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# RECENT TRENDS IN STOCHASTIC REALIZATION THEORY

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This is a survey of some recent work on Markovian representation of multivariate stationary stochastic processes. First a geometric theory in Hilbert space is developed. Next these results are translated into a Hardy space setting, and a complete set of Markov models are constructed for the given process. These models are then analyzed from a systems theoretical point of view.

## 1. Introduction

Hilbert space methods for analysis of stochastic processes have a long tradition going back to Kolmogorov [30,31,29,69,46-49,21,55,24,10]. Such an approach was taken in the important work by Masani and Wiener [69,46], as well as in [21,55], in which prediction theory for stationary processes [70] was extended to the multivariate case. A prominent feature of this body of work is the utilization of Hardy space theory, a convenient tool in spectral analysis of stochastic processes since it properly represents the underlying time structure.

Selected results from this rich mathematical theory of prediction found their application in systems engineering and were redeveloped in the particular traditions of this field [26,27], geared toward specific applications rather than mathematical completeness. In more recent years, however, there has been a tendency in stochastic systems theory to return to its original source of inspiration for methods of analysis.

The following object is a prototype of a linear stochastic system,

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having been studied extensively in the systems sciences. Let $\{x(t);t \in Z\}$ and $\{y(t);t \in Z\}$ be two purely nondeterministic stationary stochastic vector processes of dimensions $n$ and $m$ respectively satisfying the recursion

$$x(t+1) = Ax(t) + Bu(t+1) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

on all of $Z$, where $A,B,C$ are real matrices of dimensions $n\times n$, $n\times k$ and $m\times n$ respectively, and $\{u(t);t \in Z\}$ is a k-dimensional white noise sequence, i.e. $E\{u(s)u(t)'\} = I\delta_{st}$. (Here prime denotes transpose, and $\delta_{st}$ is the Kronecker symbol.) If, in addition, we take $x$ to be nondegenerate ($E\{x(t)x(t)'\}$ positive definite), $A$ must be a stability matrix, i.e. it must have all its eigenvalues inside the unit circle. The process $x$ is called the state process of the system (1) and $y$ the output process. If $u$ is Gaussian, the state process is a vector Markov process, and (1) is a Markovian representation of $y$.

The usefulness of such models in the analysis of random phenomena in engineering has lead to the study of the inverse problem: Given the process $y$, find Markovian representations (1). This is a version of the stochastic realization problem [3,14,67,33-45,53,58-63,66,68,9,17]. If $y$ has a rational spectral density, one model (1) can be easily obtained from the innovation representation of $y$, but there are others, the solution not being unique. If $y$ does not have a rational spectral density, $y$ cannot be represented by means of a finite-dimensional state process, but the problem can still make sense if we allow infinite-dimensional state processes. In order to accommodate this situation and also remove the trivial distinction between models which can be obtained from each other by coordinate-transformations in the range space of the state process, it is convenient to formulate the problem as a geometric problem in Hilbert space.

Let $\{y(t);t \in Z\}$ be a stationary m-dimensional Gaussian process which is purely nondeterministic of full rank and centered, and let $H$ be the Gaussian space [52] generated by its components. Then $H$ is a Hilbert space with inner product $\langle\xi,\eta\rangle = E\{\xi\eta\}$. Since $y$ is stationary, there is a unitary map $U: H \to H$ which acts as the shift, i.e. $y_k(t+1) = Uy_k(t)$ for all $t \in Z$ and $k = 1,2,\dots,m$ [57]. Because of the full rank assumption this shift has multiplicity $m$ [65; p.5]. A (closed) subspace $X$ of $H$ is said to be Markovian if

$$\langle E^X\alpha, E^X\beta\rangle = \langle\alpha,\beta\rangle \quad \text{for all } \alpha \in X^-,\ \beta \in X^+, \tag{2}$$

where $X^- := \bigvee_{t<0} U^t X$, $X^+ := \bigvee_{t\geq 0} U^t X$, and $E^X$ denotes the orthogonal projection onto $X$. Then, since $H$ is a Gaussian space, $X^-$ and $X^+$ are conditionally independent given $X$. A Markovian subspace $X$ will be called a state space for $y$ if

$$y_k(0) \in X \quad \text{for} \quad k = 1,2,\dots,m. \tag{3}$$

A state space is minimal if it contains no other state space as a proper subspace.

A first step toward developing a state space theory for the stochastic realization problem was taken by Akaike [1,2] and Picci [54]. These partial results were further elaborated upon in Rozanov [56]. A comprehensive theory can be found in Lindquist and Picci [37-44], Lindquist, Picci and Ruckebusch [45], and Ruckebusch [59-63], where complete characterizations of all minimal state spaces are given, and in Lindquist and Pavon [34,35], where further aspects of realization of discrete-time processes are studied. In this regard, also see Caines and Delchamps [9] and Frazho [17].

The purpose of this paper is to provide a survey of some of these results. Basically, however, we shall follow the approach developed in [40-44,34,35], and results will mostly be stated without proofs. The construction of Markovian representations consists of three steps: (i) determining all state spaces, (ii) analysis and characterization of the state spaces in terms of Hardy functions, and (iii) construction of models of type (1). As pointed out to us by S.K. Mitter, step (ii) is similar to the construction in Lax-Phillips scattering theory [32], and some interesting parallels can be drawn, although we do not yet completely understand the physical significance of this analogy. (For an early study of the connection between Wold decomposition and the fundamental representation theorem of Lax and Phillips, we refer to Masani and Robertson [47] and Masani [50].) Especially step (i), but also step (ii), has been studied from a somewhat different angle by Ruckebusch [59-63], and there has been important cross-fertilization between the approaches, as we shall see in this paper.

Several versions of the stochastic realization problem can be formulated both for continuous-time and discrete-time processes. In order to make the basic ideas the main thing and not unnecessarily obscure the mathematics, we have chosen the simplest possible formulation which still contains the main features of the problem. Consequently we consider a multivariate process $y$ rather than a scalar one. To the student of the work by Masani and Wiener [46,69] it should come as no surprise that the multivariate case leads to a much richer theory. For other formulations we refer the reader to [45,63,68].

## 2. Splitting Subspaces

Define two subspaces of $H$, the past space $H^-$ and the future space $H^+$, in the following way. Let $H^-$ be the closed linear hull of $\{y_k(t); t \le 0, k = 1,2,\dots,m\}$ and $H^+$ that of $\{y_k(t); t \ge 0, k = 1,2,\dots,m\}$. A subspace $X$ of $H$ is a splitting subspace if

$$\langle E^X\alpha, E^X\beta\rangle = \langle\alpha,\beta\rangle \quad \text{for all} \quad \alpha \in H^-, \beta \in H^+ . \tag{4}$$

This is a concept originally introduced by McKean [51], but here used in a modified way. A splitting subspace is said to be minimal if it contains no other splitting subspace as a proper subspace. Since (3) implies that $H^- \subset X^-$ and $H^+ \subset X^+$, a state space must be a splitting subspace, but the converse is not true. However, it follows from the definition (4) that, if $\eta \in H^- \cap H^+$, $\|\eta - E^X\eta\| = 0$, i.e. $\eta \in X$, and consequently (3) holds. Therefore a Markovian splitting subspace is the same as a state space, and we shall use these terms interchangeably.

Two subspaces $A$ and $B$ are said to intersect perpendicularly if $\overline{E}^A B = \overline{E}^B A = A \cap B$ (where the bar over the $E$ stands for closure). When $A \vee B = H$, this happens if and only if $A^\perp \subset B$ or, equivalently, $B^\perp \subset A$ [41].

Theorem 1. [41]. A subspace $X$ is a splitting subspace if and only if

$$X = S \cap \bar{S} \tag{5}$$

for some pair $(S,\bar{S})$ of perpendicularly intersecting subspaces such that $H^- \subset S$ and $H^+ \subset \bar{S}$. The correspondence $X \leftrightarrow (S,\bar{S})$ is one-one, $S$ and $\bar{S}$ being given by $S = H^- \vee X$ and $\bar{S} = H^+ \vee X$. Moreover, $X$ is Markovian if and only if

$$U^{-1}S \subset S \tag{6a}$$

$$U\bar{S} \subset \bar{S} \tag{6b}$$

We shall occasionally write $X \sim (S,\bar{S})$ to exhibit the correspondence between $X$ and $(S,\bar{S})$. In view of the definition of perpendicular intersection we have

Corollary 1. In Theorem 1, (5) can be replaced by $X = \overline{E}^S\bar{S}$ or $X = \overline{E}^{\bar{S}}S$.

It is not hard to see that for any two subspaces $A$ and $B$,

$$A = \overline{E}^A B \oplus (A \cap B^\perp) \tag{7}$$

where $\oplus$ denotes orthogonal direct sum and the superscript $\perp$ orthogonal complement in $H$. Then Corollary 1 and the fact that perpendicular intersection of $S$ and $\bar{S}$ is equivalent to $\bar{S}^{\perp} \subset S$ yield

Corollary 2. A subspace $X$ is a splitting subspace if and only if there are subspaces $S \supset H^{-}$ and $\bar{S} \supset H^{+}$ such that

$$H = S^{\perp} \oplus X \oplus \bar{S}^{\perp} . \tag{8}$$

The pair $(S,\bar{S})$ is the same as in Theorem 1.

Equation (8) is analogous to the decomposition in terms of incoming and outgoing subspaces in Lax-Phillips scattering theory [32]: $\bar{S}^{\perp}$ corresponds to the ingoing and $S^{\perp}$ to the outgoing subspace.

Hence, for any splitting subspace $X \sim (S,\bar{S})$, $\bar{S}$ contain both $H^{+}$ and $S^{\perp}$, i.e. $\bar{S} \supset H^{+} \vee S^{\perp}$. Similarly we must also have $S \supset H^{-} \vee \bar{S}^{\perp}$. Now equation (5) seems to suggest that, to obtain a minimal splitting subspace, $S$ and $\bar{S}$ should be reduced as far as possible without violating these conditions on $S$ and $\bar{S}$.

Theorem 2. [41]. Let $X \sim (S,\bar{S})$ be a splitting subspace, and define $\bar{S}_0 = H^{+} \vee S^{\perp}$ and $S_0 = H^{-} \vee \bar{S}_0^{\perp}$. Then $S_0$ and $\bar{S}_0$ intersect perpendicularly and $X_0 := S_0 \cap \bar{S}_0$ is a minimal splitting subspace. Moreover, if $X$ is Markovian, so is $X_0$.

The following corollaries are immediate.

Corollary 3. A splitting subspace $X \sim (S,\bar{S})$ is minimal if and only

$$\bar{S} = H^{+} \vee S^{\perp} \tag{9a}$$

$$S = H^{-} \vee \bar{S}^{\perp} . \tag{9b}$$

The next corollary insures that all minimality conditions remain the same when the class of splitting subspaces is restricted by imposing the Markov conditions (6).

Corollary 4. A minimal state space is a minimal splitting subspace.

The existence of minimal splitting subspaces is insured by

Corollary 5. Each (Markovian) splitting subspace contains a minimal (Markovian) splitting subspace.

On each splitting subspace $X$ we can define a restricted shift $U(X)\colon X \to X$ by $U(X)\xi = E^{X}U\xi$. In view of (8), the following proposition is an immediate consequence of Lemma 0 in Sarason [64], as can be seen by first considering the adjoint of (10).

Proposition 1. Let $X$ be a splitting subspace. Then $X$ is Markovian if and only if, for each $\xi \in X$,

$$E^X U^k \xi = U(X)^k \xi \tag{10}$$

for $k = 0,1,2,\dots$

## 3. Observability and Constructibility

Borrowing the terminology from deterministic systems theory [28; p. 52], we say that an element $\xi$ of a splitting subspace $X$ is unobservable if $\xi \perp H^+$ and unconstructible if $\xi \perp H^-$. Then (7) provides us with a decomposition of $X$ into the direct sum of an observable and an unobservable subspace, i.e..

$$X = \overline{E}^X H^+ \oplus [X \cap (H^+)^\perp]. \tag{11a}$$

We shall say that $X$ is (completely) observable if $X \cap (H^+)^\perp = 0$ or, equivalently, the operator $\mathcal{O}$: $H^+ \to X$ given by $\mathcal{O}\xi = E^X\xi$ has dense range. Similarly we have

$$X = \overline{E}^X H^- \oplus [X \cap (H^-)^\perp] \tag{11b}$$

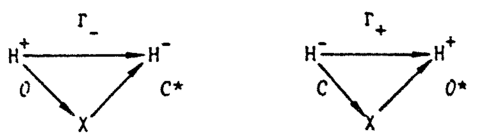
and we say that $X$ is (completely) constructible if $X \cap (H^-)^\perp = 0$ or, which is the same thing, the operator $C$ : $H^- \to X$ has full range, where $C\xi = E^X\xi$.

The following theorem is a consequence of decomposition (8).

Theorem 3. [40]. A splitting subspace is minimal if and only if it is both observable and constructible.

Ruckebusch, who was the first to use the terms observable and constructible in the sense described above, proved a version of this corollary in [61]. (Also see [45].)

Consider the Hankel operators $\Gamma_- := E^{H^-}|_{H^+}$ and $\Gamma_+ := E^{H^+}|_{H^-}$. It can be shown [42,63,43] that these operators can be factored through a subspace $X$ so that the diagrams




(12)

commute if and only if $X$ is a splitting subspace. (The two diagrams are equivalent, one being the dual of the other; $\mathcal{O}^* := E^{H^+}|_X$ and $C^* := E^{H^-}|_X$

are the adjoints of $\mathcal{O}$ and $\mathcal{C}$ respectively.) A diagram such as (12) is said to be canonical if the first factor (here $\mathcal{C}$ or $\mathcal{O}$) maps onto a dense subset of $X$ and the second factor (here $\mathcal{C}^*$ or $\mathcal{O}^*$) is one-one. Since $\ker \mathcal{O}^* = X \cap (H^+)^\perp$ and $\ker \mathcal{C}^* = X \cap (H^-)^\perp$, we have the following result.

Corollary 7. [43]. A subspace $X$ is a splitting subspace if and only if the two equivalent diagrams (12) commute and a minimal splitting subspace if and only if they are canonical.

4. The Predictor Spaces

Define the predictor space $X_- := \bar{E}^{H^-} H^+$ and the backward predictor space $X_+ := \bar{E}^{H^+} H^-$. (The reader is cautioned not to confuse these spaces with $X^-$ and $X^+$ defined in the introduction.) Then, by (7), we have the decompositions $H^- = X \oplus N^-$ and $H^+ = X_+ \oplus N^+$, where $N^- := H^- \cap (H^+)^\perp$ and $N^+ := H^+ \cap (H^-)^\perp$. Therefore, in view of Corollary 2, $X_- \sim (S_-, \bar{S}_-)$ and $X_+ \sim (S_+, \bar{S}_+)$ are splitting subspaces, and $S_- = H^-$, $\bar{S}_- = (N^-)^\perp$, $S_+ = (N^+)^\perp$ and $\bar{S}_+ = H^+$. Moreover, both $X_-$ and $X_+$ satisfy (6) and (9), so consequently they are minimal Markovian splitting subspaces.

Since $X_- \perp N^+$ and $X_+ \perp N^-$, we have the decomposition

$$H = N^- \oplus H^\square \oplus N^+, \tag{13}$$

where $H^\square$ is the frame space $H^\square := X_- \vee X_+$. Clearly $H^\square$ is a Markovian splitting subspace, but it is in general not minimal.

Proposition 2. [38]. Let $X$ be a minimal splitting subspace. Then

$$H^- \cap H^+ \subset X \subset H^\square. \tag{14}$$

Hence $H^\square$ is the closed linear hull of all minimal splitting subspaces and is therefore the only part of $H$ needed in constructing minimal splitting subspaces. It is not hard to see that an observable splitting subspace is perpendicular to $N^-$ and a constructible one is perpendicular to $N^+$. We shall say that $y$ (or, more correctly, $H^\square$) is noncyclic if $N^-$ and $N^+$ are nontrivial, i.e. $N^- \neq 0$ and $N^+ \neq 0$, and strictly noncyclic if they are both full range. Then $H^\square$ does not contain all of $H^-$ or all of $H^+$, so there is nontrivial data reduction. The frame space is finite-dimensional if and only if $y$ has a rational spectral density [40], in which case it is always strictly noncyclic.

The following proposition describes the role of $X_-$ and $X_+$ in state estimation. A weaker version of this result can be found in [61].

Proposition 3. [42,43]. Let $X$ be a splitting subspace. Then $\bar{E}^{H^-}X = X_-$ if and only if $X \perp N^-$, and $\bar{E}^{H^+}X = X_+$ if and only if $X \perp N^+$.

The other minimal state spaces can be regarded as generalized predictor spaces, as can be seen from the following theorem.

Theorem 4. [38,44]. Let $y$ be strictly noncyclic. Then a subspace $X$ is a minimal state space if and only if

$$X = \bar{E}^S H^+ \tag{15a}$$

for some $S$ satisfying (6a) and

$$S_- \subset S \subset S_+ . \tag{15b}$$

From Corollary 1 and Theorem 3 it is not hard to see that, if the condition $S \subset S_+$ is removed, (15) is equivalent to $X$ being an observable splitting subspace (and for this we need not have $y$ strictly noncyclic). The minimality, however, is much harder to prove; we refer the reader to [44] for the proof. Theorem 4 has a symmetric counterpart in which (15a) is exchanged for $X = \bar{E}^{\bar{S}}H^-$, (15b) for $\bar{S}_+ \subset \bar{S} \subset \bar{S}_-$, and (6a) for (6b).

Since $S = H^- \vee X$ (Theorem 1), Theorem 4 shows that there is a one-one correspondence between minimal state spaces $X$ and $U^{-1}$-invariant subspaces $S$ satisfying (15b). The partial ordering of these subspaces with respect to subspace inclusion induces a lattice structure on the family of minimal state spaces, which thus forms a complete lattice with minimum element $X_-$ and maximum element $X_+$.

## 5. Wold Decomposition for Proper State Spaces

A Markovian splitting subspace $X \sim (S,\bar{S})$ is said to be proper if $\cap_{t=-\infty}^{0} U^t S = 0$ and $\cap_{t=0}^{\infty} U^t \bar{S} = 0$.

Proposition 4. [40]. Let $y$ be strictly noncyclic. Then all state spaces $X \subset H^{\square}$, i.e. in particular the minimal ones, are proper.

Now, let $X \sim (S,\bar{S})$ be a proper state space. Since $X$ is Markovian, $U^{-1}S \subset S$ and $U\bar{S} \subset \bar{S}$. Then define the wandering subspaces $V := S \ominus U^{-1}S$ and $\bar{V} := \bar{S} \ominus U\bar{S}$. Since the shift $U$ has multiplicity $m$, both $V$ and $\bar{V}$ have dimension $m$ [65; p. 2], and it can be shown that $S = \oplus_{t=-\infty}^{0} U^t V$, $\bar{S} = \oplus_{t=0}^{\infty} U^t \bar{V}$, and $H = \oplus_{t=-\infty}^{\infty} U^t V = \oplus_{t=-\infty}^{\infty} U^t \bar{V}$. This is the well-known Wold decomposition; see e.g. [65]. (A continuous-time version of this decomposition was introduced by Masani [49].)

Next choose an orthonormal basis $\{v_1, v_2, \ldots, v_m\}$ in $V$ and an orthonormal basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m\}$ in $\bar{V}$, and, for each $t \in Z$, let $u(t)$

and $\bar{u}(t)$ be the m-dimensional random vectors with components $u_i(t) := U^t v_i$ and $\bar{u}_i(t) := U^t \bar{v}_i$ respectively, $i = 1,2,\ldots,m$. Then, since $U^s V \perp U^t V$ for all $s \neq t$, $\langle u_i(s), u_j(t)\rangle = \langle v_i, U^{t-s} v_j\rangle = \delta_{ij}\delta_{st}$, i.e.

$$E\{u(s)u(t)'\} = I\delta_{st} . \tag{16}$$

In the same way it is seen that also $\bar{u}$ satisfies (16). Therefore both $\{u(t);\ t \in Z\}$ and $\{\bar{u}(t);\ t \in Z\}$ are m-dimensional Gaussian white noise processes. For each such process $\{u(t);\ t \in Z\}$, we define $H(u)$ to be the Gaussian space generated by its components and $H^-(u)$ and $H^+(u)$ to be the subspaces of $H(u)$ corresponding to $\{u(t);\ t \leq 0\}$ and $\{u(t);\ t \geq 0\}$ respectively. Note that $H(u) = U^{-1}H^-(u) \oplus H^+(u)$.

Consequently, to each proper state space $X \sim (S,\bar{S})$, there corresponds a pair $(u,\bar{u})$ of white noise processes, called the generating processes of $X$, such that $S = H^-(u)$ and $\bar{S} = H^+(\bar{u})$. The pair $(u,\bar{u})$ is unique modulo trivial coordinate transformations in $V$ and $\bar{V}$. In particular, let $(u_-,\bar{u}_-)$ and $(u_+,\bar{u}_+)$ be the pairs of generating processes of $X_-$ and $X_+$ respectively. Then $H^-(u_-) = H^-$, i.e. $u_-$ is the innovation process, and $H^+(\bar{u}_+) = H^+$, i.e. $\bar{u}_+$ is the backward innovation process.

## 6. State Space Representation in Hardy Space

If $\{u(t);\ t \in Z\}$ is an m-dimensional white noise process such that $H = H(u)$, any $\eta \in H$ can be written

$$\eta = \sum_{t=-\infty}^{\infty} \langle f_{-t}, u(t)\rangle_{\mathbb{R}^m} , \tag{17}$$

where $\{f_t;\ t \in Z\}$ is an $\ell_2$-sequence of vectors in $\mathbb{R}^m$, and $\langle\cdot,\cdot\rangle_{\mathbb{R}^m}$ is the inner product in $\mathbb{R}^m$. It is well-known [57] that $u$ has the spectral representation

$$u(t) = \int_{-\pi}^{\pi} e^{i\omega t} d\hat{u}(\omega) \ ; \quad t \in Z \tag{18}$$

for a unique orthogonal stochastic vector measure $d\hat{u}$ with the property that $E\{d\hat{u}(\omega)d\hat{u}(\omega)^*\} = (2\pi)^{-1} I d\omega$. (Asterisk denotes transpose and conjugation.) Equations (17) and (18) then yield

$$\eta = \int_{-\pi}^{\pi} \langle f(e^{i\omega}), d\hat{u}(\omega)\rangle_{\mathbb{R}^m} \tag{19}$$

where

$$f(z) = \sum_{-\infty}^{\infty} f_k z^{-k} . \tag{20}$$

Here $f$ belongs to the space $L_2(T)$ of all m-dimensional vector functions square-integrable on the unit circle $T$ with respect to the Lebesgue measure $(2\pi)^{-1}d\omega$. Equation (19) defines an isometric isomorphism between $H$ and $L_2(T)$ [57]. Let $T_u : H \to L_2(T)$ be the map $T_u\eta = f$. Then $T_u$ is unitary, and $T_u^*f = \eta$.

Next define the Hardy spaces $H_2^+$ and $H_2^-$ in the following way: Let $H_2^+$ be the space of all functions (20) in $L_2(T)$ such that $f_k = 0$ for $k > 0$, and let $H_2^-$ be the functions in $L_2(T)$ for which $f_k = 0$ for $k < 0$. These Hardy functions can be extended to the complex plane, the functions in $H_2^+$ being analytic inside the unit circle and those in $H_2^-$ outside $T$ [18,22,65]. Then it follows from (17) and (19) that $T_uH^-(u) = H_2^-$ and $T_uH^+(u) = H_2^+$. Moreover, $T_uUT_u^* = z$. (We shall use $z$ to denote both $e^{i\omega}$ and the corresponding multiplication operator.) Let $L_\infty(T)$ be the space of bounded $m\times m$-matrix functions on $T$, and let $H_\infty^+$ and $H_\infty^-$ be the subspaces in $L_\infty(T)$ of functions whose columns are in $H_2^+$ and $H_2^-$ respectively. A function $F \in H_\infty^-$ will be called inner if $F(e^{i\omega})$ is a unitary matrix for each $\omega \in \mathbb{R}$. A similar function in $H_2^+$ will be called conjugate inner. (This terminology is somewhat nonstandard, the two concepts having been interchanged for later convenience.) If $F$ is conjugate inner, $F^*$ is inner. For any $F \in L_\infty(T)$, let $M_F: L_2(T) \to L_2(T)$ be the multiplication operator $M_Ff = Ff$. Then, if $F$ is inner, $M_K$ maps $H_2^-$ into $H_2^-$, and, if $F$ is conjugate inner, it maps $H_2^+$ into $H_2^+$.

<u>Lemma 1</u>. [43]. Let $X$ be a proper Markovian splitting subspace with generating processes $(u,\bar{u})$. Then there are inner functions $K$ and $Q$ and a conjugate inner function $\bar{Q}$ such that

$$z^{-1}T_uT_{\bar{u}}^* = M_K \tag{21a}$$

$$T_uT_{u_-}^* = M_Q \tag{21b}$$

$$T_{\bar{u}}T_{\bar{u}_+}^* = M_{\bar{Q}} . \tag{21c}$$

The functions $K$, $Q$ and $\bar{Q}$ are unique modulo multiplication by a constant unitary matrix.

For the sake of continuity in exposition we shall give the proof which is based on a standard technique in Hardy space theory [18,22,32].

<u>Proof</u>. Since $S$ and $\bar{S}$ intersect perpendicularly, $\bar{S}^\perp \subset S$, i.e. $U^{-1}H^-(\bar{u}) \subset H^-(u)$. Applying $T_u$ to this we obtain $z^{-1}T_uH^-(\bar{u}) \subset H_2^-$, i.e. $z^{-1}T_uT_{\bar{u}}^*H_2^- \subset H_2^-$. Hence $z^{-1}T_uT_{\bar{u}}^*$ maps $H_2^-$ into $H_2^-$. Moreover,

$T_u T_{\bar{u}}^* z^{-1} = T_u U^{-1} T_{\bar{u}}^* = z^{-1} T_u T_{\bar{u}}^*$, and therefore $z^{-1} T_u T_{\bar{u}}^*$ commutes with the shift on $H_2^-$. Consequently there is a $K \in H_\infty^-$ such that (21a) holds [18; p. 185]. Since $z^{-1} T_u T_u^*$ is unitary, $K$ must be inner. The proofs of (21b) and (21c) follow the same pattern using $H^- \subset S$ and $H^+ \subset \bar{S}$ respectively. □

By Corollary 2, any state space $X \sim (S, \bar{S})$ can be written $X = S \ominus \bar{S}^\perp$. Then, if $X$ is proper, $X = H^-(u) \ominus U^{-1} H^-(\bar{u})$, and hence $T_u X = H_2^- \ominus (K H_2^-) =: H(K)$. Moreover, (9a) can be written $H^+(\bar{u}) = H^+(\bar{u}_+) \vee U H^+(u)$, the isomorphic image of which under $T_{\bar{u}}$ is $H_2^+ = (\bar{Q} H_2^+) \vee (K^* H_2^+)$. This holds if and only if $(K^*, \bar{Q})_L = I$, i.e. $K^*$ and $\bar{Q}$ are left coprime, or, equivalently, $(K, \bar{Q}^*)_R = I$, i.e. $K$ and $\bar{Q}^*$ are right coprime [18, 22]. Similarly, it can be seen that (9b) is equivalent to $(K, Q)_L = I$.

Theorem 5. [40,43]. Let $X$ be a proper Markovian splitting subspace with generating processes $(u, \bar{u})$. Let $K$, $Q$ and $\bar{Q}^*$ be the inner functions defined in Lemma 1, and let $H(K)$ be the orthogonal complement of $H_2^- K$ in $H_2^-$. Then

$$X = T_u^* H(K), \tag{22}$$

and $X$ is observable if and only if $(K, \bar{Q}^*)_R = I$ and constructible if and only if $(K, Q)_L = I$. Moreover $d\hat{u} = Q^* d\hat{u}_-$ and $d\hat{\bar{u}} = \bar{Q}^* d\hat{\bar{u}}_+$, where $u_-$ and $\bar{u}_+$ are the forward and backward innovation processes respectively.

The inner function $K$ is called the structural function of $X$. It plays the same role as the scattering matrix in Lax-Phillips scattering theory [32]. By symmetry, (22) can be replaced by

$$X = T_{\bar{u}}^* \bar{H}(K^*), \tag{23}$$

where $\bar{H}(K^*) := H_2^+ \ominus (K^* H_2^+)$.

## 7. Spectral Factors

We have seen that any proper state space $X$ is characterized by a triplet $(K, Q, \bar{Q}^*)$ of inner functions, defined in Lemma 1. It remains to find a procedure to determine these inner functions.

To each white noise process $\{u(t);\ t \in Z\}$ such that $H(u) = H$ there corresponds a unique $m \times m$-matrix function $W$ whose columns are $T_u y_k(0)$, $k = 1, 2, \ldots, m$. Then

$$y(t) = \int_{-\pi}^{\pi} e^{i\omega t} W(e^{i\omega})' d\hat{u}(\omega), \tag{24}$$

and consequently

$$W(z)' W(\tfrac{1}{z}) = \Phi(z), \tag{25}$$

where $\Phi$ is the spectral density of $y$. By the full-rank assumption, rank $\Phi = m$. Therefore $W$ is an $m\times m$ spectral factor of $y$. Conversely, since $y$ is purely nondeterministic, it has a spectral representation

$$y(t) = \int_{-\pi}^{\pi} e^{i\omega t} d\hat{y}(\omega), \tag{26}$$

where $d\hat{y}$ is an orthogonal stochastic measure such that $E\{d\hat{y}(\omega)d\hat{y}(\omega)^*\} = \Phi(e^{i\omega})d\omega$ [57]. Then, via (18) and $d\hat{u} = (W')^{-1}d\hat{y}$, any $m\times m$ spectral factor $W$ defines an m-dimensional white noise process $u$ such that $H(u) = H$. If $u_1$ and $u_2$ are two such processes, we have

$$T_{u_1} T^*_{u_2} = M_{W_1 W_2^{-1}} \quad . \tag{27}$$

Therefore, to each proper state space $X$ there corresponds a pair $(W,\overline{W})$ of $m\times m$ spectral factors (unique modulo multiplication from the left by a constant unitary matrix) constructed as above from the generating processes $(u,\overline{u})$ of $X$. Since, for $k = 1,2,\ldots,m$, $y_k(0) \in H^- \subset H^-(u)$, the columns of $W$ belong to $H_2^-$. Such a spectral factor is called stable. In the same way we show that the columns of $\overline{W}$ belong to $H_2^+$, i.e. $\overline{W}$ is strictly unstable. Moreover, in view of Lemma 1 and (27), the structural function of $X$ is given by

$$K = z^{-1} W \overline{W}^{-1} , \tag{28}$$

and

$$W = QW_- \tag{29a}$$

$$\overline{W} = \overline{Q}\overline{W}_+ , \tag{29b}$$

where $W_-$ corresponds to the innovation process $u_-$ and $\overline{W}_+$ to the backward innovation process $\overline{u}_+$. Equations (29) are the inner-outer factorizations of $W$ and $\overline{W}$ [18,22]. Hence we call $Q$ the inner factor of $W$ and $\overline{Q}$ the conjugate inner factor of $\overline{W}$.

Corollary 8. [40]. A subspace $X$ is a proper Markovian splitting subspace if and only if

$$X = \int_{-\pi}^{\pi} H(K)(W')^{-1} d\hat{y} \tag{30}$$

for some pair $(W,\overline{W})$ of $m\times m$ spectral factors such that $W$ is stable, $\overline{W}$ is strictly unstable, and $K := z^{-1}W\overline{W}^{-1}$ is inner.

In view of Theorem 4, we say that a stable [strictly unstable] spectral factor is minimal if the corresponding $u$ [$\bar{u}$] satisfies $H^-(u) \subset S_+$ [$H^+(\bar{u}) \subset \bar{S}_-$]; cf [60]. Let $Q_+$ be the inner factor of $W_+$ and $\bar{Q}_-$ the conjugate inner factor of $\bar{W}_-$.

Proposition 5. [60,40]. A stable spectral factor $W$ is minimal if and only if its inner factor $Q$ is a right inner divisor of $Q_+$. Similarly, a strictly unstable spectral factor $\bar{W}$ is minimal if and only if its conjugate inner factor $\bar{Q}$ is a right conjugate inner divisor of $\bar{Q}_-$.

If the state space $X$ is minimal, the spectral factors $(W,\bar{W})$ in Corollary 8 both have to be minimal, but the converse is not true. (The only thing we can say is that a pair $(W,\bar{W})$ of minimal spectral factors with the properties described in Corollary 8 defines a state space $X$ contained in the frame space.) The following corollary is a Hardy space version of Theorem 4.

Corollary 9. [60,40,43]. There is a one-one correspondence between minimal state spaces $X$ and minimal stable spectral factors $W$. Under this correspondence

$$T_u X = P^{H_2^-} W \bar{W}_+^{-1} H_2^+ \tag{31}$$

where $P^{H_2^-}$ is the orthogonal projection onto $H_2^-$ and $u$ is the white noise process corresponding to $W$.

A theorem equivalent to Corollary 9 was first stated for the scalar case in [60], but the proof is incomplete, as is the corresponding vector result in [40]. A complete proof presented in [43] is based on [44]. However, whereas the vector case is decidedly nontrivial, the scalar result [60] is easy to fix up using a theorem due to Douglas, Shapiro and Shields [11]; see [44]. A vector version [12] of the last mentioned theorem can be used to see that (31) is the same as $H(K)$. See [40,43,44] for details.

## 8. Realizations

Let $X$ be a proper state space with generating processes $(u,\bar{u})$, and let $(W,\bar{W})$ be the corresponding spectral factors. Next we shall construct a model of type (1) for $X$. This construction is presented in [34,35] and follows the pattern of [42,43].

To this end, first note that, since $KH_2^-$ is a $z^{-1}$-invariant subspace of $H_2^-$,

$$P^{H(K)} z^{-t} h = A^t P^{H(K)} h \; ; \; t = 0,1,2,\ldots \tag{32}$$

for all $h \in H_2^-$, where $P^{H(K)}$ is the orthogonal projection onto the subspace

$H(K)$, and $A : H(K) \to H(K)$ is the backward restricted shift $Af = P^{H(K)}z^{-1}f$. Note that $A$ corresponds to $U(X)^*$ under the isomorphism $T_u$. Now, each $f \in H(K)$ has a representation $f(z) = \Sigma_{k=0}^{\infty} f_k z^{-k}$ on (and outside) the unit circle, in which, for any $b \in \mathbb{R}^m$, the Fourier coefficients $\langle f_k, b\rangle_{\mathbb{R}^m}$ equal $\langle f, z^{-k}b\rangle_{L_2(T)}$, $k = 0,1,2,\ldots$ . Hence, since $f \in H(K)$, we have $\langle f_k, b\rangle_{\mathbb{R}^m} = \langle f, P^{H(K)}z^{-k}b\rangle_{H(K)}$, which, in view of (32), yields

$$\langle f_k, b\rangle_{\mathbb{R}^m} = \langle f, A^k Bb\rangle_{H(K)}, \tag{33}$$

where $B: \mathbb{R}^m \to H(K)$ is defined by $Bb = P^{H(K)}b$.

The stable spectral factor $W$ has the representation

$$W(z) = \sum_{k=0}^{\infty} W_k z^{-k} \tag{34}$$

on $T$, and, since $\langle a, y(0)\rangle_{\mathbb{R}^m} \in X$ for all $a \in \mathbb{R}^m$, $Wa \in H(K)$. Therefore (33) can be applied to yield

$$\langle W_k a, b\rangle_{\mathbb{R}^m} = \langle a, CA^k Bb\rangle_{\mathbb{R}^m} \quad \text{for all } a,b \in \mathbb{R}^m \tag{35}$$

where $C: H(K) \to \mathbb{R}^m$ is defined by $Cf = (2\pi)^{-1} \int_{-\pi}^{\pi} W(e^{-i\omega})' f(\omega) d\omega$. Consequently,

$$W_k' b = CA^k Bb \quad \text{for all } b \in \mathbb{R}^m \tag{36}$$

This representation is the adjoint of the restricted shift realization of Fuhrmann [19], Helton [23], Baras [7], and Baras and Brockett [6], and it was through discussions with Professor Baras we were led to apply this technique, first in our continuous-time paper [42].

Next we define the state process

$$x(t) = \sum_{k=-\infty}^{t} A^{t-k} Bu(k) . \tag{37}$$

Since this is a $H(K)$-valued process whose covariance operator is not nuclear, it cannot be defined as a random process in the usual sense (unless $\dim H(K) < \infty$), but only in the generalized sense described in [5]. However, all linear functionals $\langle f, x(t)\rangle_{H(K)}$ are well-defined random variables in the usual sense, and

$$X = \{\langle f, x(0)\rangle_{H(K)} \mid f \in H(K)\}, \tag{38}$$

justifying the term "state process." To see this, note that, if $\xi \in X$,

$\xi = \Sigma_{k=-\infty}^{0} \langle f_{-k}, u(k)\rangle_{\mathbb{R}^m}$ where $f := T_u \xi \in H(K)$ (Theorem 5), and therefore, by (33), $\xi = \langle f, x(0)\rangle_{H(K)}$. Then (38) is a consequence of (22).

Now since

$$y(t) = \sum_{k=-\infty}^{t} W'_{t-k} u(k), \tag{39}$$

we obtain the model

$$x(t+1) = Ax(t) + Bu(t+1) \tag{40a}$$

$$y(t) = Cx(t). \tag{40b}$$

It follows from Corollary 2 that

$$H = U^{-1}H^{-}(\bar{u}) \oplus X \oplus UH^{+}(u), \tag{41}$$

and therefore $X \perp UH^{+}(u)$. This characterizes the forward evolution property of (40), the two terms in the right member of (40a) being independent; we say that (40) is a forward system. Now consider the (deterministic) reachability operator $R: H_2^- \to H(K)$ defined by

$$Rf = \sum_{k=0}^{\infty} A^k B f_k \tag{42}$$

[18; p. 243]. This operator is well-defined and surjective. In fact, by (32), $Rf = \sum_{k=0}^{\infty} P^{H(K)} z^{-k} f_k = P^{H(K)} f$, for $P^{H(K)}$ is continuous. Consequently the system (40) is exactly reachable (in the sense of deterministic systems theory) [5, 18]. Moreover, for each $t \geq 0$, $\ker CA^t$ consists of those $f \in H(K)$ for which $\langle Wa, P^{H(K)} z^{-t} f\rangle_{L_2(T)} = 0$ for all $a \in \mathbb{R}^m$, as can be seen from (32). But, since $Wa \in H(K)$, these are precisely the functions $f \in H(K)$ such that $f \perp z^t Wa$ for all $a \in \mathbb{R}^m$. Therefore, since $z^t Wa = T_u \langle a, y(t)\rangle_{\mathbb{R}^m}$ and $H(K) = T_u X$, we have

$$\bigcap_{t=0}^{\infty} \ker CA^t = T_u[X \cap (H^+)^{\perp}]. \tag{43}$$

The system (40) is said to be observable (in the sense of deterministic systems theory) if $\cap_{t=0}^{\infty} \ker CA^t = 0$; see [5,18]. Then it follows from (43) that (40) is observable if and only if $X$ is observable (in the sense of Section 2). (Cf [63] where $A$ and $C$ are defined as operators on $H$.)

Now, by a completely symmetric argument, we can use $\bar{W}$ and $\bar{u}$ to obtain

$$\bar{x}(t-1) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t-1) \tag{44a}$$

$$y(t) = \bar{C}\bar{x}(t) \tag{44b}$$

where $\bar{A}$: $\bar{H}(K^*) \to \bar{H}(K^*)$, $\bar{B}$: $\mathbb{R}^m \to \bar{H}(K^*)$ and $\bar{C}$: $\bar{H}(K^*) \to \mathbb{R}^m$ are defined by $\bar{A}f = P^{\bar{H}(K^*)}\bar{z}f$, $\bar{B}b = P^{\bar{H}(K^*)}b$ and $\bar{C}f = (2\pi)^{-1}\int_{-\pi}^{\pi} \bar{W}(e^{-i\omega})'f(\omega)d\omega$. Moreover

$$X = \{\langle f,\bar{x}(0)\rangle_{\bar{H}(K^*)} \mid f \in \bar{H}(K^*)\}, \tag{45}$$

and, by (41), $X \perp U^{-1}\bar{H}^{-}(\bar{u})$; hence (44) is a backward system. Finally, just as in the forward setting, we show that (44) is exactly reachable and that it is observable if and only if $X$ is constructible. The operators $A$ and $\bar{A}$ will be called the (forward and the backward) system generators of $X$. We summarize these results in

Theorem 6. [34,35]. Let $X$ be a proper state space for $y$, and let $(u,\bar{u})$ be its generating processes. Then $y$ is the output of an exactly reachable forward system (40) and of an exactly reachable backward system (44), both having the property that the linear functionals of the state at time zero span $X$. The forward system is observable if and only if $X$ is observable, and the backward system is observable if and only if $X$ is constructible. Moreover, $A^t$ and $\bar{A}^t$ tend strongly to zero as $t \to \infty$.

For further details on the relations between the forward and the backward systems, see [42,43].

## 9. State Space Isomorphism

From now on we shall assume that $y$ is strictly noncyclic. Recall the definitions of $U(X)$ in Section 2.

Theorem 7. [44]. Let $X_1$ and $X_2$ be two minimal state spaces. Then $U(X_1)$ and $U(X_2)$ are quasi-similar, i.e. there are quasi-invertible (injective with dense range) maps $P$: $X_1 \to X_2$ and $R$: $X_2 \to X_1$ such that

$$PU(X_1) = U(X_2)P \tag{46a}$$

$$U(X_1)R = RU(X_2)\ . \tag{46b}$$

Corollary 10. Minimal state spaces have the same dimension.

In this context it is also interesting to note the following result (see, e.g. [18]).

Proposition 6. A state space is finite-dimensional if and only if its structural functions is rational.

Since $U(X)$ and $U(X)^*$ are quasi-similar for minimal $X$ [44], we also have

<u>Corollary 11</u>. [44]. Let $(A_1,\bar{A}_1)$ and $(A_2,\bar{A}_2)$ respectively be the system generators of two minimal state spaces, $X_1$ and $X_2$. Then, $A_1$, $\bar{A}_1$, $A_2$ and $\bar{A}_2$ are pairwise quasi-similar.

Consequently all system generators of minimal state spaces have the same Jordan form [18]. This is connected with the concept of quasi-equivalence: Let $K$ be an $m\times m$ inner function. Set $\gamma_0 = 1$, and, for $i = 1,2,\ldots,m$, define $\gamma_i$ to be the greatest common inner divisor of all $i\times i$ minors of $K$. Clearly $\gamma_{i-1}$ divides $\gamma_i$ so that $k_i := \gamma_i/\gamma_{i-1}$ is a scalar inner function for $i = 1,2,\ldots,m$. The functions $k_1,k_2,\ldots,k_m$ are the invariant factors of $K$. Two inner functions are quasi-equivalent if they have the same invariant factors. This is clearly an equivalence relation.

<u>Corollary 12</u>. [44]. Minimal state spaces have quasi-equivalent structural functions.

From this it follows that, if $y$ is a scalar process $(m=1)$, minimal state spaces have the same structural function. This simplifies the analysis considerably in the scalar case.

Other state-space isomorphism results can be found in [63].

## 10. <u>Degeneracy</u>

We shall say that a (strictly noncyclic) process $y$ is degenerate if $\ker U(H^{\square}) \neq 0$. By quasi-similarity, this is equivalent to $\ker U(H^{\square})^* \neq 0$ [44]. It is easy to see that the first condition can be written $(UH^{\square}) \cap N^+ \neq 0$ and the second $(U^{-1}H^{\square}) \cap N^- \neq 0$. Now recall that the frame space $H^{\square}$ is the closed linear hull of all minimal state spaces and that a state space element in $N^-$ $(N^+)$ is unobservable (unconstructible). Consequently $N^-$ and $N^+$ are the parts of $H$ that we normally want to discard in state space construction. Degeneracy of $y$ means that, if we shift one step forward or backward in time, some elements of these discarded spaces become part of the new frame space. A process $y$ which is not degenerate will be called nondegenerate.

Let $X$ be a minimal state space. Then, in view of Proposition 3, it is reasonable to call $Z := E^{(H^-)^\perp}X$ the forward (prediction) error space of $X$. (This space plays a prominent role in the approach taken by Ruckebusch [61-63].) Since $S = H^- \vee X$, $Z = E^{(H^-)^\perp}S$. But $(H^-)^\perp$ and $S$ intersect perpendicularly, because $H^- \subset S$ and $(H^-)^\perp \vee S = H$, and therefore $Z = S \ominus H^-$. Hence $T_u Z = \mathcal{H}(Q)$, and consequently, following the recipe of Section 8, we can construct a $H(Q)$-valued random process $\{z(t);\ t \in Z\}$ (defined in the generalized sense [5]), such that

$$Z = \{\langle f, z(0)\rangle_{H(Q)} \mid f \in H(Q)\}, \tag{47}$$

which satisfies

$$z(t+1) = Fz(t) + Gu(t+1), \tag{48}$$

where $F := P^{H(Q)}z^{-1}|_{H(Q)}$ and $G := P^{H(Q)}|_{\mathbb{R}^m}$. Similarly we define the backward (prediction) error space of $X$ to be $\bar{Z} := E^{(H^+)^\perp}X$. Then $\bar{Z} = \bar{S} \ominus H^+$, and therefore we can construct a $\bar{H}(\bar{Q})$-valued random process $\{\bar{z}(t);\ t \in Z\}$ such that the linear functionals of $\bar{z}(0)$ span $Z$ and

$$\bar{z}(t-1) = \bar{F}\bar{z}(t) + \bar{G}\bar{u}(t-1), \tag{49}$$

where $\bar{F} := P^{\bar{H}(\bar{Q})}z|_{H(\bar{Q})}$ and $\bar{G} := P^{\bar{H}(\bar{Q})}|_{\mathbb{R}^m}$. As before plus (+) and minus (-) subscripts will be used to denote quantities corresponding to $X_-$ and $X_+$ respectively.

Theorem 8. [34,35]. Let $X$ be a minimal state space, and let $Z$ and $\bar{Z}$ be it error spaces. Then $y$ is nondegenerate if and only if the two conditions

(i) $\ker U(X) = 0$

(ii) $\ker U(Z) = 0$ and $\ker U(\bar{Z}) = 0$

both hold. If conditions (i) holds for one $X$, it holds for all. If (ii) holds for one $X$, it holds for all. Condition (ii) is equivalent to (ii)' $\ker U(Z_+) = 0$ and to (ii)" $\ker U(\bar{Z}_-) = 0$.

Consequently conditions (i) and (ii) are properties of y, and if $y$ is degenerate either or both of them fail. The following corollaries illustrate the main purpose of the degeneracy analysis.

Corollary 13. [34,35]. Condition (i) is equivalent to the system generators of $X$ being quasi-invertible. If this holds for one minimal state space it holds for all.

Corollary 14. [34,35]. Condition (ii) fails (for any minimal $X$) if and only if $\ker F_+^* \neq 0$, or, equivalently, $\ker \bar{F}_-^* \neq 0$, or, equivalently, either $\ker F^* \neq 0$ or $\ker \bar{F}^* \neq 0$ for any minimal state space.

Let $f \in \ker F_+^*$. Then the scalar random process $\langle f, z_+(t)\rangle_{H(Q)} = \langle G_+^* f, u_+(t)\rangle_{\mathbb{R}^m}$ is a white noise. Such an $f$ is called an invariant direction in finite-dimensional stochastic systems theory [8,20,53]. This is an important concept in connection with the solution of matrix Riccati equations for Kalman filtering [26]. Corollary 14 states that there exist invariant directions if and only if condition (ii) fails. Hence, in particular, $y$ needs to be degenerate for this to happen.

Degeneracy of type (ii) plays a key rôle also in establishing correspondence between the family of minimal state spaces in our present setting and that obtained by shifting $H^-$ to $U^{-1}H^-$, the second formulation allowing a noise component in (1b); see [34,35].

Finally, the conditions of Theorem 8 can be expressed in terms of the behavior of the inner functions $(K,Q,\bar{Q}^*)$ at infinity. Note that, say, $K(\infty)$ is well defined being the constant term in the Laurent expansion of $K$.

Corollary 15. [34,35]. Let $K_\square$ be the structural function of $H^\square$. Then $y$ is degenerate if and only if the matrix $K_\square(\infty)$ is singular. Condition (i) holds if and only if $K(\infty)$ is nonsingular, (ii) if and only if both $Q(\infty)$ and $\bar{Q}(\infty)$ are nonsingular, (ii)' if and only if $Q_+(\infty)$ is nonsingular, and (ii)" if and only if $\bar{Q}_-(\infty)$ is nonsingular.

Degeneracy can also be characterized in terms of the spectral density $\Phi$ [34,35].

## 11. Concluding Remarks

In this paper we have considered a particular version of the stochastic realization problem. However, the results reported here can be applied to a wider class of stochastic realization problems, both in discrete and continuous time, with no or minor modifications, and we refer the reader to the cited literature for further discussion. Our object here has been to survey the basic ideas of the theory rather than present a wealth of detail.

One of the assumptions made in the beginning of the paper is that the given process $y$ be Gaussian. This assumption can be dispensed with, however, if we are willing to give up the (strict sense) Markov property and exchange it for "wide sense Markov." Everything in the paper then remains valid, but the generating processes are no longer Gaussian. This is the price we need to pay to be able to stick to a linear theory. If we have a non-Gaussian process $y$ but require the strict Markov property, we must turn to the nonlinear stochastic realization problem. Extensions in this direction are presented in [33].

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