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INFINITE DIMENSIONAL STOCHASTIC REALIZATIONS OF CONTINUOUS-TIME STATIONARY VECTOR PROCESSES

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In this paper we consider the problem of representing a given stationary Gaussian process with nonrational spectral density and continuous time as the output of a stochastic dynamical system. Since the spectral density is not rational, the dynamical system must be infinite-dimensional, and therefore the continuous-time assumption leads to certain mathematical difficulties which require the use of Hilbert spaces of distributions. (This is not the case in discrete time.) We show that, under certain conditions, there correspond to each proper Markovian splitting subspace, two standard realizations, one evolving forward and one evolving backward in time.

1. INTRODUCTION

Let $\{y(t);\ t \in \mathbb{R}\}$ be an m-dimensional stationary Gaussian vector process which is mean-square continuous and purely nondeterministic and which has zero mean. Such a process has a representation

$$y(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\hat{y}(i\omega) \tag{1.1a}$$

[5,11]. Here $d\hat{y}$ is an orthogonal stochastic measure with incremental covariance

$$E\{d\hat{y}(i\omega)d\hat{y}(i\omega)^{\dagger}\} = \frac{1}{2\pi}\,\Phi(i\omega)d\omega \tag{1.1b}$$

where Φ is the $m\times m$-matrix spectral density and $\dagger$ denotes transposition *and* complex conjugation. Let $p \le m$ be the rank of Φ.

It is well-known that, if Φ is a rational function, y has a (non-unique) representation

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$$\begin{cases} dx = Ax\,dt + B\,du & (1.2a) \\ y = Cx & (1.2b) \end{cases}$$

where $\{x(t);\ t \in \mathbb{R}\}$ is a stationary vector process of (say) dimension n, $\{u(t);\ t \in \mathbb{R}\}$ is a vector Wiener process defined on all of $\mathbb{R}$ with incremental covariance

$$E\{du(t)du(t)'\} = I\,dt \tag{1.3}$$

(' denotes transposition), and A, B and C are constant matrices of appropriate dimensions. Such a representation is called a *stochastic realization* of y, and n is its *dimension*. Clearly $\{x(t),\ t \in \mathbb{R}\}$ is a Gaussian Markov process.

It is the purpose of this paper to construct stochastic realizations (with the appropriate systems-theoretical properties) of processes y with nonrational spectral densities. Such realizations are necessarily infinite-dimensional, and the analysis will require methods akin to those used in infinite-dimensional deterministic realization theory [2,3,6]. Our approach, which is based on the geometric theory of stochastic realization theory [8,9,12] and applies Hilbert space constructions common in infinite-dimensional control theory [10], is coordinate-free. These results replace those in Section 9 of [8], which contains an error.

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2. PRELIMINARIES AND NOTATIONS

Let H be the Gaussian space of y, i.e. the Hilbert space generated by the random variables $\{y_k(t);\ t \in \mathbb{R},\ k=1,2,\dots,m\}$ with the inner product $\langle\xi,\eta\rangle = E\{\xi\eta\}$. Let H^- and H^+ be the subspaces generated by $\{y_k(t);\ t \le 0,\ k=1,2,\dots,m\}$ and $\{y_k(t);\ t \ge 0,\ 1=1,2,\dots,m\}$, respectively. If $\{\eta_\theta;\ \theta \in \Theta\}$ is a set of elements in H, $\mathrm{sp}\{\eta_\theta;\ \theta \in \Theta\}$ denotes the linear span, i.e. the set of all linear combinations of these elements. The word *subspace* will carry with it the assumption that it is closed.

For any two subspaces of H, A and B, $A \vee B$ denotes the closed linear hull and $A \oplus B$ the orthogonal direct sum of A and B. Moreover, E^A denotes orthogonal projection onto A, $\bar{E}^A B$ the closure of $E^A B$, and $A^\perp$ the orthogonal complement of A in H. We write $A \perp B$ to mean that A and B are orthogonal and $A \perp B \mid X$ to mean that they are *conditionally orthogonal* given the subspace X, i.e.

$$\langle E^X\alpha, E^X\beta\rangle = \langle\alpha,\beta\rangle \qquad \text{for all } \alpha \in A,\ \beta \in B\ .$$

Since y is stationary there is a strongly continuous group $\{U_t;\ t \in \mathbb{R}\}$ of unitary operators $H \to H$ such that $U_t y_k(0) = y_k(t)$ for $k=1,2,\ldots,m$ [11]. The subspace A is said to be *full range* if $\bigvee_{t\in\mathbb{R}}(U_t A) = H$, i.e. the closed linear hull of the subspace $\{U_t A;\ t \in \mathbb{R}\}$ equals H.

3. BACKGROUND

To provide a setting for our construction we shall briefly review some basic results from our geometric theory of stochastic realization. The reader is referred to [8,9] for full details.

A subspace $X \subset H$ such that

(i) $X^- \perp X^+ \mid X$
where $X^- := \bigvee_{t\le 0}(U_t X)$ and $X^+ := \bigvee_{t\ge 0}(U_t X)$

(ii) $y_k(0) \in X$ for $k=1,2,\ldots,m$

(iii) $(X^-)^\perp$ and $(X^+)^\perp$ are full range

is called a *proper Markovian splitting subspace*, proper because of condition (iii). If, for the moment, we assume that y is the output process of the finite-dimensional system (1.2), (i)-(iii) is a coordinate-free characterization of (1.2). In fact, if $E\{x(0)x(0)'\}>0$,

$$X = \{a'x(0) \mid a \in \mathbb{R}^n\} \tag{3.1}$$

is a proper Markovian splitting subspace. Condition (i) is equivalent to $\{x(t);\ t \in \mathbb{R}\}$ being a Markov process, i.e. that it has a representation (1.2a), (ii) is equivalent to

$$y_k(t) \in \{a'x(t) \mid a \in \mathbb{R}^n\} \qquad \text{for } k=1,2,\dots,m \; , \tag{3.2}$$

i.e. to the existence of a matrix C such that (1.2b) holds, and (iii) rules out the possibility that x has a deterministic component. The coordinate-free formulation (i)-(iii) enables us to handle also the fact that X is infinite-dimensional and the concept of minimality becomes especially simple: A Markovian splitting subspace is said to be *minimal* if it has no proper subspace which is also a Markovian splitting subspace. We refer the reader to [8,9] for a discussion of what conditions (such as strict noncyclicity) are needed for (iii) to hold.

It is not hard to show that

$$X = \bar{E}^X H^+ \oplus [X \cap (H^+)^\perp] \; . \tag{3.3}$$

An element in $X \cap (H^+)^\perp$ cannot be distinguished from zero by observing the future $\{y(t);\ t \geq 0\}$ and is therefore called *unobservable*. The splitting subspace X is said to be *observable* if the unobservable subspace is trivial, i.e. $X \cap (H^+)^\perp = 0$. Likewise

$$X = \bar{E}^X H^- \oplus [X \cap (H^-)^\perp] \; , \tag{3.4}$$

and we call X *constructible* if the unconstructable subspace $X \cap (H^-)^\perp = 0$. It can be shown that X is minimal if and only if it is both observable and constructible [12].

We shall provide a complete characterization of the class of proper Markovian splitting subspaces. To this end, first consider the class of $p \times m$ matrix functions $i\omega \to W(i\omega)$ satisfying

$$W(-i\omega)'W(i\omega) = \Phi(i\omega) \; . \tag{3.5}$$

Such functions exist [11; p.114], and they are called *full-rank spectral factors*. We shall say that W is *stable (completely unstable)* if it can be extended to the right (left) complex half plane and is analytic there. (In the rational case, this means that W has all its poles in the left (right) half plane.)

To each full-rank spectral factor W we may associate a Wiener process

$$u(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}-1}{i\omega} W^{-R}(i\omega)'d\hat{y}(i\omega) \tag{3.6}$$

defined on all of $\mathbb{R}$, where W^{-R} is any right inverse of W. (It can be shown that (3.6) is independent of the choice of right inverse.) Let $\mathcal{U}$ be the class of all Wiener processes u generated in this way, and let $\mathcal{U}^-$ ($\mathcal{U}^+$) be the subclass corresponding to a stable (strictly unstable) W. For each $u \in \mathcal{U}$ let $H(du)$, $H^-(du)$ and $H^+(du)$ denote the subspaces generated by $\{u_k(t);\ t \in \mathbb{R},\ k=1,2,\dots,m\}$, $\{u_k(t);\ t \le 0,\ k=1,2,\dots,m\}$ and $\{u_k(t);\ t \ge 0,\ l=1,2,\dots,m\}$, respectively. It is not hard to see that $H(du)=H$ for all $u \in \mathcal{U}$.

THEOREM 3.1 [8]. *The subspace* X *is a proper Markovian splitting subspace if and only if there are* $u \in \mathcal{U}^-$ *and* $\bar{u} \in \mathcal{U}^+$ *such that*

$$H = H^-(d\bar{u}) \oplus X \oplus H^+(du) \quad . \tag{3.7}$$

In that case, $X = H^-(du) \cap H^+(d\bar{u})$, $H^- \vee X = H^-(du)$, *and* $H^+ \vee X = H^+(d\bar{u})$. *The splitting subspace is observable if and only if*

$$H^+(d\bar{u}) = H^+ \vee H^+(du) \tag{3.8}$$

and constructible if and only if

$$H^-(du) = H^- \vee H^-(d\bar{u}) \quad . \tag{3.9}$$

Consequently each X is completely characterized by a pair $(W,\bar{W})$ of full rank-spectral factors, W being stable and $\bar{W}$ completely unstable. The ratio $K := W\bar{W}^{-R}$ is called the *structural function* of X. It plays an important role in the systems-theoretical characterization of X.

Decomposition (3.7) should be compared with the decomposition in terms of ingoing and outgoing subspaces in Lax-Phillips scattering theory [7] and K to the scattering operator.

Next we introduce a semigroup on each X. For $t \geq 0$ define the operator $U_t(X): X \to X$ by the relation $U_t(X)\xi = E^X U_t \xi$. A proof of the first part of the following theorem can be found in [7, p.62] and a proof of the second part in [8]. (Asterisk denotes adjoint.)

THEOREM 3.2. *The family of operators* $\{U_t(X);\ t \geq 0\}$ *is a strongly continuous semigroup of contraction operators on* X *which tend strongly to zero as* $t \to \infty$. *Moreover, for all* $\xi \in X$

and $t \geq 0$,

$$E^{H^-(du)} U_t \xi = U_t(X) \xi \tag{3.10}$$

and

$$E^{H^+(d\bar{u})} U_{-t} \xi = U_t(X)^* \xi . \tag{3.11}$$

4. STOCHASTIC REALIZATIONS

The given process $\{y(t);\ t \in \mathbb{R}\}$ will not have a finite-dimensional representation (1.2) unless it has a rational spectral density. Therefore we need to decide how to interpret (1.2) in the infinite-dimensional case. It is natural to require that A is the infinitesimal generator of a strongly continuous semigroup $\{e^{At}; t \geq 0\}$ defined on some Hilbert space X and that $B: \mathbb{R}^p \to X$ and $C: X \to \mathbb{R}^m$ are bounded operators. In the finite-dimensional case, the system equations (1.2) have the unique strong solution

$$\begin{cases} x(t) = \int_{-\infty}^{t} e^{A(t-\sigma)} B\, du(\sigma) & (4.1a) \\ y(t) = Cx(t) & (4.1b) \end{cases}$$

where the integral is defined in quadratic mean (Wiener integral). In the infinite-dimensional case, however, things are more complicated.

To begin with, the integral in (4.1a) may not be well-defined. (This requires that the X-valued function $t \to e^{At}B$ be square-integrable on $[0,\infty)$; cf [13,14].) If not, we shall have to define the *state process* x in some weak sense (to be specified below). Even if the integral is well-defined so that x exists as a strong Hilbert space valued random process, (4.1) may not be a strong solution of (1.2) [14]. In this case, (1.2) will simply be defined as (4.1); this is known as a "mild solution".

If (1.2) does have a strong solution, y must satisfy the integral equation

$$y(t) = y(0) + \int_0^t CAx(\sigma)d\sigma + \int_0^t CBdu(\sigma) \tag{4.2}$$

[13,14] and therefore

$$E^{H^-(du)}[y_k(h) - y_k(0)] = O(h) \tag{4.3}$$

for all $k = 1,2,\ldots,m$. In view of (3.10), this is equivalent to

$$y_k(0) \in \mathcal{D}(\Gamma) \qquad \text{for } m=1,2,\dots,m \tag{4.4}$$

where $\mathcal{D}(\Gamma)$ is the domain of the infinitesimal generator Γ of the semigroup $\{U_t(X);\ t \geq 0\}$. From now on, we shall assume that X is a proper Markovian splitting subspace such that (4.4) holds. This turns out to be a natural assumption even in those cases that our construction does not produce a system (1.2) with a strong solution. Condition (4.4) is nontrivial only if X is infinite-dimensional, for, if $\dim X < \infty$, $\mathcal{D}(\Gamma) = X$.

Then, following a standard construction [1,4], we define the space Z to be $\mathcal{D}(\Gamma)$ equipped with the graph topology

$$<\xi,\eta>_Z = <\xi,\eta>_X + <\Gamma\xi,\Gamma\eta>_X. \tag{4.5}$$

Since Γ is a closed operator whose domain is dense in X (see eg [7, p.247]), Z is a Hilbert space, which is continuously imbedded in X. The topology of Z is stronger than that of X, and therefore all continuous linear functionals on X are continuous on Z as well. Consequently we can think of the dual space X* as imbedded in the dual space Z*. Then, identifying X* with X, we have

$$Z \subset X \subset Z^* \tag{4.6}$$

where Z is dense in X, which in turn is dense in Z*. We shall write (z,z^*) to denote the value of the functional $z^* \in Z^*$ evaluated at $z \in Z$ (or, by reflexivity, the value at z^* of z regarded as a functional on Z*). Clearly the bilinear form (z,z^*) coincides with the inner product $<z,z^*>_X$ whenever $z^* \in X$.

Since $X \subset H^-(du)$, any $\xi \in X$ has a unique representation

$$\xi = \int_{-\infty}^{0} f(-\sigma)'du(\sigma) \tag{4.7}$$

where $f \in L_2(\mathbb{R}_+, \mathbb{R}^p)$ and the integral is defined in quadratic mean. Define the (real) Hilbert space

$$\mathcal{X} = \left\{ f \,\middle|\, \int_{-\infty}^{0} f(-\sigma)'du(\sigma) \in X \right\} \tag{4.8}$$

with inner product $<f,g>_{\mathcal{X}} = \int_0^\infty f(t)'g(t)dt$. It is a well-known property of Wiener integrals that the mapping $I_u\colon \mathcal{X} \to X$ defined by (4.7) $[\xi = I_u f]$ is an isometry, and therefore we have established an isometric isomorphism between X and $\mathcal{X}$.

Clearly $\{I_u^{-1}U_t(X)^*I_u; t \geqq 0\}$ is a strongly continuous semigroup on X. Let A be its infinitesemal generator, i.e.

$$e^{At} = I_u^{-1}U_t(X)^*I_u. \tag{4.9}$$

The operator A is (in general) unbounded and densely defined in X. The adjoint A* is the infinitesimal generator of the adjoint semigroup $\{I_u^{-1}U_t(X)I_u \;;\; t \geqq 0\}$. Since

$$U_t\xi = \int_{-\infty}^{t} f(t-\sigma)'du(\sigma) \quad , \tag{4.10}$$

(3.10) yields

$$U_t(X)\xi = \int_{-\infty}^{0} f(t-\sigma)'du(\sigma) \quad . \tag{4.11}$$

Therefore,

$$\frac{1}{h}\,[U_h(X)\,\xi - \xi] = \int_{-\infty}^{0} \frac{1}{h}[f(h-\sigma) - f(-\sigma)]'du(\sigma) \tag{4.12}$$

and consequently A*f is the L_2 derivative of f (i.e. the limit in L_2 topology of the difference quotient).

Hence we have a functional representation of (4.6), namely

$$Z \subset X \subset Z^* \tag{4.13}$$

where $Z := I_u^{-1}Z$ is $\mathcal{D}(A^*)$ equipped with the inner product

$$\langle f,g\rangle_Z = \langle f,g\rangle_X + \langle A^*f,A^*g\rangle_X \tag{4.14}$$

i.e. a Hilbert space continuously imbedded in X, and Z^* is its dual, constructed as above. Here Z is a subspace of the Sobolev space $H^1(0,\infty)$, and Z^* is a space of distributions [1]. As before we write (f,f^*) to denote the scalar product between Z and Z^* extending $\langle f,f^*\rangle_X$ from $Z \times X$ to $Z \times Z^*$.

Next, define D: $Z \to X$ to be the differentiation operator. Then $Df = A^*f$ for all $f \in Z$, but, since $\|Df\|_X \leqq \|f\|_Z$, D is a bounded operator (in Z-topology). Its adjoint D^*: $X \to Z^*$ is the extension of A to X, because $(f,D^*g) = \langle A^*f,g\rangle_X$. Since $\{e^{A^*t};\, t \geqq 0\}$ is a completely continuous contraction semigroup (Theorem 3.2), D is dissipative, i.e. $\langle Df,f\rangle_X \leqq 0$ for all $f \in Z$, and I-D maps Z onto X, i.e.

$$(I - D)Z = X \tag{4.15}$$

[17; p.250]. Moreover, I - D is injective. In fact, in view of the dissipative property,

$$\| (I - D)f \|_X^2 \geq \| f \|_X^2 + \| Df \|_X^2 . \tag{4.16}$$

Consequently, $(I - D)^{-1}: X \to Z$ is defined on all of X, and, as can be seen from (4.16), it is a bounded operator. Likewise, the adjoint $(I - D^*)^{-1}$ is a bounded operator mapping Z^* onto X.

Now, assume that $f \in Z$, and let ξ be defined by (4.7), i.e. $\xi = I_u f$. Then it follows from (4.11) that $f(t+\sigma) = [I_u^{-1} U_t(X)\xi](\sigma) = (e^{A^*t}f)(\sigma)$ for $\sigma \geq 0$, i.e.

$$f(t) = (e^{A^*t}f)(0) . \tag{4.17}$$

Since Z is a *bona fide* function space and e^{A^*t} maps Z into Z, (4.17) is well-defined. In fact, as Z is a subspace of the Sobolev space $H^1(0,\infty)$, the evaluation functionals $\delta_k \in Z^*$ defined by $(f,\delta_k) = f_k(0)$, $k = 1,2,\ldots,m$, are continuous, because the evaluation operator in $H^1(0,\infty)$ is [1,4]. (Note that, since δ_k is restricted to Z, it is not the Dirac function.)

Consequently, we have

$$f_k(t) = (e^{A^*t}f,\delta_k) \tag{4.18}$$

We wish to express this in terms of the inner product in X, which from now on we shall denote $\langle\cdot,\cdot\rangle$, dropping the subscript X, whenever there is no risk for misunderstanding. To this end, note that (4.18) can be written

$$f_k(t) = \langle (I-D)e^{A^*t}f,\ (I-D^*)^{-1}\delta_k \rangle \tag{4.19}$$

Since $Df = A^*f$ and A^* and e^{A^*t} commute, this yields

$$f_k(t) = \langle (I-D)f,\ e^{At}Be_k \rangle , \tag{4.20}$$

where $B : \mathbb{R}^p \to X$ is the bounded operator

$$Ba = \sum_{k=1}^{p} (I - D^*)^{-1}\delta_k a_k \tag{4.21}$$

and e_k is the k:th axis unit vector in $\mathbb{R}^p$. Therefore, in view of (4.10), we have

$$U_t\xi = \int_{-\infty}^{t} \sum_{k=1}^{p} \langle (I-D)f, e^{A(t-\sigma)}Be_k \rangle du_k(\sigma) . \tag{4.22}$$

If the integral (4.1a) is well-defined, i.e. $t \to e^{At}B$ belongs to $L_2(\mathbb{R}_+, X)$, the usual limit argument yields

$$U_t\xi = \langle g,x(t) \rangle \tag{4.23}$$

where $g := (I-D)f = (I-D)I_u^{-1}\xi$. As can be seen from the following theorem, $x(t)$ being a strong X-valued random variable is a property of the structural function K of the splitting subspace X.

THEOREM 4.1. *Let* A *and* B *be given by (4.9) and (4.21) respectively. A necessary condition for the integral (4.1a) to define a (strong)* X*-valued random variable* $x(t)$ *is that the structural function* K *be meromorphic in the whole complex plane and analytic along the imaginary axis. A sufficient condition is that* K *be analytic in some strip* $-\alpha < \mathrm{Re}(s) \leq 0$ *(where* s *is the complex variable).*

PROOF. To establish the necessary condition, assume that (4.1a) is well-defined. It is no restriction to set $t=0$. Let $g_1, g_2 \in X$ be arbitrary, and define, for $i=1,2$, $f_i := (I-D)^{-1}g_i$ and $\xi_i := I_u f_i$. Then, since $E\{\xi_1\xi_2\} = \langle f_1, f_2\rangle$, (4.23) yields

$$E\{\langle g_1, x(0)\rangle\langle g_2, x(0)\rangle\} = \langle g_1, \Lambda g_2\rangle \tag{4.24}$$

where $\Lambda := [(I-D)(I-D^*)]^{-1}$ is the correlation operator. This operator must be nuclear [15; p.9], and therefore $(I-D)^{-1}$ is compact [4; p.34], and so is $e^{Dt}(I-D)^{-1}$ for all $t>0$, since e^{Dt} is bounded. Then the rest follows from [7; p.83]. In the sufficiency part, the assumption on K implies that the spectrum of A^* lies in the region $\mathrm{Re}(s) \leqq -\alpha < 0$ [7; p. 70]. Therefore there are positive numbers k and $\beta < \alpha$ such that $\|e^{A^*t}\| \leq k e^{-\beta t}$ [17]. But $\|e^{At}\| = \|e^{A^*t}\|$, and hence $\|e^{At}\| \in L_2(0,\infty)$. □

If the stochastic integral (4.22) is not well-defined, we can nevertheless think of x as a generalized (weakly defined) random process [4; p.242]. In fact, in view of (4.15), (4.22) assigns to each pair $(t,g) \in \mathbb{R} \times X$ a unique random variable $U_t\xi$. Thought of in this way, (4.23) makes perfect sense, and we shall take this as our definition of $\{x(t);\ t \in \mathbb{R}\}$ whenever this process is not strongly defined.

Since there is a one-one correspondence in (4.23) between $\xi \in Z$ and $g \in X$, it follows from (4.15) that $\{\langle g, x(0)\rangle \mid g \in X\} = Z$. But Z is dense in X, and therefore

$$X = \mathrm{cl}\{\langle g, x(0)\rangle \mid g \in X\} \tag{4.25}$$

where cl stands for closure (in the topology of H). This is the infinite-dimensional counterpart of (3.1). In fact, in the special case that $\dim X < \infty$, $Z = X$, and then no closure is needed.

It remains to construct a counterpart of (1.2b). In view of (4.4), $y_k(0) \in Z$, and therefore $w_k := I_u^{-1} y_k(0) \in Z$ for $k = 1,2,\dots,m$. Therefore, defining $C : X \to \mathbb{R}^m$ by

$$(Cg)_k = \langle (I-D)w_k, g\rangle \tag{4.26}$$

for $k = 1,2,...,m$, (4.22) yields

$$y(t) = \int_{-\infty}^{t} Ce^{A(t-\sigma)} B du(\sigma). \tag{4.27}$$

We may write this as

$$\begin{cases} dx = Axdt + Bdu & (4.28a) \\ y = Cx & (4.28b) \end{cases}$$

if we interpret this system as described above.

The above construction is in several respects similar to those found in (infinite-dimensional) deterministic realization theory [2, 3, 6]. Note, however, that in comparison with the shift realizations in, for example, [3], our set-up has been transposed. This is necessary in order to obtain results such as those in Theorems 5.1 and 5.2 and is quite natural if we think of stochastic realization theory as representations of *functionals* of data.

5. OBSERVABILITY AND REACHABILITY

The system (4.28) is said to be *observable* if $\cap_{t\geq 0} \ker C e^{At} = 0$ and *reachable* if $\cap_{t\geq 0} \ker B^* e^{A^* t} = 0$ [3].

THEOREM 5.1. *The system (4.28) is reachable.*

PROOF. Since $\langle (I-D)f, Ba\rangle = f(0)'a$ for all $f \in Z$, the adjoint operator $B^* : X \to \mathbb{R}^p$ is given by

$$B^* g = [(I-D)^{-1} g](0). \tag{5.1}$$

Therefore, since $e^{A^* t}$ and $(I-D)^{-1}$ commute,

$$B^* e^{A^* t} g = [e^{A^* t}(I-D)^{-1} g](0), \tag{5.2}$$

which, in view of (4.17), can be written

$$B^* e^{A^* t} g = f(t), \tag{5.3}$$

where $f := (I-D)^{-1} g$. Hence $g \in \cap_{t\geq 0} \ker B^* e^{A^* t}$ if and only if $f(t) = 0$ for all $t \geq 0$, i.e. $f = 0$, or, equivalently, $g = 0$. This establishes reachability. □

THEOREM 5.2. *The system (4.28) is observable if and only if the splitting subspace (4.25) is observable (in the sense of Section 3).*

For the proof we need a few concepts. Define M to be the vector space

$$M = \mathrm{sp}\{E^X y_k(t)\ ;\ t \geq 0,\ k = 1,2,\dots,m\}. \tag{5.4}$$

Since $E^X y_k(t) = U_t(X) y_k(0)$, M is invariant under the action of $U_t(X)$, i.e. $U_t(X)M \subset M$ for all $t \geq 0$. Moreover, $\mathcal{D}(\Gamma)$ is invariant under $U_t(X)$; this is a well-known property of a semigroup. Hence, it follows from (4.4) that $M \subset Z$. Now, if X is observable, M is dense in X, but this does not automatically imply that M is dense in Z (in graph topology). In the present case, however, this is true, as can be seen from the following lemma. In the terminology of [1; p.101], this means that the Hilbert space Z containing the vector space M and continuously embedded in the Hilbert space X is *normal*.

LEMMA 5.1. *Let* X *be observable. Then* M *is dense in* Z.

A proof of this lemma provided by A. Gombani will be given below. Setting $\mathcal{M} := I_u^{-1} M$, we have

$$\mathcal{M} = \mathrm{sp}\{e^{A^* t} w_k\ ;\ t \geq 0,\ k = 1,2,\dots,m\} \tag{5.5}$$

and therefore we may, equivalently, state Lemma 5.1 in the following way: *If* X *is observable, then* $\mathcal{M}$ *is dense in* $\mathcal{Z}$.

LEMMA 5.2. *The vector space* $\mathcal{M}$ *is dense in* $\mathcal{Z}$ *if and only if* $(I-D)\mathcal{M}$ *is dense in* $\mathcal{X}$.

PROOF. (if): Assume that $(I-D)\mathcal{M}$ is dense in $\mathcal{X}$. Then (4.15), i.e. $(I-D)\mathcal{Z} = \mathcal{X}$, and (4.16), i.e. $\|f\|_{\mathcal{Z}} \leq \|(I-D)f\|_{\mathcal{X}}$, imply that $\mathcal{M}$ is dense in $\mathcal{Z}$.
(only if): This part follows from (4.15) and the trivial relation $\|(I-D)f\|_{\mathcal{X}}^2 \leq 2\|f\|_{\mathcal{Z}}^2$. □

PROOF OF THEOREM 5.2. First note that, since e^{A^*t} and $(I-D)$ commute,

$$(Ce^{At}g)_k = \langle (I-D)e^{A^*t}w_k, g\rangle. \tag{5.6}$$

Hence $g \in \bigcap_{t\geq 0} \ker Ce^{At}$ if and only if

$$\langle h,g\rangle = 0 \quad \text{for all } h \in (I-D)\mathcal{M}. \tag{5.7}$$

Now, if (4.28) is observable, only $g=0$ satisfies (5.7). Hence $(I-D)M$ is dense in X. Therefore, M is dense in Z (Lemma 5.2) and hence in X (weaker topology), or, equivalently M is dense in X, i.e. X is observable. Conversely, assume that X is observable. Then M is dense in Z (Lemma 5.1), and consequently $(I-D)M$ is dense in X (Lemma 5.2). But then only $g=0$ can satisfy (4.34) and therefore (4.28) is observable. □

PROOF OF LEMMA 5.2. Assume that M is dense in X, and let $\bar{M}$ be the closure of M in graph topology. We know that $\bar{M}\subset Z$, and we want to show that $\bar{M}=Z$. To this end, define $\bar{D}$ to be the restriction of D to $\bar{M}$. Then $\bar{D}$ is an unbounded operator defined on a dense subset of X, and, like D, it is closed and dissipative. Hence the range of $(I-\bar{D})$ is closed [3; Thm 3.4, p.79]. Therefore, if we can show that the range of $(I-\bar{D})$ is dense in X, we know that it is all of X. This would mean that $\bar{D}$ is maximal dissipative [3; Thm 3.6, p.81]. However, D is a dissipative extension of $\bar{D}$ and hence $\bar{D}=D$. Then $\mathcal{D}(\bar{D})=\mathcal{D}(D)$, i.e. $\bar{M}=Z$ as required.

Consequently it remains to prove that $(I-\bar{D})\bar{M}$ is dense in X. Since $\bar{M}$ is dense in X, we only need to show that the equation $(I-\bar{D})f=g$, i.e.

$$\dot{f}-f=-g \tag{5.8}$$

has a solution $f\in\bar{M}$ for each $g\in\bar{M}$. But, for such a g, (5.8) has the L_2 solution

$$f(t) = \int_0^\infty e^{-\sigma}g(t+\sigma)d\sigma = \int_0^\infty (e^{A^*\sigma}g)(t)dm(\sigma) \tag{5.9}$$

where $dm=e^{-\sigma}d\sigma$, so it remains to show that this f belongs to $\bar{M}$. It follows from (5.5), that $e^{A^*\sigma}M\subset M$, and therefore, by continuity, $e^{A^*\sigma}g\in\bar{M}$ for each $\sigma\geq 0$. The function $\sigma\to e^{A^*\sigma}g$ is therefore mapping $\mathbb{R}_+$ into $\bar{M}$. It is clearly strongly measurable, and, since $e^{A^*\sigma}$ is a contraction, $\|e^{A^*\sigma}g\|_{\bar{M}}\leq\|g\|_{\bar{M}}$. Hence

$$\int_0^\infty \|e^{A^*\sigma}g\|_{\bar{M}}^2 dm(\sigma) < \infty \tag{5.10}$$

and consequently (5.9) is a Bochner integral [17, p.133]. Hence, by definition, $f\in\bar{M}$ as required. □

6. BACKWARD REALIZATIONS

As can be seen from (3.7), the splitting subspace X is orthogonal to $H^+(du)$. Therefore, in the model (4.28), the future increments of the generating process u are independent of present state. A system with this property is said to evolve *forward* in time.

Replacing condition (4.4) by

$$y_k(0) \in \mathcal{D}(\Gamma^*) \quad \text{for } m = 1,2,\dots,m \tag{6.1}$$

we can proceed as in Section 4 with obvious modifications. such as replacing $U_t(X)$, H^+ and $H^-(du)$ by $U_t(X)^*$, H^- and $H^+(d\bar{u})$ respectively, to construct a system

$$\begin{cases} d\bar{x} = \bar{A}\bar{x}dt + \bar{B}d\bar{u} & (6.2a) \\ y = \bar{C}\bar{x} & (6.2b) \end{cases}$$

having the same properties as (4.28), except that it evolves *backwards* in time. By this we mean that $X \perp H^-(d\bar{u})$, i.e. the past increments of the generating process $\bar{u}$ are independent of present state. That this is so again follows from (3.7). A backward realization (6.2) is said to be *constructible* if $\cap_{t\geq 0}\bar{C}e^{\bar{A}t} = 0$ and *controllable* if $\cap_{t\geq 0}\bar{B}^*e^{\bar{A}^*t} = 0$. Then, the backward counterparts of Theorems 5.1 and 5.2 say that (6.2) is always controllable and constructible if and only if X is constructible in the sense defined in Section 3.

Consequently, if X is such that

$$y_k(0) \in \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^*), \quad k = 1,2,\dots,m, \tag{6.3}$$

X has both a forward and a backward realization. Such an X will be called *regular*. A process y may have both regular and nonregular splitting subspaces, regularity depending on the position of X in the natural partial ordering of minimal Markovian splitting subspaces [8,9]. An investigation of these questions can be found in [16].

It sufficies to mention here that, if $(W,\bar{W})$ is the pair of spectral factors of X, (4.4) holds if and only if there is a constant matrix N such that $sW(s) - N$ is square integrable on the imaginary axis. Noting that W is the Laplace transform of

$(w_1, w_2, \ldots, w_m)$ and that (4.4) is equivalent to $w_k \in \mathcal{D}(A^*)$, this follows from Lemma 3.1 in [7]. Likewise (6.1) holds if and only if there is a constant matrix $\bar{N}$ such that $s\bar{W}(s) - \bar{N}$ is square integrable on the imaginary axis.

REFERENCES

1. J.-P. Aubin, *Applied Functional Analysis*, Wiley-Interscience, 1979.
2. J.S. Baras, R.W. Brockett, and P.A. Fuhrmann, State space models for infinite dimensional systems, *IEEE Trans. Automatic Control* AC-19 (1974), 693-700.
3. P.A. Fuhrmann, *Linear Systems and Operators in Hilbert Space*, McGraw Hill, 1981.
4. I.M.Gelfand and N. Ya. Vilenkin, *Generalized Functions*, Vol. 4, Academic Press, 1964.
5. I.I.Gikhman and A.V. Skorokod, *Introduction to the Theory of Random Processes*, Saunders, 1965.
6. J.W. Helton, Systems with infinite-dimensional state spaces: The Hilbert space approach, *Proc. IEEE* 64 (1979), 145-160.
7. F.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, 1967.
8. A. Lindquist and G. Picci, State space models for gaussian stochastic processes, *Stochastic Systems: The Mathematics of Filtering and Identification and Applications*, M. Hazewinkel and J.C. Willems (eds.), Reidel Publ. Co., 1981, 169-204.
9. A. Lindquist, M. Pavon, and G. Picci, Recent trends in stochastic realization theory, *Prediction Theory and Harmonic Analysis*, The Pesi Masani Volume, V. Mandrekar and H. Salehi (eds.), North Holland, 1983.
10. J.L. Lions, *Optimal Control of System Governed by Partial Differential Equations*, Springer Verlag, 1971.
11. Yu.A. Rozanov, *Stationary Random Processes*, Holden-Day, 1967.
12. G. Ruckebusch, Théorie géométrique de la représentation markovienne, *Ann. Inst. Henri Poincaré* XVI3 (1981), 225-297.
13. A. Bensoussan, *Filtrage Optimal des Systèmes Linéaires*, Dunod, 1971.
14. R.F. Curtain and A.J. Pritchard, *Infinite Dimensional Linear Systems*, Springer Verlag, 1978.
15. I.A. Ibraganov and Y.A. Rozanov, *Gaussian Random Processes*, Springer Verlag, 1978.
16. A. Lindquist and G. Picci, Forward and backward semimartingale representations for stationary increment processes, *Proc.*

1983 Intern. Symp. Math. Theory Networks & Systems, Beer Sheva, Israel, Springer Verlag, (to be published).

17. K. Yosida, *Functional Analysis*, Springer Verlag, 1965.

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