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On Minimal Splitting Subspaces and Markovian Representations*

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Abstract. Given a Hilbert space H, let H_1 and H_2 be two arbitrary subspaces. The problem of finding all minimal splitting subspaces of H with respect to H_1 and H_2 is solved. This result is applied to the stochastic realization problem. Each minimal stochastic realization of a given vector process y defines a family of state spaces. It is shown that these families are precisely those families of minimal splitting subspaces (with respect to the past and the future of y) which satisfy a certain growth condition.

1. Introduction

The problem of finding all minimal Markovian (state space) representations of a given random process $\{y(t);\ t\in\mathrm{T}\}$ is known as the *stochastic realization problem*. It has been studied extensively in recent years both in its deterministic [2, 3, 16] and its probabilistic [1, 4–6, 9, 10, 12–15] aspects. In this paper we extend and unify the axiomatic state space approaches presented in [5] and [15].

The notion of minimal splitting subspace, a generalization of a concept introduced in [7], was applied to this problem in [9] and [5]. This is a natural approach, for, at any given time $t\in\mathrm{T}$, a minimal splitting subspace X_t (with respect to the spaces spanned by the past and the future of y) can be interpreted as a subspace of smallest size containing all the information from the past needed in predicting the future and all the information in the future required to estimate the past, making it an obvious candidate for a state space. A dynamical state space description will then require considering families $\{X_t;\ t\in\mathrm{T}\}$ of minimal splitting subspaces. However, as we shall see below, an arbitrary such

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family will not in general yield a stochastic realization. For this to be the case, we need to impose a natural growth condition.

We begin this paper by solving the general problem of finding all minimal splitting subspaces with respect to two arbitrary subspaces. This is a geometric problem in Hilbert space, and it contains as a special case, the splitting subspace problem of the stochastic realization problem. Secondly we apply these results to the problem of finding all Markovian representations of an arbitrary stochastic vector process.

2. Some Preliminaries and Notations

Let H be a Hilbert space, whose inner product is denoted $(\cdot,\cdot)$. For two subspaces A and B of H (all subspaces are taken to be closed), $A \perp B$ means that A and B are orthogonal, $A \oplus B$ denotes direct sum, $A \ominus B$ is the subspace of A orthogonal to B, $A \vee B$ is the closed linear hull of A and B, and $A^{\perp}$ is the orthogonal complement of A in H. The orthogonal projection of $\lambda \in H$ onto A is denoted $E^{A}\lambda$ or $E\{\lambda|A\}$. Let $\bar{E}^{A}B$ or $\bar{E}\{B|A\}$ be the smallest subspace containing $E^{A}B$, i.e., the closure of $E^{A}B$.

We shall write $A \perp B|X$ if the three subspaces A, B and X satisfy the condition

$$(\alpha,\beta)=(E^{X}\alpha,E^{X}\beta) \qquad \text{for all } \alpha\in A,\ \beta\in B. \tag{1}$$

It can be seen that (1) is equivalent to each of the conditions

$$E\{\alpha|B\vee X\}=E\{\alpha|X\} \qquad \text{for all } \alpha\in A \tag{2}$$

$$E\{\beta|A\vee X\}=E\{\beta|X\} \qquad \text{for all } \beta\in B \tag{3}$$

and, if $X\subset A$, also to

$$(A\ominus X)\perp B. \tag{4}$$

3. Minimal Splitting Subspaces

Let H_1 and H_2 be two arbitrary subspaces of H and define H_0 to be the vector sum of these, i.e., $H_0=H_1\vee H_2$.

Definition 1. A subspace $X\subset H$ is a *splitting subspace* (with respect to H_1 and H_2) if

$$H_1\perp H_2|X. \tag{5}$$

A splitting subspace X is *minimal* if there is no proper subspace of X satisfying (5) and *internal* if $X\subset H_0$.

The following lemma will be useful in testing the minimality of a splitting subspace.

Lemma 1. *Let X be a splitting subspace and assume that $X = X_1 \oplus X_2$ where X_1 and X_2 are subspaces of X. Then X_1 is a splitting subspace if and only if*

$$E^{X_2}H_1 \perp E^{X_2}H_2. \tag{6}$$

Proof. (Cf. [15].) Take $\lambda \in H_1$ and $\eta \in H_2$. Then, in view of (5),

$$(\lambda, \eta) = (E^X\lambda, E^X\eta) = (E^{X_1}\lambda, E^{X_1}\eta) + (E^{X_2}\lambda, E^{X_2}\eta.)$$

Hence, (6) is equivalent to $(\lambda, \eta) = (E^{X_1}\lambda, E^{X_1}\eta)$ for all $\lambda \in H_1$ and $\eta \in H_2$, which is the same as $H_1 \perp H_2 | X_1$. □

The purpose of this section is to solve Problem P1.

Problem P1. *Find all minimal splitting subspaces.*

Let us also consider the following problem.

Problem P2. *Find all splitting subspaces X such that*

$$X \cap H_1^{\perp} = 0 \tag{7}$$

$$X \cap H_2^{\perp} = 0 \tag{8}$$

Remark 1. Using the orthogonal decomposition

$$A = \bar{E}\{B|A\} \oplus (A \cap B^{\perp}), \tag{9}$$

which holds for any subspaces A and B of H, we see that (7) and (8) are equivalent to

$$X = \bar{E}\{H_1|X\} \tag*{(7)$'$}$$

and

$$X = \bar{E}\{H_2|X\} \tag*{(8)$'$}$$

respectively.

Proposition 1. *Problems* P1 *and* P2 *are equivalent.*

Proof. (i) Let X be a solution of P1. Using (9) we can write $X = X_1 \oplus X_2$ where $X_1 = \bar{E}^X H_1$ and $X_2 = X \cap H_1^{\perp}$. Obviously $E^{X_2}H_1 = 0$, and therefore (6) holds. Hence X_1 is a splitting subspace (Lemma 1). Then, since X is a minimal splitting subspace, $X_1 = X$, and thus (7) holds. In the same way we show (8). Therefore X is a solution of P2.

(ii) Let X be a solution of P2, and let X_1 be any splitting subspace contained in X. To see that X is a solution of P1 it remains to show that $X_2 := X \ominus X_1 = 0$. By Lemma 1, (6) holds. Now, in view of $X_2 \subset X$ and condition (7)$'$, $\bar{E}^{X_2}H_1 =$

$\bar{E}^{X_2}\bar{E}^X H_1 = \bar{E}^{X_2} X = X_2$. In the same way, using (8)′, it can be shown that $\bar{E}^{X_2} H_2 = X_2$, and therefore, due to (6), $X_2 = 0$. □

From Proposition 1 we see that our main problem P1 can be replaced by the mathematically more convenient problem P2. To solve P2 we shall first study minimal splitting subspaces satisfying *one* of the conditions (7) and (8), say (7).

Lemma 2. *The subspace $X \subset H$ is a splitting subspace satisfying* (7) *if and only if $X = \bar{E}\{H_1|S\}$ for some subspace $S \supset H_2$.*

Proof. (Cf. [6].) (*if*): Let $X = \bar{E}\{H_1|S\}$ where $S \supset H_2$. Then $E^S\lambda = E^X\lambda$ for each $\lambda \in H_1$. Since $S \supset H_2 \vee X$, this implies that $E^{H_2 \vee X}\lambda = E^X\lambda$ for each $\lambda \in H_1$, i.e., in view of (2), $H_1 \perp H_2 | X$. Hence X is a splitting subspace satisfying (7)′.

(*only if*): Let X be a splitting subspace satisfying (7)′. Then, $X = \bar{E}^X H_1 = \bar{E}^{H_2 \vee X} H_1$, by condition (2), and hence $X = \bar{E}\{H_1|S\}$ with $S = H_2 \vee X \supset H_2$. □

Then, by introducing the additional condition (8), we obtain the following theorem which gives the solution of P1. For this we first need to define the *frame space*

$$H^{\square} = E\{H_1|H_2\} \vee E\{H_2|H_1\}, \tag{10}$$

which is itself a (nonminimal) internal splitting subspace [5].

Theorem 1. *The subspace X is a minimal splitting subspace if and only if $X = \bar{E}\{H_1|S\}$ for some subspace S such that $H_2 \subset S \subset (H_2 \vee H^{\square}) \oplus H_0^{\perp}$.*

We need the following lemma to prove this theorem.

Lemma 3. *Let $S \supset H_2$. Then $E^S(H_1 \cap H_2^{\perp}) = (E^S H_1) \cap H_2^{\perp}$.*

Proof. By definition, $E^S(H_1 \cap H_2^{\perp}) = \{E^S\lambda | \lambda \in H_1$ and $\lambda \perp H_2\}$. But $\lambda \perp H_2$ is equivalent to $E^{H_2}\lambda = 0$, which, in view of $S \supset H_2$, is the same as $E^{H_2}E^S\lambda = 0$ or $E^S\lambda \in H_2^{\perp}$. □

Proof of Theorem 1. In view of Lemma 2 and condition (8), it only remains to show that $S \subset (H_2 \vee H^{\square}) \oplus H_0^{\perp}$ if and only if $(E^S H_1) \cap H_2^{\perp} = 0$. But, by Lemma 3, the latter condition is equivalent to $S \subset (H_1 \cap H_2^{\perp})^{\perp}$. Now, using formula (9), it is seen that $H_0 = (H_2 \vee H^{\square}) \oplus (H_1 \cap H_2^{\perp})$, and therefore $(H_1 \cap H_2^{\perp})^{\perp} = (H_2 \vee H^{\square}) \oplus H_0^{\perp}$. □

Corollary 1. *The subspace X is an internal minimal splitting subspace if and only if $X = \bar{E}\{H_1|S\}$ for some subspace S such that $H_2 \subset S \subset H_2 \vee H^{\square}$.*

Theorem 1 provides a parameterization of the set of minimal splitting subspaces. We shall now show that the mapping $S \to X$ is one to one if S is restricted to H_0, in which case we obtain precisely the *internal* minimal splitting subspaces.

Proposition 2. *Let S be a subspace such that $S \supset H_2$, and define $X = \bar{E}\{H_1|S\}$. Then $S = (H_2 \vee X) \oplus K$ for some subspace $K \subset H_0^{\perp}$. If $S \subset H_0$, $K = 0$.*

Proof. Clearly $S \supset H_2 \vee X$, and therefore $S = (H_2 \vee X) \oplus K$ for some subspace K. We will show that $K \subset H_0^{\perp}$. To this end, as in the proof of Lemma 2, first note that $E^S\lambda = E^X\lambda$ for each $\lambda \in H_1$. Since $S \supset H_2 \vee X$, this implies that $E^S\lambda = E^{H_2 \vee X}\lambda$ for all $\lambda \in H_1$, and consequently $K \perp H_1$. Since, in addition $K \perp H_2$ (by definition), $K \perp H_0$. □

4. Applications to the Stochastic Realization Problem

Given a probability space $(\Omega, \mathcal{F}, P)$, let H be a subspace of $L_2(\Omega, \mathcal{F}, P)$ consisting of centered real Gaussian stochastic variables. Such a space is called a *Gaussian space* [8]; it is a Hilbert space with inner product $(\xi, \eta) = E\{\xi\eta\}$, where $E\{\cdot\}$ denotes mathematical expectation. For any finite-dimensional stochastic vector ζ whose components $\zeta_1, \zeta_2, \ldots, \zeta_p$ belong to H, let $H(\zeta)$ be the *Gaussian subspace generated by* ζ, i.e. the closed linear hull in H of the components of ζ. This definition immediately generalizes to the case where ζ takes values in an arbitrary real separable Hilbert space E, the components now being given by $\zeta_k = \langle \zeta, e_k \rangle$, where $\{e_k;\ k \in \mathbf{Z}^+\}$ is any basis in E and $\langle \cdot, \cdot \rangle$ is the inner product there. (Obviously $H(\zeta)$ does not depend on the choice of basis.) If $\{z(t);\ t \in \mathbf{T}\}$ is a stochastic vector process such that the components of $z(t)$ belong to H for each $t \in \mathbf{T}$, we shall write $H(z)$ to denote $\bigvee_{t \in \mathbf{T}} H(z(t))$.

Now let $\{y(t);\ t \in \mathbf{T}\}$ be a wide sense separable [11] centered m-dimensional real Gaussian stochastic process such that $H(y) \subset H$. The basic problem, to be formulated more precisely below, is to determine all possible Markovian state-space representations of y such that the state process x satisfies $H(x) \subset H$. To exploit the theory of minimal splitting subspaces developed in Section 3 for this purpose, we need to define the *past space* $H_t^-(y)$ and the *future space* $H_t^+(y)$ for each $t \in \mathbf{T}$ in such a way that $H_t^-(y)$ is nondecreasing and $H_t^+(y)$ is nonincreasing as a function of t and $H_t^-(y) \vee H_t^+(y) = H(y)$. *Here* we shall take $H_t^-(y)$ and $H_t^+(y)$ to be the closed linear hulls of $\{y(s);\ s < t\}$ and $\{y(s);\ s \geqslant t\}$ respectively, but other definitions are possible; in the definition of $H_t^-(y)$ we may take $s \leqslant t$ instead, and in some applications it is better to let $H_t^-(y)$ and $H_t^+(y)$ be generated by the past and future *increments* of y [5, 15]. In the setting of Section 3 the subspaces $H(y)$, $H_t^+(y)$ and $H_t^-(y)$ will play the roles of H_0, H_1 and H_2 respectively.

Definition 2. A (Gaussian) *stochastic dynamical system* on H is a pair (x,y) of centered (jointly Gaussian) stochastic processes $\{x(t);\ t \in \mathbf{T}\}$ and $\{y(t);\ t \in \mathbf{T}\}$, taking values in a real separable Hilbert space E and in $\mathbf{R}^m$ respectively, such that both $H(x)$ and $H(y)$ are contained in H and such that, for every $t \in \mathbf{T}$, the Gaussian subspace $X_t := H(x(t))$ generated by the random vector $x(t)$ satisfies

$$[H_t^-(y) \vee X_t^-] \perp [H_t^+(y) \vee X_t^+] \,|\, X_t \tag{11}$$

where $X_t^- := \bigvee_{s \leqslant t} X_s$ and $X_t^+ = \bigvee_{s \geqslant t} X_s$. The processes x and y are called the *state process* and the *output process* respectively, and X_t is the *state space* at time t. The stochastic system is *finite dimensional* if $\dim E < \infty$.

Remark 2. Trivially, the family $\{X_t; t\in\mathbf{T}\}$ of state spaces satisfies the conditions

(i) $\{X_t; t\in\mathbf{T}\}$ is *Markovian*, i.e. $X_t^+ \perp X_t^- | X_t$ for all $t\in\mathbf{T}$. This is equivalent to saying that $\{x(t); t\in\mathbf{T}\}$ is a *Markov process*.

(ii) for each $t\in\mathbf{T}$, X_t is a splitting subspace with respect to $H_t^-(y)$ and $H_t^+(y)$. □

We shall say that two stochastic systems (defined on the same Gaussian space) are *equivalent* if, for each $t\in\mathbf{T}$, their output processes agree a.s. and their state spaces are the same. Hence equivalent stochastic systems can have different state processes but these are related by trivial coordinate transformations in the state spaces.

As an *example* let us consider a discrete-time stochastic system with $\mathbf{T}=\mathbf{Z}^+$.

Proposition 3. *All finite dimensional stochastic systems* (x,y) *with* $\mathbf{T}=\mathbf{Z}^+$ *have a representation of type*

$$x(t+1) = A(t)x(t) + B(t)w(t); \qquad x(0) = x_0 \tag{12a}$$
$$y(t) = C(t)x(t) + D(t)w(t), \tag{12b}$$

where $\{A(t),B(t),C(t),D(t); t\in\mathbf{Z}^+\}$ *are matrices of appropriate dimensions,* x_0 *is a zero-mean Gaussian random vector, and* w *is a unitary Gaussian white noise process independent of* x_0. *Conversely, any pair* (x,y) *of stochastic processes satisfying* (12) *is a stochastic system.*

Proof. (i) Let (x,y) be a stochastic system with $\mathbf{T}=\mathbf{Z}^+$ and the state process x taking values in $\mathbf{R}^n$. We shall prove that (x,y) satisfies a representation (12). To this end first note that

$$\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} = E^{H_t^-(y)\vee X_t^-}\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} + E^{[H_t^-(y)\vee X_t^-]^\perp}\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} \tag{13}$$

Now, (11) and (2) imply that $E^{H_t^-(y)\vee X_t^-}\lambda = E^{X_t}\lambda$ for all $\lambda\in H_t^+(y)\vee X_t^+$, and consequently there are matrices $A(t)$ and $C(t)$ such that

$$E^{H_t^-(y)\vee X_t^-}\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} = \begin{pmatrix} A(t) \\ C(t) \end{pmatrix} x(t). \tag{14}$$

The second term of (13) is a white noise process; it is the innovation process of $\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix}$. By normalizing we obtain

$$E^{[H_t^-(y)\vee X_t^-]^\perp}\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} = \begin{pmatrix} B(t) \\ D(t) \end{pmatrix} w(t), \tag{15}$$

where w is a unitary Gaussian white noise and $B(t)$ and $D(t)$ are matrices such that $\begin{pmatrix} B(t) \\ D(t) \end{pmatrix}$ has full rank. Hence (x,y) satisfies (12). It remains to show that

$x_0 \perp H(w)$. But, since $\begin{pmatrix} B(t) \\ D(t) \end{pmatrix}$ has full rank, (15) implies that $w(t) \in [H_t^-(y) \vee X_t^-]^\perp$ for all $t \in \mathbf{Z}^+$. Hence $x_0 \perp H(w)$. (ii) Assume that (x,y) satisfies (12), and, for each $t \in \mathbf{Z}^+$, let X_t be the Gaussian space generated by $x(t)$. Since w is a white noise process and $x_0 \perp H_t^+(w)$, $X_0 \oplus H_t^-(w)$ is orthogonal to $H_t^+(w)$ or equivalently

$$[X_0 \oplus H_t^-(w)] \perp [H_t^+(w) \oplus X_t] | X_t \tag{16}$$

(property (4)). From (12) it is easy to see that $H_t^-(y) \vee X_t^- \subset X_0 \oplus H_t^-(w)$ and that $H_t^+(y) \vee X_t^+ \subset H_t^+(w) \oplus X_t$ and therefore (11) holds. Hence (x,y) is a stochastic system. □

Similar results hold for continuous-time processes and stationary processes defined on the whole real line.

Definition 3. A *stochastic realization* (or *Markovian representation*) of $\{y(t); t \in \mathbf{T}\}$ on H is a stochastic system (x,z) on H such that, for all $t \in \mathbf{T}, z(t) = y(t)$ a.s. The realization is said to be *finite dimensional* if the stochastic system is finite dimensional and *internal* if $X_t \subset H(y)$ for $t \in \mathbf{T}$.

Proposition 4. *Every wide sense separable centered Gaussian stochastic process* $\{y(t); t \in \mathbf{T}\}$ *has a stochastic realization on* $H(y)$.

Proof. For each $t \in \mathbf{T}$, set $X_t = H_t^-(y)$. Since trivially $H_t^-(y) \perp H(y) | H_t^-(y), X_t$, thus defined, satisfies (11) for all $t \in \mathbf{T}$. For each $t \in \mathbf{T}$, choose an orthonormal basis $\{\xi_1, \xi_2, \xi_3, \dots\}$ in X_t; this can be done due to the separability. Setting $x_k := \xi_k / k$, we have $\Sigma x_k^2 < \infty$ a.s. Now define $x(t) = \Sigma x_k e_k$, where $\{e_1, e_2, e_3, \dots\}$ is an orthonormal basis in some separable Hilbert space E. Then $x(t)$ takes values in E and (x,y) is a stochastic realization on H. Of course, we may as well set $X_t = H_t^+(y)$ or $H(y)$. □

The basic problem in realization theory is to find the smallest possible state spaces. Of course, in general, the choice in the proof of Theorem 2 will not suffice for this purpose. We need a concept of minimality. To this end first note that the family $\mathcal{H}$ of all subspaces of H is a partially ordered set with respect to vector space inclusion.

Definition 4. A stochastic realization (x,y) on H is *minimal at time* t if the corresponding state space X_t is minimal with respect to the ordering of the partially ordered set $\mathcal{H}$, i.e. if there is no proper subspace of X_t for which (11) holds. We say that (x,y) is *minimal* if it is minimal at each time $t \in \mathbf{T}$.

Proposition 5 (Ruckebusch). *A stochastic realization is minimal if and only if it is both observable and constructible, i.e., for every* $t \in \mathbf{T}$,

$$X_t \cap [H_t^+(y)]^\perp = 0 \qquad (observability) \tag{17}$$

$$X_t \cap [H_t^-(y)]^\perp = 0 \qquad (constructibility). \tag{18}$$

The proof of this result is a trivial generalization of one given in [15] for the stationary case.

Minimal splitting subspaces with respect to the past and future spaces of y are natural candidates for state spaces in the stochastic realization problem. If we consider splitting subspaces contained in the past space or in the future space only, the Markov property is a direct consequence of the splitting property. In general, however, we must impose a certain growth condition, as described in the following theorem.

Theorem 3. *A family* $\{X_t; t\in\mathbf{T}\}$ *of separable subspaces of* H *defines a minimal stochastic realization* (x,y) *(in the sense that* X_t *is the Gaussian space generated by* $x(t)$ *for each* $t\in\mathbf{T}$*) if and only if*

(i) *for each* $t\in\mathbf{T}$, X_t *is a minimal splitting subspace with respect to* $H_t^-(y)$ *and* $H_t^+(y)$;

(ii) *there exists a family* $\{Z_t; t\in\mathbf{T}\}$ *of subspaces, with the property* $Z_t\subset[H(y)\vee X_t^+]^\perp$ *for all* $t\in\mathbf{T}$ *such that the family* $\{S_t; t\in\mathbf{T}\}$*, where* $S_t:=[H_t^-(y)\vee X_t]\oplus Z_t$*, is nondecreasing, i.e.* $S_\tau\subset S_t$ *whenever* $\tau\leqslant t$.

Proof. (*if*): By Condition (i), Lemma 2, and Proposition 2, $X_t = \bar{E}\{H_t^+(y)|[H_t^-(y)\vee X_t]\oplus K_t\}$ where $K_t\perp H_t^+(y)$. Therefore, since in addition $Z_t\perp H_t^+(y)$, $X_t=\bar{E}\{H_t^+(y)|S_t\}$. Consequently, in view of (9),

$$S_t\ominus X_t = S_t\cap[H_t^+(y)]^\perp, \tag{19}$$

which is nondecreasing in t, for $\{S_t\}$ and $\{[H_t^+(y)]^\perp\}$ are. Hence, since trivially $(S_t\ominus X_t)\perp X_t$, $(S_t\ominus X_t)\perp X_t^+$. By (19), we also have $(S_t\ominus X_t)\perp H_t^+(y)$, and therefore, in view of property (4), $S_t\perp[H_t^+(y)\vee X_t^+]|X_t$. But, due to condition (ii), $H_t^-(y)\vee X_t^-\subset S_t$, and consequently (11) holds. Then, choosing a basis in X_t for each $t\in\mathbf{T}$, as in the proof of Proposition 4, produces the required state process. The stochastic realization (x,y) is minimal, for $\{X_t; t\in\mathbf{T}\}$ is a family of minimal splitting subspaces.

(*only if*): Let (x,y) be a minimal stochastic realization with spaces $\{X_t; t\in\mathbf{T}\}$. Then, Condition (i) follows from Remark 2 (ii) and, as for the minimality, from Propositions 1 and 5; note that conditions (7)–(8) and (17)–(18) are identical. To show that Condition (ii) holds, set $Z_t:=[H_t^-(y)\vee X_t^-]\ominus[H_t^-(y)\vee X_t]$. Then $S_t=H_t^-(y)\vee X_t^-$, which is nondecreasing. It remains to show that $Z_t\subset[H(y)\vee X_t^+]^\perp$. It follows from (11), (4) and the definition of Z_t that

$$Z_t\subset[H_t^-(y)\vee X_t^-]\perp\{[H_t^+(y)\vee X_t^+]\ominus X_t\}. \tag{20}$$

Since, in addition $Z_t\perp[H_t^-(y)\vee X_t]$, $Z_t\perp[H(y)\vee X_t^+]$ as required. □

We obtain a simpler version of this theorem by restricting our attention to *internal realizations.*

Corollary 3. *A family* $\{X_t; t\in\mathbf{T}\}$ *of subspaces of* $H(y)$ *defines a minimal stochastic realization of* y *if and only if Condition (i) of Theorem 3 holds and the family* $\{S_t; t\in\mathbf{T}\}$*, where here* $S_t:=H_t^-(y)\vee X_t$*, is nondecreasing.*

Proof. (*if*): Noting that $H(y)$ is separable, this part follows immediately from Theorem 3.

(*only if*): It only remains to show that $Z_t=0$ for all $t\in\mathbf{T}$. But, by definition, $Z_t\subset H_t^-(y)\vee X_t^-\subset H(y)$. Therefore, in order to satisfy the condition $Z_t\subset[H(y)\vee X_t^+]^\perp$, we must have $Z_t=0$. □

Comparing Corollaries 1 and 3 we can see from Proposition 2 that, in general, there exist families of (internal) minimal splitting subspaces which do not evolve in time in a Markovian fashion. To see this, we may choose a family $\{S_t; t\in\mathbf{T}\}$ which is not nondecreasing.

References

1. H. Akaike, Markovian representation of stochastic processes by canonical variables, *SIAM J. Control*, 13 162–173 (1975).
2. B. D. O. Anderson, The inverse problem of stationary covariance generation, *J. Statistical Physics*, 1 133–147 (1969).
3. P. Faurre, Realizations markoviennes de processus stationnaires, *Research Report no.* 13, *March* 1973, *IRIA* (*LABORIA*), Le Chesnay, France.
4. A. Lindquist and G. Picci, On the stochastic realization problem, *SIAM J. Control*, (*to be published*).
5. A Lindquist and G. Picci, On the structure of minimal splitting subspaces in stochastic realization theory, *Proc. 1977 Conf. Decision and Control*, New Orleans, 42–48 1977.
6. A Lindquist and G. Picci, A state-space theory for stationary stochastic processes, *Proc. 21st Midwest Symposium on Circuits and Systems*, Ames, Iowa, August 1978.
7. H. P. McKean, Brownian motion with a several dimensional time, *Theory Prob. Appl.* (USSR), VIII 335–354 (1963).
8. J. Neveu, *Processus Aléatoires Gaussiens*, Presses de l'Université de Montréal, 1968.
9. G. Picci, Stochastic realization of Gaussian processes, *Proc. IEEE* 64 112–122 (1976).
10. Yu. A. Rozanov, On two selected topics connected with stochastic systems theory, *Applied Mathematics & Optimization*, 3 73–80 (1976).
11. Yu. A. Rozanov, *Stationary Random Processes*, Holden-Day, San Francisco, 1967.
12. G. Ruckebusch, Representations Markoviennes de processus Gaussiens Stationnaires, *Ph.D Thesis*, Univ. of Paris VI, 1975.
13. G. Ruckebusch, Représentations Markoviennes de Processus Gaussiens Stationnaires, *C. R. Acad. Sc. Paris, Series A*, 282 649–651 (1976).
14. G. Ruckebusch, Représentations Markoviennes de Processus Gaussiens Stationnaires et Applications Statistiques, *Journées de Statistique des Processus Stochastiques*, Grenoble, June 1977, Springer Verlag Lecture Notes in Mathematics 636.
15. G. Ruckebusch, A state space approach to the stochastic realization problem, *Proc. 1978 Intern. Symp. Circuits and Systems*, New York.
16. J. C. Willems, Dissipative dynamical systems, Part II: Linear systems with quadratic supply rates, *Archive for Rational Mechanics and Analysis*, 45, 352–393 (1972).

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