to \infty$$

provided that 1^0:

$$\frac{\phi(t)}{1+t^2} \in L(-\infty, \infty)$$

and that 2^0:

$$\lim_{h\to 0} \frac{1}{h} \int_0^h |\tfrac{1}{2}[\phi(t+\tau)+\phi(t-\tau)] - \phi(t)|\,d\tau = 0.$$

Since $\phi(t)$ is bounded and

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} < \infty,$$

condition 1^0 is fulfilled. Moreover, according to a theorem by Lebesgue ([1] p. 116), condition 2^0 holds a.e. on I if $\phi(t)$ is integrable on I. Of course, this requirement is fulfilled and therefore

$$\int_a^b \delta_n(\tau - t) f(\tau)\,d\tau \to \phi(t) \text{ a.e. on } I.$$

If $f(t)$ is continuous condition 2^0 is fulfilled in every point of I. (Even in the points of discontinuity of $\phi(t)$, i.e., $t = a$ and $t = b$, this is true due to the definition of $\phi(t)$ in these points.) This concludes the proof of the lemma.

Let $|x|$ signify the Euclidean norm $(x^* x)^{1/2}$ of the vector x. Then if $x(t)$ is a stochastic process, $\|x(t)\| = (E|x(t)|^2)^{1/2}$ is a norm also. [For $\|x\| = 0 \Leftrightarrow x = 0$ and $\|\alpha x\| = |\alpha|\,\|x\|$ (α is a number) are trivially satisfied. Furthermore, $\|x+y\|^2 \leqslant E|x|^2 + E|y|^2 + 2E\{|x||y|\} \leqslant E|x|^2 + E|y|^2 + 2(E|x|^2)^{1/2}(E|y|^2)^{1/2} = (\|x\| + \|y\|)^2$, that is $\|x+y\| \leqslant \|x\| + \|y\|$, where Schwarz's inequality has been used.]

LEMMA 3.2. *$\|y(t)\|$ is a continuous function if $y(t)$ is given by* (1.1b).

The proof of this assertion follows immediately from the fact that

$$\|y(t)\|^2 = tr\{Y(t,t_0)\,S_y\,Y(t,t_0)^*\} + \int_{t_0}^{t} tr\{Y(t,\tau)\,Q_y(\tau)\,Y(t,\tau)^*\}\,d\tau$$

[where $Y(t,\tau)$ is the fundamental matrix of (1.1b)], for $Y(t,\tau)$ is continuous and Q_y is bounded.

LEMMA 3.3. *If* $y(\tau)$ *is given by* (1.1b), $y^{(n)}(\tau,t)$ *by* (2.1) *and*

$$\eta(\tau,t)=\begin{cases} y(\tau) & \text{for} \quad t_0<\tau<t \\ \frac{1}{2}y(\tau) & \text{for} \quad \tau=t_0 \quad \text{and} \quad \tau=t \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

then for a fixed t $y^{(n)}(\tau,t)$ *converges for every* τ *on* $[t-2T,t_0+2T]$ *to* $\eta(\tau,t)$ *in the norm* $\|\cdot\|=(E|\cdot|^2)^{1/2}$ *(where* $|\cdot|$ *is the Euclidean norm) as* $n\to\infty$, *that is for every* $\tau \lim_{n\to\infty}\|y^{(n)}(\tau,t)-\eta(\tau,t)\|=0$. *Moreover,* $\|y^{(n)}(t-h,t)-y(t-h)\|$ *is uniformly bounded on any bounded interval.*

Proof. For any $\sigma\in[t-2T,t_0+2T]$ we have:

$$y^{(n)}(\sigma,t)=\int_{t_0}^{t}\delta_n(s-\sigma)y(s)\,ds=\int_{t_0}^{t}\delta_n(s-\sigma)\eta(s,t)\,ds$$

$$=\int_{-2T+\sigma}^{2T+\sigma}\delta_n(s-\sigma)\eta(s,t)\,ds=\int_{-2T}^{2T}\delta_n(s)\eta(\sigma+s,t)\,ds$$

because $2T+\sigma\geqslant 2T+t-2T=t$, $-2T+\sigma\leqslant -2T+t_0+2T=t_0$ and $\eta(s,t)$ is zero outside $s\in[t_0,t]$. Since

$$\int_{-2T}^{2T}\delta_n(s)\,ds=1 \quad [1],$$

we have

$$\eta(\tau,t)=\int_{-2T}^{2T}\delta_n(s)\eta(\tau,t)\,ds,$$

and therefore:

$$\|y^{(n)}(\sigma,t)-\eta(\tau,t)\|\leqslant\int_{-2T}^{2T}\delta_n(s)\|\eta(\sigma+s,t)-\eta(\tau,t)\|\,ds \tag{3.3}$$

Now, since $\|\eta(\sigma+s,t)-\eta(\tau,t)\|\leqslant\|\eta(\sigma+s,t)\|+\|\eta(\tau,t)\|\leqslant\|y(\sigma+s)\|+\|y(\tau)\|$ which is bounded on a bounded interval due to the continuity (Lemma 3.2,) and because of

$$\int_{-2T}^{2T}\delta_n(s)\,ds=1,$$

(putting $\sigma=\tau=t-h$) the last assertion of Lemma 3.3 is true. Furthermore, the boundedness of $\|\eta(\sigma+s,t)-\eta(\tau,t)\|$ assures the validity of condition 1^0

(in the proof of Lemma 3.1). Then (considering t as a fixed parameter) for every τ,

$$\int_{-2T}^{2T} \delta_n(s) \|\eta(\sigma+s,t)-\eta(\tau,t)\| \, ds \to \|\eta(\sigma,t)-\eta(\tau,t)\|$$

in all points σ for which condition 2^0 is fulfilled (compare the proof of Lemma 3.1). This condition certainly holds for $\sigma=\tau$, for the continuity of $\|y(\tau+s)-\eta(\tau,t)\|$ with respect to s (a trivial modification of Lemma 3.2) assures continuity of $\|\eta(\tau+s,t)-\eta(\tau,t)\|$ except for the two points of discontinuity. Here, however, our choice of η makes condition 2^0 valid. Therefore, for a fixed $t \|y^{(n)}(\tau,t)-\eta(\tau,t)\| \to 0$ everywhere on the interval $[t-2T, t_0+2T]$ as $n\to\infty$, which concludes the proof of the lemma.

4. THE DETERMINISTIC PROBLEM

The problem to minimize:

$$\int_0^T [p(\tau)^* Q_1'(\tau) p(\tau) + u(\tau)^* Q_2(\tau) u(\tau)] \, d\tau + p(T)^* Q_3' p(T) \tag{4.1}$$

when

$$\frac{dp}{dt} = F(t)p(t) + B'(t)u(t) \tag{4.2}$$

has the following solution (reference [12] or any standard textbook on the subject):

$$u(t) = -Q_2(t)^{-1} B'(t)^* P(t) p(t) \tag{4.3}$$

where $P(t)$ is given by the matrix Riccati equation:

$$\begin{cases} \dfrac{dP}{dt} = -Q_1' - F^*P - PF + PB' Q_2^{-1} B'^* P \\ P(T) = Q_3'. \end{cases} \tag{4.4}$$

Writing $P(t)$ in the following way (where notations are obvious):

$$P = \begin{pmatrix} P_{xx} & P_{xy} & P_{x-n} & \dots & P_{xn} \\ P_{yx} & P_{yy} & P_{y-n} & \dots & P_{yn} \\ P_{-nx} & P_{-ny} & P_{-n-n} & \dots & P_{-nn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{nx} & P_{ny} & P_{n-n} & \dots & P_{nn} \end{pmatrix}$$

it is clear that:

$$u(t) = -Q_2(t)^{-1} B(t)^* P_{xx}(t) x(t) - Q_2(t)^{-1} B(t)^* P_{xy}(t) y(t) - \sum_{k=-n}^{n} Q_2(t)^{-1} B(t)^* P_{xk}(t) q_k(t). \tag{4.5}$$

Then since $(Q_1')_{xx} = Q_1$, $(Q_1')_{xy} = 0$, $(Q_1')_{xk} = -\alpha_k Q_1 D_1$ and $(Q_3')_{xx} = Q_3$, $(Q_3')_{xy} = 0$, $(Q_3')_{xk} = -\alpha_k(T) Q_3 D_3$ straightforward calculations of the submatrices of the matrix Riccati equation give:

$$\begin{cases} \dfrac{dP_{xx}}{dt} = -Q_1 - P_{xx} A - A^* P_{xx} + P_{xx} B Q_2^{-1} B^* P_{xx} \\ P_{xx}(T) = Q_3 \end{cases} \tag{4.6}$$

$$\begin{cases} \dfrac{dP_{xy}}{dt} = -A^* P_{xy} - P_{xy} C - \sum_{-n}^{n} \beta_k P_{xk} + P_{xx} B Q_2^{-1} B^* P_{xy} \\ P_{xy}(T) = 0 \end{cases} \tag{4.7}$$

$$\begin{cases} \dfrac{dP_{xk}}{dt} = \alpha_k Q_1 D_1 - A^* P_{xk} + P_{xx} B Q_2^{-1} B^* P_{xk} \\ P_{xk}(T) = -\alpha_k(T) Q_3 D_3 \end{cases} \tag{4.8}$$

where as before:

$$\begin{cases} \alpha_k(t) = \gamma_k e^{ki\lambda(t-h)} \\ \beta_k(t) = e^{-k\lambda t}. \end{cases}$$

It is clear from (4.7) that P_{xy} depends on n and therefore, in the sequel, we will use the notation $P_{xy}^{(n)}$ for finite n, whereas P_{xy} will signify the limit function as $n \to \infty$.

Let $\Phi(t, \tau)$ be the fundamental matrix of (4.8), that is:

$$\begin{cases} \dfrac{\partial \Phi(t, \tau)}{\partial t} = (P_{xx} B Q_2^{-1} B^* - A^*) \Phi(t, \tau) \\ \Phi(\tau, \tau) = I. \end{cases} \tag{4.9}$$

Hence:

$$P_{xk}(t) = -\alpha_k(T) \Phi(t, T) Q_3 D_3 - \int_t^T \alpha_k(\tau) \Phi(t, \tau) Q_1(\tau) D_1(\tau) \, d\tau. \tag{4.10}$$

We need an expression for $-\sum \beta_k P_{xk}$:

$$\begin{aligned} -\sum_{-n}^{n} \beta_k(t) P_{xk}(t) &= \sum_{-n}^{n} \gamma_k e^{ki\lambda(T-h-t)} \Phi(t, T) Q_3 D_3 \\ &\quad + \int_t^T \sum_{-n}^{n} \gamma_k e^{ki\lambda(\tau-h-t)} \Phi(t, \tau) Q_1(\tau) D_1(\tau) \, d\tau \\ &= \Phi(t, T) Q_3 D_3 \delta_n(T-h-t) \\ &\quad + \int_t^T \Phi(t, \tau) Q_1(\tau) D_1(\tau) \delta_n(\tau-h-t) \, d\tau. \end{aligned}$$

Letting the fundamental matrix of (1.1b) be $Y(t,\tau)$, a simple check shows that:

$$P_{xy}^{(n)}(t) = -\int_t^T \Phi(t,T)\,Q_3\,D_3\,Y(\tau,t)\,\delta_n(T-h-\tau)\,d\tau$$

$$-\int_t^T \int_\tau^T \Phi(t,s)\,Q_1(s)\,D_1(s)\,\delta_n(s-h-\tau)\,ds\,Y(\tau,t)\,d\tau. \qquad (4.11)$$

Now, since $\Phi(t,T)\,Q_3\,D_3\,Y(\tau,t)$ is continuous with respect to τ on $[0,T]$, the first term of (4.11) tends to $\Phi(t,T)\,Q_3\,D_3\,Y(T-h,t)$ for $t<T-h$ and to zero for $t>T-h$ due to Lemma 3.1. Likewise, the inner integral of the second term tends to $\Phi(t,\tau+h)\,Q_1(\tau+h)\,D_1(\tau+h)$ a.e. for $\tau<T-h$ and to zero for $\tau>T-h$, because $\Phi(t,s)\,Q_1(s)\,D_1(s)$ is bounded and measurable with respect to s on $[0,T]$. Clearly the integrand of the outer integral of the second term in (4.11) is uniformly bounded (for all matrices are bounded and

$$\int_\tau^T \delta_n(s-h-T)\,ds < 1),$$

and therefore according to the dominated convergence theorem (for instance reference [5] p. 69) as $n\to\infty$ we will have:

$$P_{xy}(t) = -\int_t^{T-h} \Phi(t,\tau+h)\,Q_1(\tau+h)\,D_1(\tau+h)\,Y(\tau,t)\,d\tau$$

$$-\Phi(t,T)\,Q_3\,D_3\,Y(T-h,t) \qquad \text{for} \quad t\leqslant T-h$$

$$P_{xy}(t)=0 \qquad \text{for} \quad T-h<t\leqslant T.$$

(Actually the first expression for P_{xy} is valid for $t<T-h$ only, but as $t=T-h$ constitutes a set of measure zero the cost functional will not be affected by this change.) It is quite clear that $P_{xy}^{(n)}(n=1,2,3\ldots)$ and P_{xy} are uniformly bounded on $[0,T]$ (this fact will be used in the sequel), for $\int \delta_n dt<1$ and all matrices involved are bounded.

Thus we can express the differential equation for P_{xy} in two different ways (the former of which will be used in Section 7):

$$\begin{cases} \dfrac{dP_{xy}}{dt} = \underline{\Phi(t,t+h)\,Q_1(t+h)\,D_1(t+h)} + \Phi(t,T)\,Q_3\,D_3\,\delta(T-h-t) \\ \qquad -A^*P_{xy} - P_{xy}C + P_{xx}BQ_2^{-1}B^*P_{xy} \qquad \text{a.e.} \\ P_{xy}(T)=0. \end{cases} \qquad (4.12)$$

(The underlined term will be missing for $t > T - h$.)

$$\begin{cases} \dfrac{dP_{xy}}{dt} = \Phi(t, t+h)\, Q_1(t+h)\, D_1(t+h) - A^* P_{xy} - P_{xy} C \\ \qquad\qquad + P_{xx} B Q_2^{-1} B^* P_{xy} \qquad \text{a.e.} \quad \text{for} \quad t \leqslant T - h \\ P_{xy}(T-h) = -\Phi(T-h, T)\, Q_3\, D_3 \\ P_{xy}(t) = 0 \qquad \text{for} \quad T - h < t \leqslant T. \end{cases} \tag{4.13}$$

[(4.13) is the interpretation of (4.12).]

5. THE FILTERING PROBLEM

In order to determine the optimal control law in the stochastic case we need the minimum variance filtering estimate $\hat{p}(t|t)$, which will replace $p(t)$ in (4.3) according to the Separation Theorem, where the second t signifies the well known fact that $\hat{p}(t|t) = E\{p(t)|z(s);\ s \in [0, t]\}$, where $\{z(s);\ s \in [0, t]\}$ stands for the minimal σ-algebra with respect to this information (really $z(s)$ should be replaced by its integral as pointed out in Section 1).

For the filtering we will have the stochastic system:

$$\frac{dp}{dt} = Fp - B' Q_2^{-1} B'^* P\hat{p}(t|t) + v.$$

Then the filtering estimate satisfies the following system of differential equations [10]:

$$\begin{cases} \dfrac{d\hat{p}}{dt} = (F - B' Q_2^{-1} B'^* P - K^* H')\hat{p} + K^* z \\ \hat{p}(0|0) = 0 \end{cases} \tag{5.1}$$

where the gain-matrix K is:

$$K(t) = Q_z(t)^{-1} H(t)' R(t). \tag{5.2}$$

$R(t)$ satisfies the matrix Riccati differential equation [10], [12]:

$$\begin{cases} \dfrac{dR}{dt} = R_1 + FR + RF^* - RH'^* Q_z^{-1} H' R \\ R(0) = E\{p(0)\, p(0)^*\} \end{cases} \tag{5.3}$$

where $R_1(t)\,\delta(t - \tau) = E\{v(t)\, v(\tau)^*\}$, from which we have $(R_1)_{xx} = Q_x$, $(R_1)_{xy} = Q_{xy}$, $(R_1)_{yx} = Q_{yx}$, $(R_1)_{yy} = Q_y$ and all other submatrices equal to zero.

Since

$$K = (Q_z^{-1} HR_{yx}, Q_z^{-1} HR_{yy}, Q_z^{-1} HR_{y-n}, \ldots Q_z^{-1} HR_{yn}),$$

$H'\hat{p} = H\hat{y}$ and $B' Q_2^{-1} B'^* P\hat{p} =$

$$((BQ_2^{-1} B^* P_{xx}\hat{x} + BQ_2^{-1} B^* P_{xy}^{(n)}\, \hat{y} + \textstyle\sum_{-n}^{n} BQ_2^{-1} B^* P_{xk}\hat{q}_k)^*, 0, 0, \ldots 0)^*$$

(this is clear from the expression (4.5)), we will have the following filtering estimates:

$$\begin{cases} \dfrac{d\hat{x}^{(n)}}{dt} = (A - BQ_2^{-1}B^*P_{xx})\hat{x}^{(n)} - (BQ_2^{-1}B^*P_{xy}^{(n)} + R_{xy}H^*Q_z^{-1}H)\hat{y} \\ \qquad - \sum_{-n}^{n} BQ_2^{-1}B^*P_{xk}\hat{q}_k + R_{xy}H^*Q_z^{-1}z \\ \hat{x}^{(n)}(0|0) = 0 \end{cases} \tag{5.4}$$

$$\begin{cases} \dfrac{d\hat{y}}{dt} = (C - R_{yy}H^*Q_z^{-1}H)\hat{y} + R_{yy}H^*Q_z^{-1}z \\ \hat{y}(0|0) = 0 \end{cases} \tag{5.5}$$

$$\begin{cases} \dfrac{d\hat{q}_k}{dt} = (\beta_k I - R_{ky}H^*Q_z^{-1}H)\hat{y} + R_{ky}H^*Q_z^{-1}z \\ \hat{q}_k(0|0) = 0. \end{cases} \tag{5.6}$$

Since obviously the x-estimate given by (5.4) depends on n, we use the notation $\hat{x}^{(n)}(t|t)$, whereas $\hat{x}(t/t)$ will signify the limit function as $n \to \infty$. Straightforward calculations of the submatrices of the matrix Riccati equation (5.3) give:

$$\begin{cases} \dfrac{dR_{xy}}{dt} = Q_{xy} + AR_{xy} + R_{xy}C^* - R_{xy}H^*Q_z^{-1}HR_{yy} \\ R_{xy}(0) = S_{xy}\,Y(0,t_0)^* \end{cases} \tag{5.7}$$

$$\begin{cases} \dfrac{dR_{yy}}{dt} = Q_y + CR_{yy} + R_{yy}C^* - R_{yy}H^*Q_z^{-1}HR_{yy} \\ R_{yy}(0) = G(0) \end{cases} \tag{5.8}$$

$$\begin{cases} \dfrac{dR_{ky}}{dt} = R_{ky}C^* + \beta_k R_{yy} - R_{ky}H^*Q_z^{-1}HR_{yy} \\ R_{ky}(0) = \displaystyle\int_{t_0}^{0} e^{-kl\lambda\tau}G(\tau)\,d\tau \end{cases} \tag{5.9}$$

where

$$G(\tau) = Y(\tau, t_0)S_y\,Y(0,t_0)^* + \int_{t_0}^{\tau} Y(\tau,s)\,Q_y(s)\,Y(0,s)^*\,ds \tag{5.10}$$

In calculating

$$R_{xy}(0) = E\{x(0)\,y(0)^*\}, \qquad R_{yy}(0) = E\{y(0)\,y(0)^*\}$$

and

$$R_{ky}(0) = E\{q_k(0)\,y(0)^*\} = \int_{t_0}^{0} e^{-kl\lambda\tau}E\{y(\tau)\,y(0)^*\}\,d\tau$$

we have used the fact that

$$y(t) = Y(t, t_0) y(t_0) + \int_{t_0}^{t} Y(t,s) v_2(s) ds.$$

Let the fundamental matrix of

$$\frac{dx}{dt} = (C - R_{yy} H^* Q_z^{-1} H) x$$

be $\Psi(t,\tau)$ that is:

$$\begin{cases} \dfrac{\partial \Psi(t,\tau)}{\partial t} = (C - R_{yy} H^* Q_z^{-1} H) \Psi(t,\tau) \\ \Psi(\tau,\tau) = I. \end{cases} \tag{5.11}$$

Then (as $\beta_k = e^{-ki\lambda t}$):

$$R_{ky}(t) = R_{ky}(0) \Psi(t,0)^* + \int_0^t e^{-ki\lambda\tau} R_{yy}(\tau) \Psi(t,\tau)^* d\tau$$

$$= \int_{t_0}^{0} e^{-ki\lambda\tau} G(\tau) d\tau \Psi(t,0)^* + \int_0^t e^{-ki\lambda\tau} R_{yy}(\tau) \Psi(t,\tau)^* d\tau. \tag{5.12}$$

If we introduce the notation:

$$\tilde{z}(t|t) = z(t) - H\hat{y}(t|t), \tag{5.13}$$

we will have the following expression for $\hat{q}_k(t|t)$:

$$\hat{q}_k(t|t) = \int_0^t e^{-ki\lambda\tau} \hat{y}(\tau|\tau) d\tau$$

$$+ \int_0^t \int_0^\tau e^{-ki\lambda s} R_{yy}(s) \Psi(\tau,s)^* ds H(\tau)^* Q_z(\tau)^{-1} \tilde{z}(\tau|\tau) d\tau$$

$$+ \int_0^t \int_{t_0}^0 e^{-ki\lambda s} G(s) ds \Psi(\tau,0)^* H(\tau)^* Q_z(\tau)^{-1} \tilde{z}(\tau|\tau) d\tau. \tag{5.14}$$

We will wait until Section 7 to determine the term

$$\sum_{-n}^{n} BQ_2^{-1} B^* P_{xk} \hat{q}_k$$

in (5.4).

6. THE SMOOTHING ESTIMATE

It will be shown in Section 7 that the optimal control includes an integral of the minimum variance smoothing (interpolation) estimate $\hat{y}(\tau|t) = E\{y(\tau)|z(s);\ s \in [0,t]\}(\tau < t)$. For this reason we will determine a suitable expression for $\hat{y}(\tau|t)$ and at the same time we will get a verification of the basic equations given by Zachrisson [11] and others.

From Section 2 we have:

$$y^{(n)}(\tau,t) = \int_{t_0}^{t} \delta_n(\tau - s)y(s)\,ds = \sum_{-n}^{n} \gamma_k e^{ki\lambda\tau} q_k(t). \tag{6.1}$$

Let

$$\hat{y}^{(n)}(\tau|t) = E\{y^{(n)}(\tau,t)|z(\sigma);\ \sigma \in [0,t]\}$$

and

$$\hat{\eta}(\tau|t) = E\{\eta(\tau,t)|z(\sigma);\sigma \in [0,t]\}$$

where $\eta(\tau,t)$ is defined in Lemma 3.3. Then we have:

LEMMA 6.1. *$\hat{y}^{(n)}(\tau|t) \to \hat{\eta}(\tau|t)$ on $[t_0,T]\times[0,T]$ in the norm of Lemma 3.3 as $n \to \infty$.*

Proof (compare reference [6] p. 348).

$$\begin{aligned}\|\hat{y}^{(n)}(\tau|t) - \hat{\eta}(\tau|t)\|^2 &= E|E\{y^{(n)}(\tau,t) - \eta(\tau,t)|z(\sigma);\ \sigma \in [0,t]\}|^2\\ &\leqslant EE\{|y^{(n)}(\tau,t) - \eta(\tau,t)|^2|z(\sigma);\ \sigma \in [0,t]\}\\ &= E|y^{(n)}(\tau,t) - \eta(\tau,t)|^2 = \|y^{(n)}(\tau,t) - \eta(\tau,t)\|^2 \to 0\end{aligned}$$

when $n \to \infty$ according to Lemma 3.3. Furthermore for $t \in [0,T]$, $\tau \in [t-2T, t_0+2T] \supseteq [-T, t_0+2T] \supset [t_0,T]$, because $t_0 > -T$ and hence the lemma is true.

Thus according to Lemma 6.1:

$$\hat{y}^{(n)}(\tau|t) \to \begin{cases}\hat{y}(\tau|t) & \text{for}\quad t_0 < \tau < t\\ \tfrac{1}{2}\hat{y}(t|t) & \text{for}\quad \tau = t\\ 0 & \text{for}\quad t < \tau \leqslant T\end{cases} \tag{6.2}$$

where $\to$ stands for convergence in the above sense.†

† As this is convergence in the mean, there is a subsequence that converges a.s.

Thus we need an expression for $\hat{y}^{(n)}(\tau|t)$:

$$\hat{y}^{(n)}(\tau|t) = \sum_{-n}^{n} \gamma_k e^{ki\lambda\tau} E\{q_k(t)|z(s);\, s \in [0,t]\} = \sum_{-n}^{n} \gamma_k e^{ki\lambda\tau} \hat{q}(t|t)$$

$$= \int_0^t \delta_n(\tau - s)\, \hat{y}(s|s)\, ds$$

$$+ \int_0^t \int_0^\sigma \delta_n(\tau - s)\, R_{yy}(s)\, \Psi(\sigma,s)^*\, ds H(\sigma)^*\, Q_z(\sigma)^{-1}\, \tilde{z}(\sigma|\sigma)\, d\sigma$$

$$+ \int_0^t \int_{t_0}^0 \delta_n(\tau - s)\, G(s)\, ds \Psi(\sigma,0)^*\, H(\sigma)^* Q_z(\sigma)^{-1}\, \tilde{z}(\sigma|\sigma)\, d\sigma$$

where we have used the expression (5.14) for $\hat{q}(t|t)$ and (2.3) for $\delta_n(t)$.

Since $\hat{y}(s|s)$ is continuous a.s. (for according to reference [9] p. 165 $\hat{y}(s|s)$ is a Gaussian diffusion process) and $R_{yy}(s)$, $\Psi(\sigma,s)$ and $G(s)$ are continuous, we can apply Lemma 3.1 and therefore:

$$\hat{y}(\tau|t) = \hat{y}(\tau|\tau) + \int_\tau^t R_{yy}(\tau)\, \Psi(\sigma,\tau)^*\, H(\sigma)^*\, Q_z(\sigma)^{-1}\, \tilde{z}(\sigma|\sigma)\, d\sigma \quad \text{for} \quad 0 \leqslant \tau < t \tag{6.3}$$

$$\hat{y}(\tau|t) = \int_0^t G(\tau)\, \Psi(\sigma,0)^*\, H(\sigma)^*\, Q_z(\sigma)^{-1}\, \tilde{z}(\sigma|\sigma)\, d\sigma \quad \text{for} \quad t_0 < \tau \leqslant 0. \tag{6.4}$$

As the two expressions are the same for $\tau = 0$ we have replaced $<$ with $\leqslant$. Moreover, we have used the dominated convergence theorem.

By differentiating (6.3) and (6.4) with respect to t we will have the following differential equations:

$$\frac{\partial \hat{y}(\tau|t)}{\partial t} = R_{yy}(\tau)\, \Psi(t,\tau)^*\, H(t)^*\, Q_z(t)^{-1}\, \tilde{z}(t|t)$$
$$\hat{y}(\tau|\tau) \text{ given by (5.5)} \qquad \text{for} \quad 0 \leqslant \tau < t \tag{6.5}$$

$$\frac{\partial \hat{y}(\tau|t)}{\partial t} = G(\tau)\, \Psi(t,0)^*\, H(t)^*\, Q_z(t)^{-1}\, \tilde{z}(t|t)$$
$$\hat{y}(\tau|0) = 0 \qquad \text{for} \quad t_0 < \tau \leqslant 0. \tag{6.6}$$

(Note that $\hat{y}(\tau|t)$ is a Gaussian diffusion, for $\tilde{z}(\sigma|\sigma)d\sigma$ is a differential of a Wiener process (compare reference [9] p. 165), and therefore (6.5) and (6.6) are *formal* notations for (6.3) and (6.4) respectively.)

Define $U(t)$ as the matrix solution to:

$$\begin{cases} \dfrac{dU}{dt} = -(C^* - H^* Q_z^{-1} HR_{yy}) U \\ U(0) = I. \end{cases} \tag{6.7}$$

Then from (5.11) it is quite clear that the fundamental matrix of (6.7) is $\Psi(\tau, t)^*$ (where t and τ have been reversed):

$$U(t) = \Psi(\tau, t)^* U(\tau). \tag{6.8}$$

Furthermore, define the matrix $V(t)$:

$$V(t) = R_{yy}(t) U(t). \tag{6.9}$$

Then the differential equations for $\hat{y}(\tau|t)$ when $0 \leqslant \tau < t$ can be formulated in the following manner, as it is clear that $U(t)^{-1}$ exists [12]:

$$\begin{cases} \dfrac{\partial \hat{y}(\tau|t)}{\partial t} = V(\tau) U(t)^{-1} H(t)^* Q_z(t)^{-1} \tilde{z}(t|t) \\ \hat{y}(\tau|\tau) \text{ given by (5.5).} \end{cases} \tag{6.10}$$

By differentiating (6.9) we will determine a matrix differential equation for $V(t)$:

$$\begin{aligned} \frac{dV}{dt} &= \frac{dR_{yy}}{dt} U + R_{yy} \frac{dU}{dt} \\ &= Q_y U + CR_{yy} U + R_{yy} C^* U - R_{yy} H^* Q_z^{-1} HR_{yy} U \\ &\quad - R_{yy} C^* U + R_{yy} H^* Q_z^{-1} HR_{yy} U \\ &= Q_y U + CV \end{aligned}$$

where we have used (5.8), (6.7), and (6.9).

Thus the matrix functions $U(t)$ and $V(t)$ satisfy the linear system of differential equations:

$$\begin{cases} \dfrac{dU}{dt} = -C^* U + H^* Q_z^{-1} HV \\ \dfrac{dV}{dt} = Q_y U + CV \\ U(0) = I; \qquad V(0) = G(0). \end{cases} \tag{6.11}$$

Equations (6.10) and (6.11) have been derived by Zachrisson [11] by other means.

We will need the following lemma in the sequel:

LEMMA 6.2. *$\|\hat{\eta}(\tau|t)\|$ and $\|\hat{y}^{(n)}(\tau|t)\|$ are uniformly bounded on $[t_0, T] \times [0, T]$, where $\|\cdot\|$ is the norm of Lemma* 3.3.

Proof.

$$\|\hat{\eta}(\tau|t)\|^2 = E|\hat{\eta}(\tau|t)|^2 = E|E\{\eta(\tau,t)|z(s); s \in [0,t]\}|^2$$
$$\leqslant EE\{|\eta(\tau,t)|^2 \,|\, z(s); s \in [0,t]\} = E|\eta(\tau,t)|^2 = \|\eta(\tau,t)\|^2$$

where $|\cdot|$ stands for Euclidean norm. Now, since according to Lemma 3.2 $\|y(\tau)\|$ is continuous and thus bounded, $\|\eta(\tau, t)\|$ is bounded too, and so is $\|\hat{\eta}(\tau|t)\|$. Furthermore,

$$\|\hat{y}^{(n)}(\tau|t)\| \leqslant \int_{t_0}^{t} \delta_n(\tau-s)\|\hat{y}(s|t)\|\,ds \leqslant \max_{s\in[t_0,t]} \|\hat{y}(s|t)\| \leqslant \max_{s\in[t_0,t]} \|y(s)\|,$$

and therefore $\|\hat{y}^{(n)}(\tau|t)\|$ is uniformly bounded.

7. THE OPTIMAL CONTROL LAW

According to the Separation Theorem and (4.5) the optimal control of the approximate problem is:

$$\hat{u}^{(n)}(t) = -Q_2(t)^{-1} B(t)^*[P_{xx}(t)\hat{x}^{(n)}(t|t) + P_{xy}^{(n)}(t)\hat{y}(t|t) - g^{(n)}(t)] \quad (7.1)$$

where:

$$g^{(n)}(t) = -\sum_{k=-n}^{n} P_{xk}(t)\hat{q}_k(t|t).$$

We wish to determine $g^{(n)}(t)$, for this term is present in both (7.1) and (5.4). It is convenient to determine $\sum_{-n}^{n} \alpha_k(s)\hat{q}(t|t)$ first. Therefore, remembering (2.5) and (6.1), we obtain:

$$\sum_{-n}^{n} \alpha_k(s)\hat{q}_k(t|t) = \sum_{-n}^{n} \gamma_k e^{ki\lambda(s-h)} E\{q_k(t)|z(\sigma);\ \sigma \in [0,t]\}$$
$$= E\{y^{(n)}(s-h,t)|z(\sigma);\ \sigma \in [0,t]\} = \hat{y}^{(n)}(s-h|t).$$

By using (4.10) we can now determine $g^{(n)}(t)$:

$$g^{(n)}(t) = \Phi(t,T)\,Q_3\,D_3\,\hat{y}^{(n)}(T-h|t) + \int_t^T \Phi(t,\tau)\,Q_1(\tau)D_1(\tau)\,\hat{y}^{(n)}(\tau-h|t)\,d\tau.$$

Let $g(t)$ signify the formal limit of $g^{(n)}(t)$ when $n \to \infty$. Then using the norm $\|\cdot\| = (E|\cdot|^2)^{1/2}$ of Lemma 3.3 we have:

$$\|g - g^{(n)}\| \leqslant \|\Phi(t,T)\,Q_3\,D_3\|\,\|\hat{\eta}(T-h|t) - \hat{y}^{(n)}(T-h|t)\|$$
$$+ \int_t^T \|\Phi(t,\tau)\,Q_1(\tau)\,D_1(\tau)\|\,\|\hat{\eta}(\tau-h|t) - \hat{y}^{(n)}(\tau-h|t)\|\,d\tau.$$

Now, since the integrand is uniformly bounded for every n (Lemma 6.2) and $\|\hat{\eta}-\hat{y}^{(n)}\| \to 0$ as $n \to \infty$ (Lemma 6.1), $g^{(n)}(t) \to g(t)$ in the sense described above (reference [5] p. 69). Then using (6.2) we have:

$$g(t) = \int_{t-h}^{t} \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h)\, \hat{y}(s|t)\, ds \qquad \text{for} \quad t < T-h \quad (7.2)$$

and

$$g(t) = \Phi(t,T)\, Q_3\, D_3\, \hat{y}(T-h|t) + \int_{t-h}^{T-h} \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h)\, \hat{y}(s|t)\, ds \qquad \text{for} \quad t > T-h. \quad (7.3)$$

Since $t = T-h$ is a set of measure zero, the choice of $g(t)$ on this set affects neither the cost function nor $\hat{x}(t|t)$, and therefore we might as well let (7.2) be valid for $t = T-h$, although Lemma 6.1 gives another result.

It is quite clear that $\|g(t)\|$ and $\|g^{(n)}(t)\|$ are uniformly bounded on $[0, T]$, for according to Lemma 6.2, $\|\hat{y}(s|t)\|$ and $\|\hat{y}^{(n)}(s|t)\|$ are uniformly bounded on $[-h, T] \times [0, T]$. (This fact will be used below.)

By differentiating we obtain a differential (vector-) equation for $g(t)$:

$$\frac{dg}{dt} = \Phi(t, t+h)\, Q_1(t+h)\, D_1(t+h)\, \hat{y}(t|t) - Q_1(t)\, D_1(t)\, \hat{y}(t-h|t) + (P_{xx} B Q_2^{-1} B^* - A^*) g(t) + \int_{t-h}^{t} \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h) \frac{\partial \hat{y}(s|t)}{\partial t}\, ds$$

for $t \leqslant T-h$; for $t > T-h$ the first term will be missing and instead $\Phi(t,T)\, Q_3\, D_3[\partial\hat{y}(T-h|t)/\partial t]$ will be added, and finally $\int_{t-h}^{t}$ will be replaced by $\int_{t-h}^{T-h}$. [As according to (6.3) and (6.4) the t-dependence of $\hat{y}(\tau|t)$ is introduced by the upper limit of a stochastic integral, differentiating $g(t)$ with respect to this parameter is permitted provided the order of integration is immaterial. Now, this is clearly the case since all (deterministic) functions involved are bounded on $[t_0, T]$ (reference [3] p. 431).]

Using (6.5) and (6.6) we have:

$$\begin{cases} \dfrac{dg}{dt} = (P_{xx} B Q_2^{-1} B^* - A^*) g + \underline{\Phi(t, t+h)\, Q_1(t+h)\, D_1(t+h)\, \hat{y}(t|t)} \\ \qquad - Q_1(t)\, D_1(t)\, \hat{y}(t-h|t) + \Gamma(t)\, H(t)^*\, Q_z(t)^{-1}\, \tilde{z}(t|t) \\ g(0) = 0 \end{cases} \qquad (7.4)$$

($g(0) = 0$ because, according to (6.6), $\hat{y}(\tau|0) = 0$ for $\tau \leqslant 0$; the underlined term is missing for $t > T - h$) where $\Gamma(t)$ is an $m_1 \times m_2$ matrix given by:

$$\Gamma(t) = \begin{cases} = \int_0^t \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h)\, R_{yy}(s)\, \Psi(t,s)^*\, ds \\ \quad + \int_{t-h}^0 \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h)\, G(s)\, \Psi(t,0)^*\, ds \qquad \text{for} \quad 0 \leqslant t \leqslant h \\ = \int_{t-h}^t \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h)\, R_{yy}(s)\, \Psi(t,s)^*\, ds \qquad \text{for} \quad h \leqslant t \leqslant T-h \\ = \int_{t-h}^{T-h} \Phi(t, s+h)\, Q_1(s+h)\, D_1(s+h)\, R_{yy}(s)\, \Psi(t,s)^*\, ds \\ \quad + \Phi(t,T)\, Q_3\, D_3\, R_{yy}(T-h)\, \Psi(t, T-h)^* \qquad \text{for} \quad T-h < t \leqslant T. \end{cases} \tag{7.5}$$

Using (4.9) and (5.11) we have:

$$\begin{aligned} \frac{d\Gamma}{dt} = (P_{xx} B Q_2^{-1} B^* - A^*)\,\Gamma + \Gamma(C^* - H^* Q_z^{-1} H R_{yy}) \\ + \Phi(t, t+h)\, Q_1(t+h)\, D_1(t+h)\, R_{yy}(t) \\ - Q_1(t)\, D_1(t)\, R_{yy}(t-h)\, \Psi(t, t-h)^* \end{aligned} \tag{7.6}$$

This equation is valid for $h \leqslant t < T - h$ only. For $0 \leqslant t \leqslant h$ the last term is replaced by $-Q_1(t)\, D_1(t)\, G(t-h)\, \Psi(t,0)^*$ (for $t = h$ the two terms are the same) and for $T - h \leqslant t \leqslant T$ the next to last term is replaced by

$$\Phi(t,T)\, Q_3\, D_3\, R_{yy}(t)\, \delta(T-h-t).$$

Furthermore:

$$\Gamma(0) = \int_0^h \Phi(0,\tau)\, Q_1(\tau)\, D_1(\tau)\, G(\tau - h)\, d\tau. \tag{7.7}$$

It remains to give a differential equation for $\hat{x}(t|t)$ (the limit function of $\hat{x}^{(n)}(t|t)$ as $n \to \infty$). Formally we will have the following equation:

$$\begin{cases} \dfrac{d\hat{x}}{dt} = (A - BQ_2^{-1} B^* P_{xx})\,\hat{x} - (BQ_2^{-1} B^* P_{xy} + R_{xy} H^* Q_z^{-1} H)\,\hat{y} \\ \qquad + BQ_2^{-1} B^* g + R_{xy} H^* Q_z^{-1} z \\ \hat{x}(0|0) = 0. \end{cases} \tag{7.8}$$

Letting $\hat{x}(t|t)$ be given by (7.8), we have:

$$\frac{d}{dt}(\hat{x}^{(n)}-\hat{x}) = (A - BQ_2^{-1}B^*P_{xx})(\hat{x}^{(n)}-\hat{x}) - BQ_2^{-1}B^*(P_{xy}^{(n)}-P_{xy})\hat{y} + BQ_2^{-1}B^*(g^{(n)}-g)$$

$$\|\hat{x}^{(n)}(t|t)-\hat{x}(t|t)\| \leqslant \int_0^t \|\Phi(\tau,t)^* B(\tau)Q_2(\tau)^{-1}B(\tau)^*\| \, \|P_{xy}^{(n)}(\tau)-P_{xy}(\tau)\| \times \|\hat{y}(\tau|\tau)\| \, d\tau + \int_0^t \|\Phi(\tau,t)^* B(\tau)Q_2(\tau)^{-1}B(\tau)^*\| \, \|g^{(n)}(\tau)-g(\tau)\| \, d\tau.$$

Since the integrands are uniformly bounded for all n and tends to zero as $n \to \infty$, it is clear that $\|\hat{x}^{(n)}(t|t)-\hat{x}(t|t)\|$ tends to zero, too (reference [5] p. 69). Moreover, uniform boundedness of $\|P_{xy}^{(n)}-P_{xy}\|$ and $\|g^{(n)}-g\|$ implies that $\|\hat{x}^{(n)}(t|t)-\hat{x}(t|t)\|$ is uniformly bounded on $[0,T]$.

If $\hat{u}(t)$ is the formal limit of $\hat{u}^{(n)}(t)$ as $n \to \infty$, we obtain:

$$\|\hat{u}^{(n)}-\hat{u}\| \leqslant \|Q_2^{-1}B^*\| \{ \|P_{xx}\| \, \|\hat{x}^{(n)}-\hat{x}\| + \|P_{xy}^{(n)}-P_{xy}\| \, \|\hat{y}\| + \|g^{(n)}-g\| \},$$

and therefore we have proved the following lemma:

LEMMA 7.1. *$\hat{u}^{(n)}(t) \to \hat{u}(t)$ and $\hat{x}^{(n)}(t|t) \to \hat{x}(t|t)$ in the norm of Lemma 3.3 as $n \to \infty$. Moreover, $\|\hat{u}^{(n)}(t)-\hat{u}(t)\|$ is uniformly bounded on $[0,T]$ for all n.*

In order to obtain a closed loop control system, it is more convenient to write (7.8) in the following way:

$$\begin{cases} \dfrac{d\hat{x}}{dt} = A\hat{x} + B\hat{u} + R_{xy}H^*Q_z^{-1}(z - H\hat{y}) \\ \hat{x}(0|0) = 0. \end{cases} \tag{7.9}$$

In Section 8 we will show that $\hat{u}$ is the optimal control of our problem. Define the m_3-vector $\omega(t)$ (not to be confused with the sample variable):

$$\omega(t) = P_{xx}(t)\hat{x}(t|t) + P_{xy}(t)\hat{y}(t|t) - g(t). \tag{7.10}$$

Then the optimal control can be written:

$$\hat{u}(t) = -Q_2(t)^{-1}B(t)^*\omega(t). \tag{7.11}$$

By differentiating (7.10) and using (4.6), (4.12), (5.5), (7.4), and (7.8) we obtain a differential equation for $\omega(t)$:

$$\begin{aligned}\frac{d\omega}{dt} &= P_{xx} A\hat{x} - P_{xx} BQ_2^{-1} B^* P_{xx} \hat{x} - P_{xx} BQ_2^{-1} B^* P_{xy} \hat{y} \\ &\quad + P_{xx} BQ_2^{-1} B^* g + P_{xx} R_{xy} H^* Q_z^{-1} \tilde{z} - Q_1 \hat{x} - P_{xx} A\hat{x} \\ &\quad - A^* P_{xx} \hat{x} + P_{xx} BQ_2^{-1} B^* P_{xx} \hat{x} + P_{xy} C\hat{y} \\ &\quad + P_{xy} R_{yy} H^* Q_z^{-1} \tilde{z} + \underline{\Phi(t, t+h)\, Q_1(t+h)\, D_1(t+h)\, \hat{y}} \\ &\quad + \Phi(t, T)\, Q_3\, D_3\, \hat{y}\delta(T-h-t) - A^* P_{xy} \hat{y} - P_{xy} C\hat{y} \\ &\quad + P_{xx} BQ_2^{-1} B^* P_{xy} \hat{y} - P_{xx} BQ_2^{-1} B^* g + A^* g \\ &\quad - \underline{\Phi(t, t+h)\, Q_1(t+h)\, D_1(t+h)\, \hat{y}} + Q_1\, D_1\, \hat{y}(t-h|t) - \Gamma H^* Q_z^{-1} \tilde{z} \\ &= -A^* \omega - Q_1[\hat{x} - D_1 \hat{y}(t-h|t)] + \Phi(t, T)\, Q_3\, D_3\, \hat{y}\delta(T-h-t) \\ &\quad + (P_{xx} R_{xy} + P_{xy} R_{yy} - \Gamma)\, H^* Q_z^{-1} \tilde{z}.\end{aligned}$$

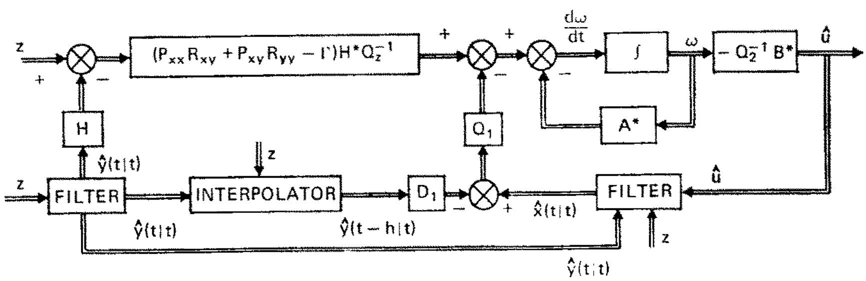

FIG. 7.1.

(The underlined terms are missing for $t > T-h$; $\hat{x} = \hat{x}(t|t)$, $\hat{y} = \hat{y}(t|t)$ and $\tilde{z} = z(t) - H\hat{y}(t|t)$.)

Furthermore $\omega(0) = 0$, for $\hat{x}(0|0) = \hat{y}(0|0) = g(0) = 0$. Then we have (observing that $\Phi(t,T)\, Q_3\, D_3\, \hat{y}(t|t)$ is a continuous function):

$$\begin{cases} \dfrac{d\omega}{dt} = -A(t)^* \omega(t) - Q_1(t)[\hat{x}(t|t) - D_1(t)\, \hat{y}(t-h|t)] \\ \qquad + [P_{xx}(t)\, R_{xy}(t) + P_{xy}(t)\, R_{yy}(t) - \Gamma(t)]\, H(t)^*\, Q_z(t)^{-1}\, \tilde{z}(t|t) \\ \omega(0) = 0 \\ \omega(T-h+0) = \omega(T-h-0) + \Phi(T-h, T)\, Q_3\, D_3\, \hat{y}(T-h|T-h). \end{cases} \tag{7.12}$$

A matrix block diagram of the optimal controller is given in Fig. 7.1. The filters are the usual Kalman filters ([4], [9], [12]) as given by (7.9) and (5.5) (for $\hat{x}$ there is a feedback loop) and the interpolator is given by Section 6 (for a more detailed discussion of this interpolator and a block diagram for it, we refer to Zachrisson [11]). Furthermore P_{xx}, R_{xy} and R_{yy} (given by (4.6), (5.7) and (5.8)) are the same as if the time argument of $y(t)$ in (1.2) were not delayed. However, this is not true for P_{xy} which is given by (4.12) or (4.13). In addition, we need $\Gamma(t)$ as given by (7.6) and the paragraph following it. Also note that $\omega(t)$ is discontinuous for $t = T - h$ (and so is $\Gamma(t)$).

Equation (7.12) can in a certain sense be considered to be an adjoint equation of (1.1a). It seems as though (7.12) can be derived by means of Pontryagin's maximum principle, although we are not prepared to give an unobjectionable solution to this problem at present. We hope to return to this problem in another paper. Nevertheless, we will give the basic ideas of this nonrigorous and formal solution in the Appendix.

8. OPTIMALITY OF THE FORMAL SOLUTION

We will use the following notations:

$$L[u] = \int_0^T \{[x(t) - D_1(t)y(t-h)]^* Q_1(t)[x(t) - D_1(t)y(t-h)] + u(t)^* Q_2(t)u(t)\}\,dt + [x(T) - D_3 y(T-h)]^* Q_3[x(T) - D_3 y(T-h)] \tag{8.1}$$

$$L_n[u] = \int_0^T \{[x(t) - D_1(t)y^{(n)}(t-h,t)]^* Q_1(t)[x(t) - D_1(t)y^{(n)}(t-h,t)] + u(t)^* Q_2(t)u(t)\}\,dt + [x(T) - D_3 y^{(n)}(T-h,T)]^* Q_3[x(T) - D_3 y^{(n)}(T-h,T)] \tag{8.2}$$

$$V[u] = E\{L[u]\} \tag{1.2}$$

$$V_n[u] = E\{L_n[u]\} \tag{2.10}$$

where of course $x(t)$ is a function of $u(t)$ given by the differential equation (1.1a).

Let U be any class of (t,ω)-measurable functions $u(t,\omega)$ (ω is the sample variable) such that for a fixed $t \in [0,T]$ the vector $u(t,\omega)$ is a linear function $M_t[z]$ of the observation vector $z(\tau,\omega)$ on the interval $0 \leqslant \tau \leqslant t$ and such that

$$\int_0^T E|u|^2\,dt = \int_0^T \|u\|^2\,dt < \infty.$$

(With this assumption $V[u] < \infty$ and $V_n[u] < \infty$.) Furthermore, $\hat{u}^{(n)}$ and $\hat{u}$ should belong to U. (In the sequel ω will be suppressed from notation.) We know that $\min_{u \in U} V_n[u]$ is provided by $\hat{u}^{(n)}$ (Separation Theorem†) and we will

† Cf. [8] or [14]. Wonham [10] does not employ the class U.

show that there exists a $u \in U$ that minimizes $V[u]$ and that this optimal control is precisely $\hat{u}(t)$.

LEMMA 8.1. *For every* $u \in U$, $|V_n[u] - V[u]| \to 0$ *as* $n \to \infty$.

Proof ($|\cdot|$ signifies Euclidean norm when applied to vectors and matrices; $\|\cdot\|$ is the norm of Lemma 3.3; for nonstochastic elements: $|\cdot| = \|\cdot\|$).

As $\int_0^T \|u\|^2 dt < \infty$, $\int_0^t \|u\| dt \leqslant \int_0^T \|u\| dt$ is bounded on $[0,T]$, because $L_2[0,T] \subseteq L_1[0,T]$.

Now, $\|y(t)\|$ is bounded on $[0,T]$ due to the continuity (Lemma 3.2), and the same is true for $\|x(t)\|$. For let $X(t,\tau)$ be the fundamental matrix of (1.1a) and we will have:

$$\|x(t)\| \leqslant \|X(t,0)x(0) + \int_0^t X(t,\tau)v_1(\tau)d\tau\| + \int_0^t |X(t,\tau)B(\tau)|\,\|u(\tau)\|\,d\tau.$$

The first term is bounded (and continuous) for the same reason as $\|y(t)\|$, and the second term is smaller than $K' \int_0^t \|u\| d\tau$ (where K' is the upper bound of $|X(t,\tau)B(\tau)|$) and thus bounded on $[0,T]$. Then we have:

$$\begin{aligned}|L_n[u] - L[u]| \leqslant \int_0^T &\{2|x(t) - D_1(t)y(t-h)|\,|Q_1(t)D_1(t)| \times \\ &\times |y(t-h) - y^{(n)}(t-h,t)| \\ &+ |D_1(t)^* Q_1(t) D_1(t)|\,|y(t-h) - y^{(n)}(t-h,t)|^2\}\,dt \\ &+ 2|x(T) - D_3 y(T-h)|\,|Q_3 D_3|\,|y(T-h) - y^{(n)}(T-h,T)| \\ &+ |D_3^* Q_3 D_3|\,|y(T-h) - y^{(n)}(T-h,T)|^2.\end{aligned}$$

Using Schwarz's inequality we obtain (for convenience arguments are suppressed from notation):

$$\begin{aligned}|V_n[u] - V[u]| = |E\{L_n[u] - L[u]\}| &\leqslant E|L_n[u] - L[u]| \\ &\leqslant \int_0^T \{2|Q_1 D_1|\,\|x - D_1 y\|\,\|y - y^{(n)}\| \\ &+ |D_1^* Q_1 D_1|\,\|y - y^{(n)}\|^2\}\,dt \\ &+ 2|Q_3 D_3|\,\|x - D_3 y\|\,\|y - y^{(n)}\| + |D_3^* Q_3 D_3|\,\|y - y^{(n)}\|^2.\end{aligned}$$

Now, $|Q_1 D_1|$, $|D_1^* Q_1 D_1|$, and $\|x - D_i y\| \leqslant \|x\| + |D_i|\|y\|$, $i = 1,3$ are bounded. Furthermore, according to Lemma 3.3, $\|y - y^{(n)}\|$ is uniformly bounded for all n and tends to zero as $n \to \infty$. Therefore $|V_n[u] - V[u]| \to 0$ as $n \to \infty$ (reference [5] p. 69), which was to be proved.

LEMMA 8.2. $|V[\hat{u}^{(n)}] - V[\hat{u}]| \to 0$ *as* $n \to \infty$.

Proof. Let $x^{(n)}(t)$ and $x(t)$ signify the solutions of (1.1a) when $u(t)$ equals $\hat{u}^{(n)}(t)$ and $\hat{u}(t)$ respectively. In the same way as in Lemma 8.1 we have (arguments are suppressed from notation):

$$\begin{aligned}|V[\hat{u}^{(n)}] - V[\hat{u}]| \leqslant \int_0^T \{&2|Q_1|\,\|x^{(n)} - x\|\,\|x - D_1 y\| \\ &+ |Q_1|\,\|x^{(n)} - x\|^2 + 2|Q_2|\,\|\hat{u}^{(n)} - \hat{u}\|\,\|\hat{u}\| \\ &+ |Q_2|\,\|\hat{u}^{(n)} - \hat{u}\|^2\}\,dt + 2|Q_3|\,\|x^{(n)} - x\|\,\|x - D_3 y\| \\ &+ |Q_3|\,\|x^{(n)} - x\|^2.\end{aligned}$$

But
$$\|x^{(n)}(t) - x(t)\| \leqslant \int_0^t |X(t,\tau)\,B(\tau)|\,\|\hat{u}^{(n)}(\tau) - \hat{u}(\tau)\|\,d\tau.$$

Now, $\|\hat{u}^{(n)} - \hat{u}\|$ is uniformly bounded on $[0,T]$ for all n and tends to zero as $n \to \infty$ (Lemma 7.1). Then $\|x^{(n)} - x\|$ has these properties, too.

Moreover, as pointed out earlier in this section, $\|x\|$ and $\|y\|$ are bounded on $[0,T]$. Since $\|\hat{u}\| \leqslant |Q_2^{-1} B^*|\{|P_{xx}|\|\hat{x}\| + |P_{xy}|\|\hat{y}\| + \|g\|\}$, the same is true for $\|\hat{u}\|$, because $|Q_2^{-1} B^*|$, $|P_{xx}|$, $\|\hat{x}\|$, $|P_{xy}|$, $\|\hat{y}\|$ and $\|g\|$ are all bounded. As for $\|\hat{y}\|$ and $\|g\|$ this is clear from Lemma 6.2 and Section 7 respectively, and in the same way as in Lemma 6.2 we can show that $\|\hat{x}\| \leqslant \|x\|$. Therefore, $|V[\hat{u}^{(n)}] - V[\hat{u}]| \to 0$ as $n \to \infty$, for the integrand is uniformly bounded and $\to 0$ as $n \to \infty$ (reference [5] p. 69).

LEMMA 8.3. $|V_n[\hat{u}^{(n)}] - V[\hat{u}^{(n)}]| \to 0$ *as* $n \to \infty$.

Proof (this is a simple modification of Lemma 8.1). The only change in the proof of Lemma 8.1 is that $x^{(n)}$ should replace x. Then $\|x^{(n)}\|$ should be uniformly bounded. But this is clearly the case as $\|x^{(n)}\| \leqslant \|x\| + \|x^{(n)} - x\|$ (cf. the proof of Lemma 8.2).

THEOREM 8.1. *$\hat{u}(t)$ is an optimal solution in the class U to the problem posed in Section* 1, *that is to minimize $V[u]$.*

Proof. Suppose that $\hat{u}(t)$ is not optimal. Then there is a $u'(t) \in U$ such that $V[u'] < V[\hat{u}]$. Let $\epsilon > 0$ be the difference, that is $V[u'] = V[\hat{u}] - \epsilon$. Now according to Lemma 8.1, there is an N_1 such that $|V_n[u'] - V[u']| < \frac{1}{3}\epsilon$ for every $n > N_1$ and, according to Lemma 8.3, an N_2 such that $|V_n[\hat{u}^{(n)}] - V[\hat{u}^{(n)}]| < \frac{1}{3}\epsilon$ for every $n > N_2$. Furthermore according to Lemma 8.2, there is an N_3 such that $|V[\hat{u}^{(n)}] - V[\hat{u}]| < \frac{1}{3}\epsilon$ when $n > N_3$. Then for $n > \max(N_1, N_2, N_3)$ we have:

$$V_n[u'] < V[u'] + \tfrac{1}{3}\epsilon = V[\hat{u}] - \tfrac{2}{3}\epsilon < V[\hat{u}^{(n)}] - \tfrac{1}{3}\epsilon < V_n[\hat{u}^{(n)}].$$

But according to the Separation Theorem $\hat{u}^{(n)}$ is optimal in the class U for the problem of minimizing $V_n[u]$. Thus we have established a contradiction and the theorem must be true.

ACKNOWLEDGMENT

I wish to express my gratitude to my teacher Lars Erik Zachrisson, who suggested the problem and gave important hints for its solution.

APPENDIX

In this Appendix we will derive the basic equations of our paper in a *formal* manner using the maximum principle of Pontryagin and the following assumption.

We assume that at each time t (or at least at almost every t) the optimal control $\hat{u}(t)$ is the same as if no further observations other than $\{z(s);\ s \in [0,t]\}$ were to be received in the future. This is a natural assumption since we cannot anticipate future observations. That is, determining the control $\tilde{u}(\tau|t)$ which for every $\tau \geqslant t$ is measurable with respect to the σ-ring $\{z(s);\ s \in [0,t]\}$ and that minimizes (for convenience we put $D_1 = D_3 = I$ and $Q_3 = 0$):

$$E\left\{\int_t^T \{[x(\tau) - y(\tau - h)]^* Q_1(\tau)[x(\tau) - y(\tau - h)] + \tilde{u}(\tau|t)^* Q_2(\tau)\tilde{u}(\tau|t)\} d\tau | z(s);\ s \in [0,t]\right\}$$

the optimal control minimizing $V[u]$ is obtained as:

$$\hat{u}(t) = \tilde{u}(t|t).$$

[In order that this procedure be well defined (we can get a control equivalent to $\tilde{u}(\tau|t)$ by changing its value for instance at $\tau = t$, which would give a different $\hat{u}(t)$) we must impose some restriction like continuity.]

Putting $x = x_1 + x_2$ (where x_1 like u is $\{z(s);\ s \in [0,t]\}$-measurable) we have:

$$\begin{cases} \dfrac{dx_1}{dt} = Ax_1 + Bu & x_1(0) = 0 \\[2ex] \dfrac{dx_2}{dt} = Ax_2 + v_1 & x_2(0) = x(0). \end{cases}$$

Then we have:

$$\begin{aligned} &E\{[x(\tau) - y(\tau - h)]^* Q_1(\tau)[x(\tau) - y(\tau - h)]|z(s); s \in [0,t]\} \\ &\quad = [\hat{x}(\tau|t) - \hat{y}(\tau - h|t)]^* Q_1(\tau)[\hat{x}(\tau|t) - \hat{y}(\tau - h|t)] \\ &\qquad + E\{[x_2(\tau) - y(\tau - h)]^* Q_1(\tau)[x_2(\tau) - y(\tau - h)]|z(s); s \in [0,t]\} \\ &\qquad - [\hat{x}_2(\tau|t) - \hat{y}(\tau - h|t)]^* Q_1(\tau)[\hat{x}_2(\tau|t) - \hat{y}(\tau - h|t)] \qquad \text{(A1)} \end{aligned}$$

where $\hat{x}(\tau|t) = E\{x(\tau)|z(s);\ s \in [0,t]\} = x_1(\tau) + \hat{x}_2(\tau|t)$.

Since only the first term of (A1) depends on the choice of $u(\tau)$, we have the following control problem (recognizing the fact that $E\{u^* Q_2 u|z(s); s \in [0,t]\} = u^* Q_2 u$):

$$\min \int_t^T \{[\hat{x}(\tau|t) - \hat{y}(\tau - h|t)]^* Q_1(\tau)[\hat{x}(\tau|t) - \hat{y}(\tau - h|t)] + u(\tau)^* Q_1(\tau) u(\tau)\} \, d\tau$$

when:

$$\begin{cases} \dfrac{\partial \hat{x}(\tau|t)}{\partial \tau} = A(\tau)\hat{x}(\tau|t) + B(\tau)u(\tau) \qquad \tau > t \\ \hat{x}(t|t) \text{ given} \end{cases} \tag{A2}$$

[In deriving (A2) we have used the fact that (for $\tau > t$):

$$E\left\{\int_t^\tau X(\tau,s)\, v_1(s)\, ds | z(s); s \in [0,t]\right\} = E\left\{\int_t^\tau X(\tau,s)\, v_1(s)\, ds\right\} = 0$$

and that $E\{u(\tau)|z(s);\ s \in [0,t]\} = u(\tau)$ according to our assumption.]

Applying Pontryagin's maximum principle to this problem (cf. reference [12]) and letting $\lambda(\tau|t)$ signify the adjoint vector, we will have the following Hamiltonian:

$$\mathscr{H} = \lambda^* A\hat{x} + \lambda^* Bu + (\hat{x} - \hat{y})^* Q_1(\hat{x} - \hat{y}) + u^* Q_2 u \tag{A3}$$

where $\hat{x} = \hat{x}(\tau|t)$ and $\hat{y} = \hat{y}(\tau - h|t)$. Hence:

$$\tilde{u}(\tau|t) = -\tfrac{1}{2} Q_2(\tau)^{-1} B(\tau)^* \lambda(\tau|t) \tag{A4}$$

where:

$$\begin{cases} \dfrac{\partial \lambda}{\partial \tau} = -A^* \lambda - 2Q_1(\hat{x} - \hat{y}) \\ \lambda(T|t) = 0. \end{cases} \tag{A5}$$

Define $\tilde{\omega}(\tau|t) = \tfrac{1}{2}\lambda(\tau|t)$. Then according to our assumption:

$$\hat{u}(t) = -Q_2(t)^{-1} B(t)^* \tilde{\omega}(t|t)$$

or [cf. (7.11)]:

$$\omega(t) = \tilde{\omega}(t|t) \tag{A6}$$

where $\tilde{\omega}(\tau|t)$ is given by the following system:

$$\begin{cases} \dfrac{\partial \hat{x}(\tau|t)}{\partial \tau} = A(\tau)\hat{x}(\tau|t) - B(\tau) Q_2(\tau)^{-1} B(\tau)^* \tilde{\omega}(\tau|t) & \text{(A7)} \\ \dfrac{\partial \tilde{\omega}(\tau|t)}{\partial \tau} = -A(\tau)^* \tilde{\omega}(\tau|t) - Q_1(\tau)\hat{x}(\tau|t) + Q_1(\tau)\hat{y}(\tau - h|t) & \text{(A8)} \\ \hat{x}(t|t) \text{ given}; \qquad \tilde{\omega}(T|t) = 0. \end{cases}$$

Now, it is reasonable to assume that $\tilde{\omega}(\tau|t)$ includes a term $P_{xx}(\tau)\hat{x}(\tau|t)$, for if $\hat{y}(\tau-h|t)\equiv 0$ the solution of the above system is $\tilde{\omega}(\tau|t)=P_{xx}(\tau)\hat{x}(\tau|t)$, where (for instance reference [12]):

$$\begin{cases} \dfrac{dP_{xx}}{dt} = -Q_1 - P_{xx}A - A^* P_{xx} + P_{xx} B Q_2^{-1} B^* P_{xx} \\ P_{xx}(0)=0. \end{cases} \tag{A9}$$

For this reason, we will write (A8) in the following way:

$$\begin{cases} \dfrac{\partial\tilde{\omega}}{\partial\tau} = (P_{xx} B Q_2^{-1} B^* - A^*)\tilde{\omega} - P_{xx} B Q_2^{-1} B^* \tilde{\omega} - Q_1 \hat{x}(\tau|t) + Q_1 \hat{y}(\tau-h|t) \\ \tilde{\omega}(T|t)=0. \end{cases}$$

Then, $\Phi(t,\tau)$ being defined by (4.9):

$$\tilde{\omega}(t|t) = \int_t^T \Phi(t,s) P_{xx}(s) B(s)\, Q_2(s)^{-1} B(s)^* \tilde{\omega}(s|t)\, ds$$

$$+ \int_t^T \Phi(t,s)\, Q_1(s)\hat{x}(s|t)\, ds - \int_t^T \Phi(t,s)\, Q_1(s)\hat{y}(s-h|t)\, ds. \tag{A10}$$

But, for $s \geqslant t$:

$$\begin{cases} \hat{x}(s|t) = X(s,t)\hat{x}(t|t) - \displaystyle\int_t^s X(s,\sigma) B(\sigma)\, Q_2(\sigma)^{-1} B(\sigma)^* \tilde{\omega}(\sigma|t)\, d\sigma \\ \hat{y}(s|t) = Y(s,t)\hat{y}(t|t). \end{cases}$$

Hence, using (A10):

$$\tilde{\omega}(t|t) = \int_t^T \Phi(t,s)\, Q_1(s) X(s,t)\, ds\, \hat{x}(t|t)$$

$$- \int_t^{T-h} \Phi(t,s+h)\, Q_1(s+h)\, Y(s,t)\, ds\, \hat{y}(t|t)$$

$$- \int_{t-h}^{t} \Phi(t,s+h)\, Q_1(s+h)\hat{y}(s|t)\, ds$$

$$- \int_t^T \Phi(t,s)\, Q_1(s) \int_t^s X(s,\sigma) B(\sigma)\, Q_2(\sigma)^{-1} B(\sigma)^* \tilde{\omega}(\sigma|t)\, d\sigma\, ds$$

$$+ \int_t^T \Phi(t,s) P_{xx}(s) B(s)\, Q_2(s)^{-1} B(s)^* \tilde{\omega}(s|t)\, ds \qquad \text{for} \quad t \leqslant T-h.$$

For $t > T-h$ the second term will be missing and the upper limit of the third term will be $T-h$ instead of t.

Here $\int_t^T \Phi(t,s)\,Q_1(s)\,X(s,t)\,ds = P_{xx}(t)$, and *defining* $P_{xy}(t)$ in the following way:

$$\begin{cases} P_{xy}(t) = -\int_t^{T-h} \Phi(t,s+h)\,Q_1(s+h)\,Y(s,t)\,ds & \text{for} \quad t \leqslant T-h \\ P_{xy}(t) = 0 \quad \text{for} \quad t > T-h \end{cases} \tag{A11}$$

or:

$$\begin{cases} \dfrac{dP_{xy}}{dt} = \Phi(t,t+h)\,Q_1(t+h) + P_{xx}\,BQ_2^{-1}\,B^*\,P_{xy} - A^*\,P_{xy} - P_{xy}\,C \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t \leqslant T-h \\ P_{xy}(t) = 0 \quad \text{for} \quad t > T-h. \end{cases} \tag{A12}$$

Then we will have:

$$\omega(t) = \tilde{\omega}(t\,|\,t) = P_{xx}(t)\,\hat{x}(t\,|\,t) + P_{xy}(t)\,\hat{y}(t\,|\,t) - g(t) \tag{A13}$$

where:

$$\begin{aligned} g(t) = & \int_{t-h}^{\min(t,\,T-h)} \Phi(t,s+h)\,Q_1(s+h)\,\hat{y}(s\,|\,t)\,ds \\ & + \int_t^T \Phi(t,s)\,P_{xx}(s)\,B(s)\,Q_2(s)^{-1}\,B(s)^*\,\tilde{\omega}(s\,|\,t)\,ds \\ & - \int_t^T \Phi(t,\sigma) \underbrace{\int_\sigma^T \Phi(\sigma,s)\,Q_1(s)\,X(s,\sigma)\,ds}_{P_{xx}(\sigma)} B(\sigma)\,Q_2(\sigma)^{-1}\,B(\sigma)^*\,\tilde{\omega}(\sigma\,|\,t)\,d\sigma \end{aligned}$$

where we have changed the order of integration in the last term. Since the last two terms cancel out we have:

$$g(t) = \int_{t-h}^{\min(t,\,T-h)} \Phi(t,s+h)\,Q_1(s+h)\,\hat{y}(s\,|\,t)\,ds. \tag{A14}$$

These are exactly the equations received in the paper (putting $D_1 = D_3 = I$ and $Q_3 = 0$).

A rigorous proof using these ideas would as well prove the Separation Theorem.

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