On a condition for minimality of Markovian splitting subspaces

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Given a stationary Gaussian vector process, consider a Markovian splitting subspace X contained in the frame space which is either observable or constructible. Such an X will be called *reduced*. In this paper we show that a Markovian splitting subspace is minimal if and only if it is reduced. This was claimed in some earlier papers but there are nontrivial gaps in the proofs presented there. The proof is based on a lemma stating that all reduced X have quasi-equivalent structural functions. This property is also important in isomorphism theory for minimal splitting subspaces.

Keywords: Stochastic realization, Minimal state spaces, Quasi-equivalence, Quasi-similarity, State space isomorphism.

1. The main result

Let $\{y(t); t \in \mathbb{R}\}$ be a real m-dimensional stationary Gaussian vector process which is centered and mean-square continuous, and let H be the Gaussian space [10] generated by its components. This is a Hilbert space with inner product $\langle \xi, \eta \rangle = E\langle \xi\eta \rangle$. Let H^- and H^+ be the subspaces of H generated by $\{y(t); t \leq 0\}$ and $\{y(t); t \leq 0\}$ respectively. The stationarity of y implies that there is a strongly continuous group $\{U_t; t \in \mathbb{R}\}$ of unitary operators in H, called the *shift*, such that $y_k(t) = U_t y_k(0)$ for all $t \in \mathbb{R}$ and $k = 1, 2, \ldots, m$ [12]. Now, in addition, we assume that y is purely nondeterministic, i.e. $\bigcap_{t \in \mathbf{R}} U_t H^- = 0$, and strictly noncyclic, i.e. the subspaces $H^- \cap (H^+)^\perp$ and $H^+ \cap (H^-)^\perp$ are full range. (Here the superscript $\perp$ denotes orthogonal complement in H. A subspace Z of H is *full range* if $\bigvee_{t \in \mathbf{R}} U_t Z = H$.)

A (closed) subspace X of H is called a *splitting subspace* if

$$\langle E^X\alpha, E^X\beta \rangle = \langle \alpha, \beta \rangle \quad \text{for all } \alpha \in H^-, \beta \in H^+, \tag{1}$$

where $E^X\lambda$ denotes the orthogonal projection of $\lambda \in H$ on X. A subspace X is *Markovian* if $\bigvee_{t \leq 0} U_t X$ and $\bigvee_{t \geq 0} U_t X$ are conditionally independent given X. A Markovian splitting subspace X is *minimal* if it contains no other Markovian splitting subspace as a proper subspace, *observable* if $X \cap (H^+)^\perp = 0$, and *constructible* if $X \cap (H^-)^\perp = 0$. It can be shown [13] that X is minimal if and only if it is both observable and constructible.

Any minimal splitting subspace is contained in the *frame space*

$$H^\square := (E^H \; H^+) \vee (E^{H'} H^-). \tag{2}$$

In fact, $H^\square$ is the closed linear hull of all minimal splitting subspaces [6–8]. A Markovian splitting subspace is said to be *reduced* if $X \subset H^\square$ and it is either observable or constructible. Clearly any minimal X is reduced. It is the purpose of this note to show that the converse is also true.

Theorem. *A Markovian splitting subspace is reduced if and only if it is minimal.*

The problem of determining all minimal Markovian splitting subspaces has been studied extensively by Lindquist and Picci [5 – 8], Ruckebusch [13–15] and others. The theorem stated above is useful, for the reduced X are in one-one correspondence to the minimal stable spectral factors [14]. (Also see [6].)

The theorem has an important corollary. To state it we need another definition. On each Markovian splitting subspace X there is a strongly continuous semigroup $\{U_t(X); t \geq 0\}$ of contractive operators given by $U_t(X)\xi = E^X U_t \xi$; we shall call it the *Markov semigroup* of X. (See [6–8].)

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Corollary 1. *Let X_1 and X_2 be two minimal Markovian splitting subspaces. Then $U_t(X_1)$ and $U_t(X_2)$ are quasi-similar, i.e. there are quasi-invertible (injective with dense range) maps M: $X_1 \to X_2$ and N: $X_2 \to X_1$ such that*

$$\begin{cases} MU_t(X_1) = U_t(X_2)M, & (3a) \\ U_t(X_1)N = NU_t(X_2) & (3b) \end{cases}$$

for all $t \geq 0$.

This strengthens the state-space isomorphism results of [15].

2. Background material

It is shown in [6–8] that a subspace X is a splitting subspace if and only if $X = S \cap \bar{S}$ for some pair $(S, \bar{S})$ of subspaces such that $S \supset H^-$, $\bar{S} \supset H^+$, and S and $\bar{S}$ intersect perpendicularly, i.e. $S^\perp \perp \bar{S}^\perp$. The correspondence $X \leftrightarrow (S, \bar{S})$ is one–one, S and $\bar{S}$ being given by $S = H^- \vee X$ and $\bar{S} = H^+ \vee X$. To exhibit this correspondence we write $X \sim (S, \bar{S})$. Moreover, X is Markovian if and only if the S and $\bar{S}$ satisfy the invariance conditions

$$\begin{cases} U_t S \subset S \quad \text{for } t \leq 0, & (4a) \\ U_t \bar{S} \subset \bar{S} \quad \text{for } t \geq 0, & (4b) \end{cases}$$

it is observable if and only if $\bar{S} = H^+ \vee S^\perp$ and constructible if and only if $S = H^- \vee \bar{S}^\perp$.

From now on we shall only consider splitting subspaces contained in the frame space $H^\square$, for all reduced X belong to this class. Clearly we must then have

$$\begin{cases} S_- \subset S \subset S_+, & (5a) \\ \bar{S}_+ \subset \bar{S} \subset \bar{S}_-, & (5b) \end{cases}$$

where $S_- := H^-$, $S_+ := H^- \vee H^\square$, $\bar{S}_+ := H^+$, and $\bar{S}_- := H^+ \vee H^\square$. It is not hard to see that $(S_-, \bar{S}_-)$ and $(S_+, \bar{S}_+)$ are pairs of perpendicularly intersecting subspaces satisfying (3) and the observability and constructibility conditions. Hence they define minimal Markovian splitting subspaces X_- and X_+ respectively. It can be shown [6–8] that $X_- = \bar{E}^{H^-} H^+$ and $X_+ = \bar{E}^{H^+} H^-$, where the bar over E denotes closure.

Next we shall reformulate these facts in terms of Hardy functions. Since the process y is mean-square continuous and purely nondeterministic, it has a spectral representation

$$y(t) = \int_{-\infty}^{\infty} e^{i\omega t} \, d\hat{y}(i\omega) \tag{6a}$$

where $d\hat{y}$ is an orthogonal stochastic measure such that

$$E\{d\hat{y}(i\omega) d\hat{y}(i\omega)^*\} = \frac{1}{2\pi} \Phi(i\omega) d\omega \tag{6b}$$

(* denotes transposition and conjugation). The $m \times m$-matrix function Φ is called the spectral density; suppose it has rank $p \leq m$. Now let $\mathcal{H}_2^+$ ($\mathcal{H}_2^-$) be the Hardy space of p-dimensional row vector functions which are (double-sided) Laplace-transforms of functions in $L_2^p(\mathbb{R})$ which vanish on the negative (positive) real line. Any $m \times p$-matrix solution W of

$$W(s)W(-s)' = \Phi(s) \tag{7}$$

($'$ denotes transpose) is called a (full-rank) *spectral factor*; we say that W is *stable* if its rows belong to $\mathcal{H}_2^+$ and *strictly unstable* if they belong to $\mathcal{H}_2^-$. Any stable spectral factor has a unique decomposition

$$W(s) = W_-(s)Q(s), \tag{8}$$

where W_- is the *outer* spectral factor, i.e. the closed linear hull of $\{e^{i\omega t}W_-;\, t \leq 0\}$ equals $\mathcal{H}_2^+$, and Q is *inner*, i.e. Q is bounded and analytic in the open right half-plane and its values on the imaginary axis I are unitary matrices. Likewise any strictly unstable spectral factor W has the decomposition

$$\bar{W}(s) = \bar{W}_+(s)\bar{Q}(s), \tag{9}$$

defined analogously to (8) only exchanging $\mathcal{H}_2^+$, $t \leq 0$ and right half-plane for $\mathcal{H}_2^-$, $t \geq 0$, and left half-plane. Then $\bar{Q}^*$ will be inner. Now, to any spectral factor W, there is a p-dimensional vector Wiener process $\{u(t);\, t \in \mathbb{R}\}$ defined by

$$u(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1}{i\omega} d\hat{u}(i\omega); \quad d\hat{u} = W^{-L} d\hat{y} \tag{10}$$

(where W^{-L} is any left inverse of W). Let $\mathcal{U}$ be the class of all such u, and let $\mathcal{U}^+$ and $\mathcal{U}^-$ be the subclasses of $\mathcal{U}$ corresponding to stable respectively strictly unstable W.

It is well known [12] that, for each $f \in L_2^p(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(-t) \, du(t) = \int_{-\infty}^{\infty} \hat{f}(i\omega) \, d\hat{u}(i\omega) \tag{11}$$

where $\hat{f} \in L_2^p(I)$ is the (double-sided) Laplace-transform of f restricted to I. If $u \in \mathcal{U}$, the Gaussian space generated by the components of u coincides with H so that for each $\eta \in H$, there is a unique f such that η equals (11). Then, by standard arguments, there is a unitary operator $T_u: H \to L_2^p(I)$ such that $T_u \eta = \hat{f}$. Let $H^-(du)$ $[H^+(du)]$ be the subspace of H consisting of those random variables (11) for which f vanishes on the negative (positive) real line. Then $T_u H^-(du) = \mathcal{H}_2^+$ and $T_u H^+(du) = \mathcal{H}_2^-$.

Modulo trivial coordinate-transformations in $\mathbb{R}^p$, there is a one–one correspondence between $\mathcal{U}^+$ and the class of subspaces $S \supset H^-$ satisfying (4a), and for each such pairing $S = H^-(du)$. Likewise, the subspaces $\bar{S} \supset H^+$ satisfying (4b) are in one–one correspondence with the $\bar{u} \in \mathcal{U}^-$ in such a way that $\bar{S} = H^-(d\bar{u})$. The two subspaces S and $\bar{S}$ intersect perpendicularly if and only if the *structural function*

$$K(s) := \overline{W}(s)^{-L} W(s) \tag{12}$$

is inner, where W corresponds to u and $\overline{W}$ to $\bar{u}$. (It can be shown that (12) is independent of the choice of left inverse.) Due to perpendicular intersection $X = S \ominus \bar{S}^{\perp}$, i.e. $X = H^-(du) \ominus H^-(d\bar{u})$. But $T_u H^-(d\bar{u}) = \mathcal{H}_2^+$ and $T_u H^-(d\bar{u}) = \mathcal{H}_2^+ K$. Hence $T_u X = \mathcal{H}(K)$, where $\mathcal{H}(K)$ is the orthogonal complement of $\mathcal{H}_2^+ K$ in $\mathcal{H}_2^+$.

Consequently any Markovian splitting subspace $X \subset H^{\square}$ is characterized by a triplet $(K, Q, \bar{Q}^*)$ of inner functions. By applying T_u to the observability condition $\bar{S} = H^+ \vee S^{\perp}$ and $T_{\bar{u}}$ to the constructibility condition $S = H^- \vee \bar{S}^{\perp}$ it is not hard to see that X is observable if and only if $(K, \bar{Q}^*)_L = I$, i.e. K and $\bar{Q}^*$ are left coprime, and constructible if and only if $(K, Q)_R = I$ i.e. K and Q are right coprime. In the same way, applying the T_u-map of X_+ to (5a) shows that Q must be a left inner divisor of Q_+, the Q-factor of X_+, i.e. there is an inner function J such that $QJ = Q_+$. An analogous statement holds for $\bar{Q}$ and $\bar{Q}_-$, the $\bar{Q}$-factor of X_-. (See Section 7 in [8].)

We shall denote by $\{\Sigma_t(K); t \geq 0\}$ the semigroup on $\mathcal{H}(K)$ obtained from $\{U_t(x); t \geq 0\}$ under the image of T_u, i.e. $\Sigma_t(K) = T_u U_t(X) T_u^*$. Then $\Sigma_t(K) f = P^{\mathcal{H}(K)} e^{i\omega t} f$, where $P^{\mathcal{X}}$ denotes orthogonal projection on the subspace $\mathcal{X}$, i.e. $\Sigma_t(K)$ is the restricted shift on $\mathcal{H}(K)$.

3. The main lemma

In the scalar case ($m = 1$) it is easy to prove the theorem, and this was implicitly done in Lindquist and Picci [5]. (See Section 6). The proof is based on the following observation.

Proposition 1. *Let $m = 1$. Then all reduced Markovian splitting subspaces have the same structural function K.*

It will be instructive to give an independent proof of this result, to pinpoint precisely what fails in the vector case. This will be done below. Now, given Proposition 1, the proof of the theorem is immediate: Let X be reduced. Then $K = K_+$, the structural function of X_+. Since X_+ is constructible, K and Q_+ are coprime. But Q is an inner divisor of Q_+, and therefore K and Q are also coprime. Hence X is constructible. By a similar argument, observability of X_- implies observability of X.

In the vector case things are more complicated. Then Proposition 1 has to be replaced by

Main lemma. *Reduced Markovian splitting subspaces have quasi-equivalent structural functions.*

Quasi-equivalence, a concept originally introduced by Nordgren [11], can be defined in the following way for inner functions. Let K be a $p \times p$ inner function. Set $\gamma_0 = 1$, and, for $i = 1, 2, \dots, p$, define γ_i to be the greatest common inner divisor of all $i \times i$ minors of K. Clearly γ_{i-1} divides γ_i so that $k_i := \gamma_i / \gamma_{i-1}$ is inner for $i = 1, 2, \dots, p$. The scalar inner functions $k_0, k_1, \dots, k_p$ are the *invariant factors* of K. We shall say that two inner functions are *quasi-equivalent* if they have the same invariant factors. This is clearly an equivalence relation. (Quasi-equivalence is usually defined in a different manner. The equivalence between our definition and the original one is a theorem. See Fuhrmann [2, p. 214].)

For the proof of this result we shall need a series of lemmas.

Lemma 1. *Let $X \subset H^{\square}$ be an observable Markovian splitting subspace with structural function K. Then there are inner functions G and J such that $GK_+ = KJ$, where $(K, G)_L = I$ and $(K_+, J)_R = I$.*

Proof. Let J be the inner function defined at the end of Section 2. Then $\bar{Q}^* K_+ = KJ$. Since X_+ is constructible, $(K_+, Q_+)_R = I$. But $Q_+ = QJ$, and therefore $(K_+, J)_R = I$. From observability of X it follows that $(K, \bar{Q}^*)_L = I$. Then the lemma follows by setting $G = \bar{Q}^*$. □

Lemma 2. *Let $X \subset H^{\square}$ be a constructible Markovian splitting subspace with structural function K. Then there are inner functions R and P such that $RK = K_- P$, where $(K_-, R)_L = I$ and $(K, P)_R = I$.*

Proof. Follows by an argument which is symmetric to that of Lemma 1, replacing X_+ by X_-. □

Proposition 1 now follows from a theorem by Douglas, Shapiro and Shields [1]. Let F be inner and set $T := F^* K$. Then it is shown in [1] that $\mathcal{H}(K) = \bar{P}^{\mathcal{H}_2^+} \mathcal{H}_2^- T$ if and only if F and K are coprime. (Also see [2].) Therefore, if $X \subset H^{\square}$ is observable, setting $T = G^* K$ it follows from Lemma 1 that $\mathcal{H}(K) = \mathcal{H}(K_+)$, for we also have $T = J^* K_+$. Hence $K = K_+$. A symmetric argument based on Lemma 2 shows that $K = K_-$ for all constructible $X \subset H^{\square}$, including X_+. Hence Proposition 1 follows. However this proof does not work in the vector case ($m > 1$), since then the inner factors do not commute.

Instead we shall use the fact that the restricted shifts $\Sigma_t(K_1)$ and $\Sigma_t(K_2)$ are quasi-similar if and only if K_1 and K_2 are quasi-equivalent [2;p. 215]. Let $\mathcal{H}^\infty$ be the space of all $p \times p$-matrix functions which are bounded and analytic in the open right half-plane. Then, in particular all $p \times p$ inner functions belong to $\mathcal{H}^\infty$. An inner function will be called *real* if it takes real values on the real axis.

Lemma 3. *Let K_1 and K_2 be real inner functions. Then K_1 and K_2 are quasi-equivalent if and only if there are $A, B \in \mathcal{H}^\infty$ such that*

$$AK_1 = K_2 B \tag{13}$$

where $(K_1, B)_R = I$ and $(K_2, A)_L = I$.

Proof. (*if*): By Theorem 14-8 in [2], which is a version of the Nagy–Foias Lifting Theorem, there is a quasi-invertible N such that

$$N\Sigma_t(K_1) = \Sigma_t(K_2)N. \tag{14}$$

The transpose of a real inner function is also inner. Therefore, since $K_1' A' = B' K_2'$, there is also a quasi-invertible M such that

$$\Sigma_t(K_1')M = M\Sigma_t(K_2'). \tag{15}$$

But an inner function and its transpose have the same invariant factors and are therefore quasi-equivalent. Hence, for $i = 1, 2$, $\Sigma_t(K_i)$ and $\Sigma_t(K_i')$ are quasi-similar, and consequently there are quasi-invertible functions R_1 and R_2 so that, in particular, $\Sigma_t(K_1)R_1 = R_1\Sigma_t(K_1')$ and $R_2\Sigma_t(K_2) = \Sigma_t(K_2')R_2$. The operator $R := R_1 M R_2$ is quasi-invertible [16,p. 70]. Moreover

$$\begin{aligned}\Sigma_t(K_1)R &= R_1\Sigma_t(K_1')MR_2\\ &= R_1 M\Sigma_t(K_2')R_2\\ &= R\Sigma_t(K_2),\end{aligned} \tag{16}$$

which together with (14) implies that $\Sigma_t(K_1)$ and $\Sigma_t(K_2)$ are quasi-similar. Then K_1 and K_2 are quasi-equivalent.

(*only if*): Follows from the first part of the proof of Theorem 15-9 in [2, p. 215].

Proof of main lemma. It follows from Lemmas 1 and 3 that, for any observable $X \subset H^{\square}$, the structural function K is quasi-equivalent to K_+. In particular K_- and K_+ are quasi-equivalent. Moreover, from Lemmas 2 and 3 it follows that any constructible $X \subset H^{\square}$ has a K which is quasi-equivalent to K_-. □

4. Proof of the theorem

Suppose that $X \sim (S, \bar{S})$ is a constructible Markovian splitting subspace such that $X \subset H^{\square}$, and let $X_1 \sim (S_1, \bar{S}_1)$ be a minimal Markovian splitting subspace contained in X. The existence of such an X_1 is insured by Prop. 4.3 in Lindquist and Picci [7]. We want to show that $X_1 = X$, which implies that X is minimal. To this end, note that $X_1 \subset X$ implies that $S_1 \subset S$ and $\bar{S}_1 \subset \bar{S}$. But by constructibility $S = H^- \vee \bar{S}^\perp$ and $S_1 = H^- \vee \bar{S}_1^\perp$, and therefore, since $\bar{S}^\perp \subset \bar{S}_1^\perp$, $S \subset S_1$. Hence $S_1 = S$ so that X_1 and X have the same T_u-map. Therefore it follows from $X_1 \subset X$ that $\mathcal{H}(K_1) \subset \mathcal{H}(K)$, and consequently $\mathcal{H}_2^+ K \subset \mathcal{H}_2^+ K_1$. But then there must be an inner function θ such that $K = \theta K_1$ [4, p. 69]. Now, since K and K_1 are quasi-equivalent (main lemma), $\det K = \det K_1$, and consequently $\det\theta = 1$. However, an inner function with this property must be a constant unitary matrix, and therefore $\mathcal{H}(K_1)=\mathcal{H}(K)$, which is equivalent to $X_1 = X$. By a symmetric argument we show that all observable $X \subset H^{\square}$ must also be minimal.

5. Corollaries

The proof of Corollary 1 is an immediate consequence of the fact that X_1 and X_2 have quasi-equivalent structural functions K_1 and K_2, i.e. that $\Sigma_t(K_1)$ and $\Sigma_t(K_2)$ are quasi-similar.

There is a natural canonical form for $\Sigma_t(K)$ corresponding to minimal X, namely

$$\hat{\Sigma}_t(K) := \Sigma_t(k_1) \oplus \Sigma_t(k_2) \oplus \cdots \oplus \Sigma_t(k_p) \quad (17)$$

which defines a semigroup on

$$\mathcal{H}(k_1) \oplus \mathcal{H}(k_2) \oplus \cdots \oplus \mathcal{H}(k_p).$$

Here $k_1, k_2, \ldots, k_p$ are the invariant factors of K. This is the *Jordan form* of $\Sigma_t(K)$ [2, p. 214].

Corollary 2. *All semigroups $\{\Sigma_t(K); t \geqslant 0\}$ corresponding to minimal X have the same Jordan form.*

Since $\{\Sigma_t(K)^*; t \geqslant 0\}$ may be used as the basic semigroup in the differential equation realization of X [8], this is a useful fact.

It can be shown that $\Sigma_t(K)^*$ and $\Sigma_t(K')$ are unitarily equivalent [2;Th. 13-2, p. 191]. This leads to the following useful fact.

Corollary 3. *Let K be a real inner function. Then $\Sigma_t(K)^*$ and $\Sigma_t(K)$ are quasi-similar. If $m = 1$, they are unitarily equivalent.*

In this paper it has been convenient and natural to restrict attention to Markovian splitting subspaces $X \sim (S, \bar{S})$ contained in the frame space $H^{\square}$. However to insure that there are representations $S = H^-(\mathrm{d}u)$ and $\bar{S} = H^+(\mathrm{d}\bar{u})$ so that isomorphism with $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ can be established it suffices to assume that X is *proper*, i.e. $\bigcap U_t S = 0$ and $\bigcap U_t \bar{S} = 0$. All $X \subset H^{\square}$ are proper under the assumption of strict non-cyclicity [6–8].

Corollary 4. *Let X be a proper Markovian splitting subspace. Then $U_t(X)^*$ and $U_t(X)$ are quasi-similar. If $m = 1$, they are unitarily equivalent.*

6. Remarks

The theorem presented in this note was first stated in [9], but there is a nontrivial gap in the proof which invalidates it for infinite-dimensional X. The same incomplete argument has also been used in [14,15]. As pointed out to us by A.E. Frazho, the proof in [6] of the same theorem (Lemma 3.7) is incorrect: A counter-example to the assertion of the second sentence of the if-part of this proof can be constructed based on Problem 9 in Halmos' Hilbert Space Problem Book [3]. However, all these results are valid if attention is restricted to finite-dimensional X.

A proof for the scalar case ($m = 1$), valid also for infinite-dimensional X, is given in [5]. However, since at the time of its appearance we were not aware of the error in [9], this proof is a bit implicit. To interpret [5] correctly, exchange 'minimal' for 'reduced' in Section 4. Then it is shown that all reduced X have the same $\mathcal{H}(K)$ much along the lines of Section 3 of this note.

One contribution of this paper is to extend the above results to the infinite-dimensional case. This could not be achieved by merely patching up the incomplete proofs of [9] and [6] (which incidentally are just a few lines), but a completely new approach had to be taken. This approach also spreads some further light on the structure of minimal Markovian splitting subspaces.

7. An example

To illustrate the main lemma we give a simple example. Let y be a 2-dimensional process with the rational spectral density

$$\Phi(s) = \frac{1}{(s^2-1)(s^2-4)} \times \begin{bmatrix} 17-2s^2 & -(s+1)(s-2) \\ -(s-1)(s+2) & 4-s^2 \end{bmatrix}.$$

Then the structural functions of X_- and X_+ can be seen to be

$$K_-(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s-1.2 & 1.6 \\ 1.6 & s+1.2 \end{bmatrix}$$

and

$$K_+(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2-70/37 & 24/37 \\ 24/37 & s+70/37 \end{bmatrix}$$

respectively. Both X_- and X_+ have dimension 3, but it can be shown that $\mathcal{H}(K_-) \cap \mathcal{H}(K_+) = 0$. However, as required by the main lemma, K_- and K_+ have the same invariant factors, namely

$$k_1(s) = \frac{s-1}{s+1} \quad \text{and} \quad k_2(s) = \frac{(s-1)(s-2)}{(s+1)(s+2)}.$$

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