OPTIMAL DAMPING OF FORCED OSCILLATIONS IN DISCRETE-TIME SYSTEMS BY OUTPUT FEEDBACK*

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Abstract. In this paper we consider optimal control by output feedback of a linear discrete-time system corrupted by an additive harmonic vector disturbance with known frequencies but unknown amplitudes and phases. We consider both a deterministic and a stochastic version of the problem. The object is to design a robust optimal regulator which is universal in the sense that it does not depend on the unknown amplitudes and phases and is optimal for all choices of these values. We show that, under certain natural technical conditions, an optimal universal regulator (OUR) exists in a suitable class of stabilizing and realizable linear regulators, provided the dimension of the output is no smaller than the dimension of the harmonic disturbance. When this dimensionality condition is not satisfied, the existence of an OUR is not a generic property, and consequently it does not exist from a practical point of view. For the deterministic problem we also show that, under slightly stronger technical conditions, any linear OUR is also optimal in a very wide class of nonlinear regulators. In the stochastic case we are only able to show optimality in the linear class of regulators.

1. Introduction

Many important engineering problems can be formulated mathematically as a linear-quadratic regulator problem with the added complication of an unobserved harmonic additive disturbance, for which only the frequencies are known. Some examples, among many others, are vibration damping in industrial machines and helicopters [5, 6, 8, 9, 2, 19], noise reduction in vehicles and transformers [18], control of aircraft in the presence of wind shear [14, 17, 21], and control of the roll motion of a ship [10]. Such a harmonic disturbance adds critically stable dynamics which is unobservable and unstabilizable, and therefore traditional linear-quadratic methods cannot be used. Nor can one in general use a discrete-time version of the methods proposed in [3, 4]. In [16] this problem was solved in the case of complete state information. In the present paper, the same methodology is extended to take care of the case of output feedback.

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Specifically, we consider a discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t + Ew_t \tag{1.1a}$$

$$y_t = Cx_t \tag{1.1b}$$

$(t = 0, 1, 2, \dots)$ with state $x_t \in \mathbb{R}^n$, output $y_t \in \mathbb{R}^m$ and two vector inputs, namely a control $u_t \in \mathbb{R}^k$ an unobserved disturbance $w_t \in \mathbb{R}^\ell$ which we shall take to be harmonic with known frequencies but unknown amplitudes and phases. More precisely,

$$w_t = \sum_{j=1}^{N} w^{(j)} e^{i\theta_j t}, \tag{1.2}$$

where the frequencies

$$-\pi < \theta_1 < \theta_2 < \cdots < \theta_N \leq \pi \tag{1.3}$$

are known, but the complex vector amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, in which the phases have been absorbed, are not. Moreover, A, B, C, E are constant real matrices of appropriate dimensions such that (A, B) is stabilizable and (C, A) is detectable, and without loss of generality

$$\operatorname{rank} C = m \quad \text{and} \quad \operatorname{rank} E = \ell. \tag{1.4}$$

In fact, if the first condition is not satisfied, some components of y_t could be eliminated. Moreover, if $\hat{\ell} := \operatorname{rank} E < \ell$, Ew_t may be exchanged by $\hat{E}\hat{w}_t$, where $\hat{w}_t \in \mathbb{R}^{\hat{\ell}}$ by an obvious reformulation. Of course, (1.4) implies that $m \leq n$ and that $\ell \leq n$.

The deterministic problem to be considered in this paper is to damp the forced oscillation in the system (1.1) by output feedback. This is to be done so as to minimize a cost functional

$$\Phi = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \Lambda(x_t, u_t), \tag{1.5}$$

where $\Lambda(x, u)$ is a real quadratic form

$$\Lambda(x, u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{1.6}$$

with properties to be specified in Section 2. This functional is a measure of the forced oscillations in the closed-loop system and, for the classes of admissible regulators to be defined next, it does not depend on initial conditions.

What we want to construct is a regulator which is optimal in some suitable class and which does not depend on the unknown complex vector amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ and consequently is *universal* in the sense that it simultaneously solves the complete family of optimization problems corresponding to different choices of these complex amplitudes. Such a regulator will be referred to as an *optimal universal regulator* (OUR). Moreover, this optimal regulator must be robust with respect to possible estimation errors in the known frequencies $\theta_1, \theta_2, \dots, \theta_N$ in the sense that the cost Φ is continuous in the estimation errors and tends to its true optimal value as the errors tend to zero. It is not hard to see that there are optimal regulators which depend on $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, but that there actually exist universal ones is perhaps surprising.

At first sight it might be tempting to try to apply standard linear-quadratic regulator theory to an extended control system obtained by amending to (1.1) the critically stable autonomous system

$$\begin{cases} z_{t+1} = F z_t \\ w_t = H z_t \end{cases} \tag{1.7}$$

where F and H are matrices of dimensions $N\ell \times N\ell$ and $\ell \times N\ell$ respectively given by

$$F = \begin{bmatrix} e^{i\theta_1} I_\ell & 0 & \dots & 0 \\ 0 & e^{i\theta_2} I_\ell & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\theta_N} I_\ell \end{bmatrix} \qquad H = \begin{bmatrix} I_\ell & I_\ell & \dots & I_\ell \end{bmatrix} \tag{1.8}$$

and $z_0 := \mathrm{col}(w^{(1)}, w^{(2)}, \dots, w^{(N)}) \in \mathbb{C}^{N\ell}$. In this context, universality of a regulator would imply that it does not depend on z_0 and that it is optimal for all z_0. However, since this would add uncontollable, critically stable modes to the system, standard linear-quadratic regulator theory does not apply.

If we were to consider the simple optimization problem to find a *process* (x_t, u_t) minimizing Φ subject to the constraints (1.1a), it would, as pointed out in [16], be necessary to assume that

$$\frac{1}{\sqrt{t}} |x_t| \to 0 \quad \text{as } t \to \infty, \tag{1.9}$$

in order to insure that the cost is finite. We shall denote by $\mathcal{A}$ the class of all processes (x_t, u_t) satisfying (1.1a) and this stability condition. (To insure that the infimum of Φ over it is not $-\infty$, we must of course introduce some condition on the quadratic form (1.6). This will be done in Section 2.)

However, this class of admissible processes is much too large since we would like to optimize by feedback and in such a way that the optimal regulator is universal. Therefore we shall consider two classes of regulators, one that is linear and one that allows for nonlinear regulators.

Let $\mathfrak{L}$ be the class of all linear realizable and stabilizing regulators

$$\mathfrak{L}: \quad D(\sigma) u_t = N(\sigma) y_t, \tag{1.10}$$

where $D(\lambda)$ and $N(\lambda)$ are real matrix polynomials of dimensions $k \times k$ and $k \times m$ respectively, and σ is the forward shift $\sigma y_t = y_{t+1}$. Here *realizable* means that the leading coefficient matrix of $D(\lambda)$ is nonsingular and $\deg N \leq \deg D$ so that $D(\lambda)^{-1} N(\lambda)$ is a proper rational matrix function. By *stabilizing* we mean that the coefficient matrix of the closed-loop system is asymptotically stable, i.e., the matrix polynomial

$$\Xi(\lambda) = \begin{bmatrix} \lambda I_n - A & -B \\ -N(\lambda) C & D(\lambda) \end{bmatrix} \tag{1.11}$$

is such that $\det \Xi(\lambda) \neq 1$ for $|\lambda| \geq 1$. (Here, of course, I_n is the $n \times n$ identity matrix.)

Secondly, we consider a class $\mathfrak{N}$ of in general nonlinear regulators

$$\mathfrak{N}: \quad u_t = f_t(y_t, y_{t-1}, \dots, y_0, u_{t-1}, \dots, u_0), \tag{1.12}$$

which are stabilizing in the sense that the processes (x_t, u_t), generated by such feedback controls all belong to $\mathcal{A}$. Clearly, $\mathfrak{L} \subset \mathfrak{N}$. It is trivial but useful for the subsequent analysis to note that, if $\mathcal{A}_{\mathfrak{L}}$ and $\mathcal{A}_{\mathfrak{N}}$ are the classes of processes (x_t, u_t) which are generated by the regulators in $\mathfrak{L}$ and $\mathfrak{N}$ respectively, then

$$\mathcal{A}_{\mathfrak{L}} \subset \mathcal{A}_{\mathfrak{N}} \subset \mathcal{A}. \tag{1.13}$$

In this paper we show that, under suitable technical conditions, there exists an optimal universal regulator in the linear class $\mathfrak{L}$, provided $\ell \leq m$, and that this regulator is also OUR in the class $\mathfrak{N}$ under slightly stronger conditions. The optimal universal regulator is not unique so a general description of all such regulators is obtained. If $\ell > m$, an OUR will exist only when the system parameters satisfy certain equations, making the existence of an OUR a nongeneric property. From a practical point of view this implies that there is no OUR if $\ell > m$. Nonuniversal optimal regulators are given in Section 6 in the case of nonexistence of an OUR.

We stress that our solutions are optimal in the sense stated in this paper only, and that other desirable design specifications may not be satisfied for an arbitrary universal optimal regulator. Therefore it is an important property of our procedure that it allows for a considerable degree of design freedom.

Next, let us consider a stochastic version of this problem. Merely replacing the amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ by (jointly distributed) random vectors and the cost function (1.5) by

$$\Phi = \lim_{T\to\infty} \mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T}\Lambda(x_t, u_t)\}, \tag{1.14}$$

where $\mathrm{E}\{\cdot\}$ denotes mathematical expectation, would, as explained in Section 8, amount to a trivial extension of the deterministic problem formulation described above. It turns out that the appropriate stochastic problem is obtained by replacing (1.7) by the stochastic system

$$\begin{cases} z_{t+1} = Fz_t + Gv_t \\ w_t = Hz_t + Kv_t \end{cases} \tag{1.15}$$

where z_0 a random vector and where $v_0, v_1, v_2, \dots$ is a white noise process with an intensity V_t tending sufficiently quickly to zero to insure that $\mathrm{E}\{|w_t|^2\}$ remains bounded. An optimal regulator is *universal* for this problem if it does not depend on z_0, G, K and V_t, and it is optimal for all values of these quantities. In Section 8 we show that any optimal universal regulator for the deterministic control problem is an optimal universal regulator for the stochastic problem, at least in the class $\mathfrak{L}$. Whether it is also optimal in the class $\mathfrak{N}$ is still an open question.

The outline of our paper is as follows. Section 2 is a preliminary section in which we introduce some technical conditions to be used in different contexts later. Sections 3–5 are devoted to the underlying optimization problem over $\mathcal{A}_{\mathfrak{L}}$. This is done in a simpler way, but somewhat more limited context, than in [16], and therefore we shall need the more general result of [16] later in Section 7 for the case of nonlinear regulators. In Section 4 a parameterization of all regulators in $\mathfrak{L}$, akin to that of Youla but especially adapted to our present problem, is introduced. Section 5 presents the design of the OUR in $\mathfrak{L}$ in the case that $\ell \leq m$, and Section 6 considers the case

$\ell > m$. In Section 7 the existence of optimal universal regulators in the class $\mathcal{N}$ is studied. The theorems stated there could be regarded as our main result. Finally, Section 8 is devoted to the stochastic case.

2. Assumptions and definitions

In this section we introduce some technical conditions to be referred to later in this paper.

First we need to specify required properties of the quadratic form (1.6). It could be indefinite, but in order to insure that the cost function (1.5) is bounded from below we must introduce some positivity condition.

Strong frequency domain condition (SFDC). There is a $\delta > 0$ such that

$$\Lambda(\tilde{x}, \tilde{u}) \geq \delta(|\tilde{x}|^2 + |\tilde{u}|^2) \tag{2.1}$$

for all $\tilde{x} \in \mathbb{C}^n$, $\tilde{u} \in \mathbb{C}^k$, $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and

$$\lambda\tilde{x} = A\tilde{x} + B\tilde{u}. \tag{2.2}$$

Since $\tilde{x}$ and $\tilde{u}$ are complex, * in (1.6) will of course have to be taken as Hermitian conjugation instead of merely transposition.

For the linear case the following weaker condition will suffice.

Weak frequency domain condition (WFDC). There is a $\delta > 0$ such that (2.1) holds for all $\tilde{x} \in \mathbb{C}^n$, $\tilde{u} \in \mathbb{C}^k$ and

$$\lambda = e^{i\theta_j} \quad j = 1, 2, \ldots, N \tag{2.3}$$

satisfying (2.2).

Note that both of these conditions are invariant under the action of the feedback group

$$(A, B) \to (TAT^{-1} + TBK, TB), \tag{2.4}$$

where T is a nonsingular matrix and K is an arbitrary matrix of appropriate dimensions. Moreover, if A does not have any eigenvalues on the unit circle, SFDC is equivalent to

$$\Lambda(\tilde{x}, \tilde{u}) > 0 \quad \text{for all } \tilde{u} \neq 0, \quad \tilde{x} = (\lambda I - A)^{-1}B\tilde{u} \tag{2.5}$$

and λ on the unit circle, and WFDC is equivalent to (2.5) for all $\lambda = e^{i\theta_j}, j = 1, 2, \ldots, N$. Therefore, writing

$$\Lambda(\tilde{x}, \tilde{u}) = \tilde{u}^*\Pi(\lambda)\tilde{u} \quad \text{where } \tilde{x} = (\lambda I - A)^{-1}B\tilde{u} \tag{2.6}$$

and where the Hermitian $k \times k$ matrix function

$$\Pi(\lambda) = B^*(\bar{\lambda}I - A^*)^{-1}Q(\lambda I - A)^{-1}B + B^*(\bar{\lambda}I - A^*)^{-1}S + S^*(\lambda I - A)^{-1}B + R, \tag{2.7}$$

SFDC may be written

$$\Pi(\lambda) > 0 \quad \text{for all } \lambda \text{ on the unit circle} \tag{2.8}$$

and WFDC as

$$\Pi(e^{i\theta_j}) > 0 \quad \text{for } j = 1, 2, \dots, N. \tag{2.9}$$

Secondly, without loss of generality, we also assume that the matrix A is stable, i.e.,

$$\det(\lambda I - A) \neq 0 \quad \text{for } |\lambda| \geq 1. \tag{2.10}$$

In fact, if it is not, we can always stabilize by dynamic feedback, in general at the price of increased dimension of the system. Under very special conditions (see, for example, [11]), there is a matrix K such that $\Gamma := A + BKC$ is a stable matrix, and then the feedback law

$$u_t = KCx_t + v_t \tag{2.11}$$

allows us to exchange (1.1a) for a similar system where A and u_t are exchanged for $A + BKC$ and v_t respectively. In general, however, an observer must be used. As is well-known, one can always use the controller

$$\begin{cases} \hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y_t - C\hat{x}_t) \\ u_t = K\hat{x}_t + v_t \end{cases} \tag{2.12}$$

leading to a closed-loop system

$$\begin{bmatrix} x_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} A & BK \\ LC & A + BK - LC \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v_t + \begin{bmatrix} E \\ 0 \end{bmatrix} w_t \tag{2.13a}$$

$$y_t = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} \tag{2.13b}$$

which has precisely the form (1.1). Pre- and postmultiplying the new "A-matrix" with $\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$ we see that it has the characteristic polynomial

$$\det(\lambda I - A - BK) \det(\lambda I - A + LC), \tag{2.14}$$

as is well-known. Due to stabilizability of (A, B) and detectability of (C, A), by the Pole Assignment Theorem, K and L can be chosen so that (2.14) has all its roots in the open unit disc as required. The dynamic regulator (2.12) is, however, never minimal, and observers of lower dimension can be found in any standard textbook on the subject (see, e.g., [1, 13]).

3. The auxiliary optimization problem

We now consider the problem of minimizing the cost function (1.5) subject to (1.1) over the class $\mathcal{A}_{\mathfrak{L}}$ of admissible processes (x_t, u_t) corresponding to regulators (1.10) in the linear class $\mathfrak{L}$.

To this end, let Ψ_x, Ψ_u and Ψ_y be the transfer functions from the harmonic input Ew_t to x_t, u_t and y_t respectively. Clearly, these rational matrix functions are determined by

$$(\lambda I - A)\Psi_x = B\Psi_u + I \tag{3.1a}$$
$$D(\lambda)\Psi_u = N(\lambda)\Psi_y \tag{3.1b}$$
$$\Psi_y = C\Psi_x \tag{3.1c}$$

Since a regulator in $\mathfrak{L}$ is realizable by definition, $W(\lambda) := D(\lambda)^{-1}N(\lambda)$ is a proper rational matrix function, i.e., $W(\infty)$ is finite. But it follows from (3.1) that

$$[I - \lambda^{-1}A - \lambda^{-1}BW(\lambda)C]\Psi_x(\lambda) = \lambda^{-1}I,$$

and consequently $\Psi_x(\infty) = 0$. i.e., Ψ_x is strictly proper. Then, by (3.1c) the same is true for Ψ_y, and by (3.1b) for Ψ_u, i.e., $\Psi_y(\infty) = 0$ and $\Psi_u(\infty) = 0$.

Due to the stability of (1.11), the process (x_t, u_t, y_t) tends asymptotically to the unique harmonic solution

$$x_t = \sum_{j=1}^{N} x^{(j)} e^{i\theta_j t}, \quad u_t = \sum_{j=1}^{N} u^{(j)} e^{i\theta_j t}, \quad y_t = \sum_{j=1}^{N} y^{(j)} e^{i\theta_j t}, \tag{3.2}$$

where

$$x^{(j)} = \Psi_x(\lambda_j)Ew^{(j)}, \quad u^{(j)} = \Psi_u(\lambda_j)Ew^{(j)}, \quad y^{(j)} = \Psi_y(\lambda_j)Ew^{(j)} \tag{3.3}$$

and

$$\lambda_j = e^{i\theta_j}, \quad j = 1, 2, \ldots, N. \tag{3.4}$$

Therefore, the usual limit (rather than just limsup) does exist in (1.5), and it is given by

$$\Phi = \sum_{j=1}^{N} \Lambda(x^{(j)}, u^{(j)}). \tag{3.5}$$

To see this, observe that, if f_t and g_t are two harmonic sequences

$$f_t = \sum_{j=1}^{N} f^{(j)} e^{i\theta_j t} \quad \text{and} \quad g_t = \sum_{j=1}^{N} g^{(j)} e^{i\theta_j t}, \tag{3.6}$$

and M is an arbitrary matrix of appropriate dimensions, then

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{j=1}^{T} f_t^* M g_t = \sum_{j=1}^{N} \sum_{k=1}^{N} f^{(j)*} M g^{(k)} \varphi_{jk}, \tag{3.7}$$

where

$$\varphi_{jk} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} e^{i(\theta_k - \theta_j)t}. \tag{3.8}$$

The limit (3.8) does exist and equals one if $j = k$ and zero otherwise. Consequently,

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f_t^* M g_t = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f_t^* M g_t = \sum_{j=1}^{N} f^{(j)*} M g^{(j)}. \tag{3.9}$$

Using this formula (3.5) follows.

Now, in view of the constraint (1.1a),

$$x^{(j)} = (\lambda_j I - A)^{-1}(Bu^{(j)} + Ew^{(j)}), \tag{3.10}$$

and therefore

$$\Lambda(x^{(j)}, u^{(j)}) = u^{(j)*}\Pi(\lambda_j)u^{(j)} + p_j^* u^{(j)} + u^{(j)*}p_j + q_j, \tag{3.11}$$

where Π is given by (2.7),

$$p_j = [Q(\lambda_j I - A)^{-1}B + S]^*(\lambda_j I - A)^{-1}Ew^{(j)} \tag{3.12}$$

and

$$q_j = w^{(j)*}E^*(\bar{\lambda}_j I - A^*)^{-1}Q(\lambda_j I - A)^{-1}Ew^{(j)}. \tag{3.13}$$

Next, consider the auxiliary optimization problem to minimize Φ, given by (3.5) and (3.11)–(3.13), where $u^{(1)}, u^{(2)}, \dots, u^{(N)}$ are regarded as independent variables in $\mathbb{C}^k$. Since A is stable, $(\lambda_j I - A)^{-1}$ does exist, and WFDC implies that

$$\Pi(\lambda_j) > 0 \quad \text{for } j = 1, 2, \dots, N. \tag{3.14}$$

Therefore

$$\Lambda(x^{(j)}, u^{(j)}) = (u^{(j)} - \hat{u}^{(j)})^*\Pi(\lambda_j)(u^{(j)} - \hat{u}^{(j)}) + \Phi_{\min}^{(j)}, \tag{3.15}$$

where

$$\hat{u}^{(j)} = -\Pi(\lambda_j)^{-1}p_j \quad \text{and} \quad \Phi_{\min}^{(j)} = q_j - p_j^*\Pi(\lambda_j)^{-1}p_j, \tag{3.16}$$

so $u^{(j)} = \hat{u}^{(j)}$ is the solution of the problem to minimize $\Lambda(x^{(j)}, u^{(j)})$ given (3.10). More precisely, $\hat{u}^{(j)}$ has the form

$$\hat{u}^{(j)} = U(\lambda_j)w^{(j)}, \tag{3.17}$$

where

$$U(\lambda) = -\Pi(\lambda)^{-1}[Q(\lambda I - A)^{-1}B + S]^*(\lambda I - A)^{-1}E. \tag{3.18}$$

Consequently,

$$\Phi = \sum_{j=1}^{N}(u^{(j)} - \hat{u}^{(j)})^*\Pi(\lambda_j)(u^{(j)} - \hat{u}^{(j)}) + \sum_{j=1}^{N}\Phi_{\min}^{(j)}, \tag{3.19}$$

and therefore the solution to the auxiliary optimization problem is given by $u^{(j)} = \hat{u}^{(j)}, j = 1, 2, \dots, N$.

However, the variables $u^{(1)}, u^{(2)}, \dots, u^{(N)}$ are really not independent since the optimization should be done over the class $\mathcal{A}_{\mathfrak{L}}$ of admissible processes and hence tied together via (3.3) and a regulator (1.10) in $\mathfrak{L}$ which must be universal and thus not depend on the unknown vector amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$. Therefore, we next proceed to characterizing the class of all regulators in $\mathfrak{L}$.

First, however, let us observe for later reference that, in view of (3.3), (3.17), (3.16), (3.12) and (3.13), the cost function (3.19) takes the form

$$\Phi = w^*\Omega w, \quad \text{where } w := \begin{bmatrix} w^{(1)} \\ w^{(2)} \\ \vdots \\ w^{(N)} \end{bmatrix} \tag{3.20}$$

for some symmetric Hermitian matrix Ω which depends on the choice of regulator in $\mathfrak{L}$.

4. Parameterization of the class $\mathfrak{L}$ of realizable and stabilizing linear regulators

Next, we shall present a parameterization of all regulators in $\mathfrak{L}$, akin to the Youla parameterization but more suitable for our purposes. To this end, we first need a definition of equivalence.

Definition 4.1. Two regulators

$$D_1(\sigma)u_t = N_1(\sigma)y_t \quad \text{and} \quad D_2(\sigma)u_t = N_2(\sigma)y_t$$

are equivalent if there exist matrix polynomials D_0 and N_0, of dimensions $k \times k$ and $k \times m$ respectively, such that

$$D_1 = M_1 D_0, \quad N_1 = M_1 N_0 \quad D_2 = M_2 D_0, \quad N_2 = M_2 N_0$$

for some stable $k \times k$ matrix polynomials M_1 and M_2. We recall that a square matrix polynomial is *stable* if $\det M(\lambda) \neq 0$ for $|\lambda| \geq 1$.

Clearly, as can be seen from (3.1), Ψ_x, Ψ_u and Ψ_y are invariant under this equivalence.

Lemma 4.2. *(i) Let A be a stable matrix with characteristic polynomial $\chi(\lambda)$, and let $V(\lambda)$ be the matrix polynomial*

$$V(\lambda) = \chi(\lambda) C(\lambda I_n - A)^{-1}. \tag{4.1}$$

Let $\rho(\lambda)$ be an arbitrary stable scalar polynomial and let $R(\lambda)$ be an arbitrary $k \times m$ polynomial such that

$$\deg(RV) < \deg \rho. \tag{4.2}$$

Then the regulator

$$D(\sigma)u_t = N(\sigma)y_t \tag{4.3}$$

with

$$D(\lambda) = \rho(\lambda) I_k + R(\lambda) V(\lambda) B, \quad N(\lambda) = \chi(\lambda) R(\lambda) \tag{4.4}$$

is realizable and stabilizable, and for this regulator

$$\Psi_u(\lambda) = \frac{R(\lambda)}{\rho(\lambda)} V(\lambda), \quad \det \Xi(\lambda) = \chi(\lambda) \left[\rho(\lambda)\right]^k \tag{4.5}$$

where Ξ is given by (1.11).

(ii) Conversely, any realizable and stabilizable regulator (4.3) *belongs to the class of regulators* (4.3)–(4.4) *in the sense that it is equivalent to one in this class.*

This is a overparameterization of $\mathfrak{L}$ which is an advantage in our application. We can for example chose $\rho = \chi\rho_1$ for some stable scalar polynomial ρ_1 and take the degree of R to be at most $\deg \rho_1$.

Proof. (i) Let $D(\lambda)$ and $N(\lambda)$ be defined by (4.4). It is evident that the realizability condition holds. For stabilizability we need to show that $\Xi(\lambda)$, as defined by (1.11), is a stable matrix polynomial. We have

$$\det \Xi(\lambda) = \det(\lambda - A) \det[D - NC(\lambda I - A)^{-1}B]. \tag{4.6}$$

But, in view of (4.1) and (4.4), $D - NC(\lambda I - A)^{-1}B = \rho I_k$, and hence the second of equations (4.5) follows. To prove the first of equations (4.5), note that, in view of (3.1) and (4.4),

$$\begin{aligned} N\Psi_y &= NC(\lambda I - A)^{-1}(B\Psi_u + I) \\ &= RVB\Psi_u + RV \end{aligned}$$

and

$$D\Psi_u = \rho\Psi_u + RVB\Psi_u.$$

Therefore, the first of equations (4.5) follows from the second of equations (3.1).

(ii) Next, let (4.3) be an arbitrary regulator in $\mathfrak{L}$. Then, Ξ, defined by (1.11), is stable, and, by (4.6),

$$\det \Xi = \chi^{1-k} \det P,$$

where P is the $k \times k$ matrix polynomial

$$P = \chi D - NVB, \tag{4.7}$$

which is stable and nontrivial since $\det P = \chi^{k-1} \det \Xi$ is stable. In view of (3.1),

$$D\Psi_u = NC(\lambda I - A)^{-1}(B\Psi_u + I). \tag{4.8}$$

Solving (4.8), taking (4.1) and (4.7) into account, yields $P\Psi_u = NV$. Then,

$$\Psi_u = P^{-1}NV = \frac{P_a NV}{\det P}, \tag{4.9}$$

where P_a is the adjoint matrix polynomial $P_a := P^{-1} \det P$. Now, choose $\rho := \det P$, which has just been shown to be stable, and $R := P_a N$. Then the first of equations (4.5) holds. Since $\Psi_u(\infty) = 0$ for all regulators in $\mathfrak{L}$, this in turn implies that (4.2) holds. Moreover, Ψ_x is given by (3.1a), from which it also follows that $\det \Psi_x \not\equiv 0$. Next, define the matrix polynomials

$$\hat{D} = \rho I_k + RVB \quad \text{and} \quad \hat{N} = \chi R.$$

Then, by the first part of the lemma, the linear regulator

$$\hat{D}(\sigma)u_t = \hat{N}(\sigma)y_t \tag{4.10}$$

belongs to $\mathfrak{L}$, and the corresponding transfer functions, $\hat{\Psi}_u$ and $\hat{\Psi}_x$, have the properties $\hat{\Psi}_u = \Psi_u$ and $\hat{\Psi}_x = \Psi_x$. Consequently, (3.1b) and (3.1c) yield

$$D^{-1}NC = \Psi_u\Psi_x^{-1} = \hat{\Psi}_u\hat{\Psi}_x^{-1} = \hat{D}^{-1}\hat{N}C. \tag{4.11}$$

Since $\det CC^* \neq 0$, it follows that

$$D^{-1}N = \hat{D}^{-1}\hat{N}. \tag{4.12}$$

Let the $k \times k$ matrix polynomial $\hat{M}$ be the greatest common left divisor of $\hat{D}$ and $\hat{N}$, i.e.

$$\hat{D} = \hat{M}D_0, \qquad \hat{N} = \hat{M}N_0, \tag{4.13}$$

where D_0 and N_0 are left coprime matrix polynomials. Since (4.10) belongs to $\mathfrak{L}$, $\hat{M}$ is stable, and, since $\det \hat{D} \not\equiv 0$, we see that $\det \hat{M} \not\equiv 0$ and $\det D_0 \not\equiv 0$. From (4.11) we have $N = DD_0^{-1}N_0$, so setting $M := DD_0^{-1}$, we obtain

$$D = MD_0, \qquad N = MN_0. \tag{4.14}$$

Since D_0 and N_0 are left coprime, there exist matrix polynomials Π_1 and Π_2 such that

$$D_0\Pi_1 + N_0\Pi_2 = I.$$

(See, e.g., [7].) Therefore

$$M = M(D_0\Pi_1 + N_0\Pi_2) = D\Pi_1 + N\Pi_2$$

is a matrix polynomial. Since $Dy_t = Ny_t$ belongs to $\mathfrak{L}$, M is stable. From (4.13) and (4.14) we now see that the regulators $Dy_t = Nx_t$ and $\hat{D}y_t = \hat{N}x_t$ are equivalent, as claimed. □

5. Design of a linear optimal universal regulator: The case $m \geq \ell$

Let ρ and R be real polynomials defined as in Lemma 4.2. Then, by (4.5), the harmonic component of the control u_t, as defined in (3.2) and (3.3), is given by

$$u^{(j)} = \frac{R(\lambda_j)}{\rho(\lambda_j)}V(\lambda_j)Ew^{(j)}. \tag{5.1}$$

We recall that the harmonic components of x_t and u_t are the only parts that contribute to the cost functional (1.5), and, as explained in Section 3, optimality is achieved if

$$u^{(j)} = \hat{u}^{(j)} \quad \text{for } j = 1, 2, \dots, N, \tag{5.2}$$

where $\hat{u}^{(j)} = U(\lambda_j)w^{(j)}$, as seen from (3.19) and (3.17).

The question now is whether there are real polynomials ρ and R, satisfying the conditions of Lemma 4.2, such that (5.2) holds for all choices of $w^{(1)}, w^{(2)}, \dots, w^{(N)}$. If this is so, there does exist an optimal universal regulator in $\mathfrak{L}$, and it is given by Lemma 4.2 in terms of R and ρ. If not, an optimal universal regulator may not exist, and we shall see in Section 6 that it does not exist as a rule, but an optimal regulator which is not universal may exist.

Consequently, an optimal universal regulator does exist, if

$$R(\lambda_j)F_j = \rho(\lambda_j)U(\lambda_j), \quad j = 1, 2, \dots, N, \tag{5.3}$$

where $F_1, F_2, \dots, F_N$ are $m \times \ell$ complex matrices defined by

$$F_j := V(\lambda_j)E = \chi(\lambda_j)C(\lambda_jI - A)^{-1}E. \tag{5.4}$$

Now, if $m \geq \ell$ and

$$\det F_j^* F_j \neq 0, \quad j = 1, 2, \dots, N, \tag{5.5}$$

then it is easy to see that

$$R(\lambda_j) = \rho(\lambda_j) U(\lambda_j) (F_j^* F_j)^{-1} F_j^* + \tilde{R}_j, \quad j = 1, 2, \dots, N \tag{5.6}$$

is a solution of (5.3) for all $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_N$ such that $\tilde{R}_j F_j = 0$, for $j = 1, 2, \dots, N$, and that these are precisely all solutions to (5.3).

Therefore, for any stable scalar polynomial ρ, of sufficiently large degree, there is an R satisfying (5.6) such that the corresponding regulator (4.3) is an optimal universal regulator. In fact, for this regulator, (5.2) holds so that (3.19) implies that

$$\Phi = \Phi_{\min} := \sum_{j=1}^{N} \Phi_{\min}^{(j)} \tag{5.7}$$

and that $\Phi \geq \Phi_{\min}$ for all other regulators in $\mathfrak{L}$, proving optimality. Since, in addition, D and N do not depend on $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, the optimal regulator is universal.

Let us summarize the results obtained so far. We consider the problem of finding an optimal universal regulator in the linear class $\mathfrak{L}$. This corresponds to the optimization problem to minimize the cost function Φ, defined by (1.5), over the class $\mathcal{A}_{\mathfrak{L}}$ of admissible processes, subject to the constraint (1.1), under the condition that $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ are unknown and hence must not affect the regulator or the optimality of it.

Theorem 5.1. *Suppose that*

(1) $m := \dim y_t \geq \ell := \dim w_t$,
(2) *A is a stable matrix with characteristic polynomial χ,*
(3) $\det F_j^* F_j \neq 0, \quad j = 1, 2, \dots, N$,
(4) *WFDC holds, i.e., $\Pi(\lambda_j) > 0$ for $j = 1, 2, \dots, N$.*

Then:

(i) *There exists an optimal universal regulator in the class $\mathfrak{L}$, and it is defined by formulas* (4.4) *where ρ is a stable scalar polynomial and R is a $k \times m$ matrix polynomial such that* $\deg(RV) < \deg \rho$ *which satisfies the interpolation conditions* (5.6).
(ii) *Any optimal universal regulator in $\mathfrak{L}$ is equivalent to one of the type mentioned in (i).*
(iii) *The optimal value of the cost function Φ is*

$$\Phi_{min} := \sum_{j=1}^{N} \Phi_{min}^{(j)}, \tag{5.8}$$

where $\Phi_{min}^{(1)}, \Phi_{min}^{(2)}, \dots, \Phi_{min}^{(N)}$ are defined by (3.16).

Proof. Statements (i) and (iii) have already been proven above, so it only remains to prove (ii). To this end, let

$$\hat{D}(\sigma) u_t = \hat{N}(\sigma) y_t \tag{5.9}$$

be an optimal universal regulator. Then (5.9) is optimal for any choice of the vector amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, for example, for the choice that all these vector amplitudes are zero except the j:th one so that $w_t = w^{(j)} e^{i\theta_j t}$. Then

$$\min_{\mathcal{A}_{\mathfrak{L}}} \Phi = \Phi^{(j)}_{\min},$$

and the optimal control is $u^{(j)} = \hat{u}^{(j)}$, where $\hat{u}^{(j)} = U(\lambda_j) w^{(j)}$. Since this should hold for all values of $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, this implies that

$$\Psi_u(\lambda_j) E = U(\lambda_j). \tag{5.10}$$

Since j is arbitrary, this holds for each $j = 1, 2, \dots, N$. By Lemma 4.2, the regulator (5.9) is equivalent to some regulator (4.3) with D and N given by (4.4), and thus it has the same transfer function Ψ_u so that (5.10) holds for $j = 1, 2, \dots, N$. This is equivalent to the interpolation condition (5.6). This proves statement (ii). □

6. Design of linear optimal regulators, universal and nonuniversal: The case $m < \ell$

In this section we show that an optimal universal regulator in $\mathfrak{L}$ does not in general exist in the case that

$$m := \dim y_t < \ell := \dim w_t \tag{6.1}$$

unless certain algebraic relations on the system parameters are satisfied, and therefore it does not exist in practice.

Suppose for the moment that $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ are fixed, and let us determine all optimal, possibly nonuniversal, regulators in the class $\mathfrak{L}$. To this end, recall that the cost function is

$$\Phi = \sum_{j=1}^{N} (u^{(j)} - \hat{u}^{(j)})^* \Pi(\lambda_j)(u^{(j)} - \hat{u}^{(j)}) + \sum_{j=1}^{N} \Phi^{(j)}_{\min}, \tag{6.2}$$

for any regulator in $\mathfrak{L}$, where $\hat{u}^{(j)} := U(\lambda_j) w^{(j)}$ and $u^{(j)} = \Psi_u(\lambda_j) E w^{(j)}$ are given by (3.17) and (3.3) respectively. Consider first those regulators in $\mathfrak{L}$ which are defined by Lemma 4.2(i) and formulas (4.4) via appropriate polynomials R and ρ. For such a choice of polynomials

$$u^{(j)} = X_j F_j w^{(j)}, \tag{6.3}$$

where the $m \times \ell$ matrices $F_1, F_2, \dots, F_N$ are given by (5.4) and

$$X_j = \frac{R(\lambda_j)}{\rho(\lambda_j)}. \tag{6.4}$$

Let $\mathcal{J}' := \{j \mid F_j w^{(j)} \neq 0\}$ and $\mathcal{J}'' := \{j \mid F_j w^{(j)} = 0\}$ so that $\mathcal{J}' \cup \mathcal{J}'' = \{1, 2, \dots, N\}$. Then (6.2) implies that

$$\Phi = \sum_{j \in \mathcal{J}'} (u^{(j)} - \hat{u}^{(j)})^* \Pi(\lambda_j)(u^{(j)} - \hat{u}^{(j)}) + \Phi_{\min}, \tag{6.5}$$

where

$$\Phi_{\min} = \sum_{j\in\mathcal{J}''} (\hat{u}^{(j)})^* \Pi(\lambda_j) \hat{u}^{(j)} + \sum_{j=1}^{N} \Phi_{\min}^{(j)}. \tag{6.6}$$

Recall that $\Pi(\lambda_j) > 0$. It follows from (6.5) that if we find a regulator $D(\sigma)u_t = N(\sigma)y_t$ with R, ρ such that

$$u^{(j)} = \hat{u}^{(j)} \quad \text{for } j \in \mathcal{J}', \tag{6.7}$$

then it is optimal. The corresponding polynomials ρ and R must satisfy relations

$$X_j F_j w^{(j)} = U(\lambda_j) w^{(j)}, \quad j \in \mathcal{J}'. \tag{6.8}$$

In this case the infimum of Φ is attained in $\mathcal{L}$ and

$$\inf_{\mathcal{L}} \Phi = \Phi_{\min}. \tag{6.9}$$

It is easy to see that all solutions X_j of (6.8) are given by

$$X_j = \frac{U(\lambda_j) w^{(j)} (w^{(j)})^* F_j^*}{|F_j w^{(j)}|^2} + \tilde{X}_j \quad \text{where } \tilde{X}_j F_j w^{(j)} = 0 \text{ and } j \in \mathcal{J}'. \tag{6.10}$$

Obviously there exist a matrix polynomial $R(\lambda)$ and a scalar stable polynomial $\rho(\lambda)$ such that

$$R(\lambda_j) = \rho(\lambda_j) X_j, \quad j \in \mathcal{J}' \tag{6.11}$$

and condition $\deg(RV) < \deg \rho$ of Lemma 4.2 holds. Here ρ may be any stable scalar polynomial of sufficiently high degree. The corresponding regulator is optimal, and we have also proved relation (6.9). (Note that in the case $m \geq \ell$ considered in Section 5 the conditions $F_j w^{(j)} = 0$ imply that $w^{(j)} = 0$ under the assumption that $\det F_j^* F_j \neq 0$. Therefore $\hat{u}^{(j)} = 0$ for $j \in \mathcal{J}''$ and (6.6) coincides with (5.8).)

Now, consider an arbitrary optimal regulator in $\mathcal{L}$ obtained via formulas (4.5) of Lemma 4.2. Then $\Phi = \Phi_{\min}$, and, because of (6.5), we obtain first (6.7) and then (6.8), (6.10) and (6.11). Hence the regulator is determined in the way mentioned above.

Consider next an arbitrary optimal regulator

$$\hat{D}(\sigma) u_t = \hat{N}(\sigma) y_t$$

in $\mathcal{L}$, not necessarily obtained via formulas (4.5). By Lemma (4.1) it is equivalent to a regulator $Du_t = Ny_t$ determined via formulas (4.5). Since Φ depends only on Ψ_u, which is the same for these regulators, and $\Phi = \Phi_{\min}$ for the regulator $\hat{D}u_t = \hat{N}y_t$, we have $\Phi = \Phi_{\min}$ also for the regulator $Du_t = Ny_t$. Therefore, $Du_t = Ny_t$ is optimal also and consequently it is of the kind discussed above.

Let us formulate the results obtained so far. Suppose $m < \ell$, and let the complex amplitudes $w^{(1)}, w^{(2)}, \ldots, w^{(N)}$ be fixed. Then an optimal regulator exists. Any regulator defined by formulas (4.5) with ρ, R satisfying the conditions of Lemma 4.2 and the interpolation conditions (6.11) is optimal. Conversely, any regulator which is optimal in $\mathcal{L}$ is equivalent to one obtained in this way.

Now suppose that a *universal* optimal regulator exists and that it is defined by formulas (4.5) in Lemma 4.2. Then (6.8) must hold, i.e.,

$$X_j F_j w^{(j)} = U(\lambda_j) w^{(j)} \quad \text{if } F_j w^{(j)} \neq 0.$$

But universality implies that this must hold for all values of $w^{(1)}, w^{(2)}, \ldots, w^{(N)}$, and consequently, as seen from a continuity argument, we must have

$$X_j F_j = U(\lambda_j), \quad j = 1, 2, \ldots, N \tag{6.12}$$

precisely as in Section 5. The difference from the situation in Section 5 is that, in the case $m < \ell$, (6.12) does not in general have a solution since it is an overdetermined system of $k\ell N$ linear equations with kmN unknown variables, the components of the $k \times m$ matrices $X_1, X_2, \ldots, X_N$. Let us find the conditions for the existence of such a solution.

Suppose that the rows of the $m \times \ell$ matrices $F_1, F_2, \ldots, F_N$ are linearly independent, i.e., that

$$\det F_j F_j^* \neq 0, \quad j = 1, 2, \ldots, N. \tag{6.13}$$

Then postmultiplying (6.12) by $F_j^*(F_j F_j^*)^{-1}$ for $j = 1, 2, \ldots, N$, we obtain

$$X_j = U(\lambda_j) F_j^* (F_j F_j^*)^{-1}, \quad j = 1, 2, \ldots, N. \tag{6.14}$$

It follows from (6.12) that

$$U(\lambda_j) \left[F_j^* (F_j F_j^*)^{-1} F_j - I \right] = 0, \quad j = 1, 2, \ldots, N. \tag{6.15}$$

Conversely, (6.14) and (6.15) imply (6.12), and hence (6.15) is equivalent to the existence of a solution $X_1, X_2, \ldots, X_N$ in (6.12). This is of course a very strict condition, showing that the existence of a universal regulator in the case $m < \ell$ is nongeneric. For example, if $m = k = 1$, this condition implies that the ℓ-dimensional row vectors F_j and $U(\lambda_j)$ are proportional, i.e.,

$$U(\lambda_j) = \kappa_j F_j, \quad j = 1, 2, \ldots, N,$$

where $\kappa_1, \kappa_2, \ldots, \kappa_N$ are scalars.

We have thus established that (6.15) is a necessary condition for the existence of an optimal universal regulator. It is also sufficient, because, under this condition, equations

$$R(\lambda_j) = \rho(\lambda_j) U(\lambda_j) F_j^* (F_j F_j^*)^{-1}, \quad j = 1, 2, \ldots, N, \tag{6.16}$$

i.e., equations (6.14), satisfy (6.8) for all choices of $w^{(1)}, w^{(2)}, \ldots, w^{(N)}$, and consequently appropriate ρ and R do exist so that the corresponding regulator (4.3), determined by (4.4), is an optimal universal regulator. Any other optimal universal regulator in $\mathfrak{L}$ is equivalent to one constructed in this way. In fact, by Lemma 4.2(ii), any regulator in $\mathfrak{L}$ is equivalent to one constructed via (4.2)–(4.4) and thus has the same closed-loop transfer functions Ψ_u, Ψ_x, Ψ_y, and hence the same $x^{(j)}, u^{(j)}, y^{(j)}$ in (3.1), and consequently the same value of the cost function Φ. Moreover, we just showed that any optimal universal regulator constructed via (4.2)–(4.4) must satisfy (6.16).

We summarize the results of this section in the following theorem. First, however, let us recall the problem under consideration. Find a regulator (1.10) in the class

$\mathfrak{L}$ such that the overall closed-loop system consisting of (1.1) and (1.10) generates a process (x_t, u_t) minimizing the cost function (1.5). The external harmonic disturbance (1.2) is such that (1.3) holds. We say that the regulator (1.10) is an optimal universal regulator (OUR) in $\mathfrak{L}$ if D and N do not depend on $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ and it is optimal for all values of $w^{(1)}, w^{(2)}, \dots, w^{(N)}$.

Theorem 6.1. *Suppose that*

(1) $m := \dim y_t < \ell := \dim w_t$,
(2) *A is a stable matrix with characteristic polynomial $\chi(\lambda)$,*
(3) *WFDC holds, i.e., $\Pi(\lambda_j) > 0$ for $j = 1, 2, \dots, N$.*

Then:

(i) Let $\det F_j F_j^* \neq 0, \quad j = 1, 2, \dots, N$. Then there exists an optimal universal regulator in the class $\mathfrak{L}$ if and only if conditions (6.15) hold, where $U(\lambda)$ is defined by (3.18), $\lambda_1, \lambda_2, \dots, \lambda_N$ by (3.4) and $F_1, F_2, \dots, F_N$ by (5.4). In this case, the OUR is defined by formulas (4.4) where ρ is an arbitrary stable scalar polynomial of sufficiently high degree, and R is a $k \times m$ matrix polynomial such that $\deg(RV) < \deg \rho$ which satisfies the interpolation conditions (6.16). Any other OUR in $\mathfrak{L}$ is equivalent to some regulator of this type. The transfer function Ψ_u from Ew_t to u_t for the corresponding optimal closed-loop system is

$$\Psi_u(\lambda) = \frac{R(\lambda)}{\rho(\lambda)} V(\lambda). \tag{6.17}$$

(ii) If (6.15) fails for some $j = 1, 2, \dots, N$, then there is no OUR. Then there is an optimal nonuniversal regulator $D(\sigma)u_t = N(\sigma)y_t$ in $\mathfrak{L}$ defined via (4.4), where ρ is an arbitrary stable (real) scalar polynomial of sufficiently high degree, and R is a $k \times m$ matrix polynomial such that $\deg(RV) < \deg \rho$ which satisfies the interpolation conditions

$$R(\lambda_j) = \frac{\rho(\lambda_j)}{|F_j w^{(j)}|^2} U(\lambda_j) w^{(j)} (w^{(j)})^* F_j^* + \tilde{R}_j \quad \textit{where } \tilde{R}_j F_j w^{(j)} = 0, \tag{6.18}$$

for all $j \in \mathfrak{I}'$, i.e., for all j for which $F_j w^{(j)} \neq 0$. Any other optimal regulator is equivalent to one constructed in this way. The transfer matrix Ψ_u from Ew_t to u_t is given by (6.17).

Consequently, the existence of an optimal universal regulator is a highly nongeneric property when $m < \ell$, so, from a practical point of view, OUR does not exist in this case.

Remark 6.2. If (6.15) holds and $F_j w^{(j)} \neq 0$ for some j, then we may replace (6.18) by

$$R(\lambda_j) = \rho(\lambda_j) U(\lambda_j) F_j^* (F_j F_j^*)^{-1}. \tag{6.19}$$

For this j we have $u_j = \hat{u}_j$ for all $w^{(j)} \in \mathbb{C}^\ell$.

7. Main results

In this section we show that an optimal universal regulator in $\mathcal{L}$ is also an OUR in the larger class $\mathcal{N}$, defined in Section 1, under conditions which are only slightly stronger than those in Theorems 5.1 and 6.1. This fact is a corollary of the following main lemma.

Lemma 7.1. *Suppose that the strong frequency domain condition (SFDC) holds, i.e.,*

$$\Pi(\lambda) > 0 \quad \textit{for all } \lambda \textit{ on the unit circle.} \tag{7.1}$$

Let $\mathcal{A}$ and $\mathcal{A}_{\mathcal{L}}$ be the classes of admissible processes (x_t, u_t) defined in Section 1*. Then, if the problem to minimize the cost function* (1.5) *over all processes in $\mathcal{A}_{\mathcal{L}}$ has an optimal solution satisfying the interpolation condition*

$$u^{(j)} = \hat{u}^{(j)} \quad \textit{for } j \in \mathcal{J}', \tag{7.2}$$

this solution is also optimal for the problem to minimize (1.5) *over $\mathcal{A}$.*

Proof. Define $\hat{\mathcal{L}}$ to be the class of all linear stabilizing regulators

$$\hat{\mathcal{L}}: \quad \hat{D}(\sigma)u_t = \hat{N}(\sigma)y_t,$$

obtained by setting $C := I$ in the definition of $\mathcal{L}$. Then, any D and N corresponding to a regulator in $\mathcal{L}$ define a regulator in $\hat{\mathcal{L}}$ by setting $\hat{D} = D$ and $\hat{N} = NC$, and consequently

$$\mathcal{L} \subset \hat{\mathcal{L}}. \tag{7.3}$$

Now, the quantities $\Pi(\lambda_j)$, $\hat{u}^{(j)}$ and $\Phi^{(j)}_{\min}$ do not depend on C (see (2.7), (3.12), (3.13) and (3.16)–(3.18)) and therefore $\Phi_{\min}$ in (6.6) does not depend on C either, although $u^{(j)}$ does. Consequently, it follows from (6.5) that

$$\Phi \geq \Phi_{\min} \tag{7.4}$$

for all processes (x_t, u_t) in $\mathcal{A}_{\hat{\mathcal{L}}}$. But, since a process which is optimal in $\mathcal{A}_{\mathcal{L}}$ satisfies (7.2) so that $\Phi = \Phi_{\min}$, it is optimal also in $\mathcal{A}_{\hat{\mathcal{L}}}$. Moreover, under the strong frequency domain condition (SFDC), it was proven in [16, Theorem 5.1 and Remark 5.2] that a minimum of Φ over all $(x_t, u_t) \in \mathcal{A}$ can be obtained by choosing a process in $\mathcal{A}_{\hat{\mathcal{L}}}$. Consequently, a process which is optimal in $\mathcal{A}_{\mathcal{L}}$ is optimal in $\mathcal{A}$ also, as claimed. □

Consequently, we have established the main result of this paper, namely the following extension of Theorem 5.1 to the larger class $\mathcal{N}$ of nonlinear regulators. We recall that $\mathcal{L} \subset \mathcal{N}$.

Theorem 7.2. *Suppose that*

(1) $m := \dim y_t \geq \ell := \dim w_t$,
(2) *A is a stable matrix,*
(3) $\det F_j^* F_j \neq 0, \quad j = 1, 2, \dots, N$,
(4) *SFDC holds, i.e.,* $\Pi(\lambda) > 0$ *for all λ on the unit circle.*

Then there is an optimal universal regulator in the class $\mathcal{N}$, which actually belongs to $\mathcal{L} \subset \mathcal{N}$, and it can be determined as in Theorem 5.1.

Proof. By Theorem 5.1, there exists an optimal universal regulator in $\mathcal{L}$ under the stated conditions, and it follows from the proof of Theorem 5.1 that the interpolation conditions (5.2), and consequently also (7.2), holds for this regulator. Consequently, by Lemma 7.1, the corresponding process (x_t, u_t) is also optimal in $\mathcal{A}$ and hence in the class of all admissible processes generated by $\mathcal{N}$. Therefore the optimal universal regulator in $\mathcal{L}$ is an OUR also in $\mathcal{N}$. □

Similarly we also have the following extension of Theorem 6.1.

Theorem 7.3. *Suppose that*

(1) $m := \dim y_t < \ell := \dim w_t$,
(2) *A is a stable matrix,*
(3) $\det F_j F_j^* \neq 0, \quad j = 1, 2, \dots, N$,
(4) *SFDC holds, i.e., $\Pi(\lambda) > 0$ for all λ on the unit circle.*

Then, provided condition (6.15) *holds, there is an optimal universal regulator in the class $\mathcal{N}$, which actually belongs to $\mathcal{L} \subset \mathcal{N}$, and it can be determined as in point (i) of Theorem* 6.1. *If condition* (6.15) *fails, there is a nonuniversal optimal regulator in $\mathcal{N}$ which belongs to $\mathcal{L}$ and is given in point (ii) of Theorem* 6.1.

The proof of Theorem 7.3 follows the same principles as that of Theorem 7.2.

8. The stochastic case

It is interesting to note that an optimal universal regulator in $\mathcal{L}$ for the (deterministic) control problem discussed in the previous sections is optimal in $\mathcal{L}$ for the stochastic control problem obtained by taking $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ to be (jointly distributed) random vectors and replacing the cost function (1.5) by

$$\Phi = \lim_{T\to\infty} \mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t)\}, \tag{8.1}$$

where $\mathrm{E}\{\cdot\}$ denotes mathematical expectation. In fact, universality implies that the same regulator (1.11) is optimal for all values of $w^{(1)}, w^{(2)}, \dots, w^{(N)}$, so summing with respect to the appropriate probability measure shows that this regulator is optimal also for the stochastic problem (Lemma 8.2).

Therefore it is natural to ask whether there is a more nontrivial stochastic version of the problem previously discussed in this paper for which the optimal universal regulator is optimal. As pointed out in Section 1, the appropriate stochastic problem is obtained by replacing (1.7) by the stochastic system

$$\begin{cases} z_{t+1} = F z_t + G v_t \\ w_t = H z_t + K v_t \end{cases} \tag{8.2}$$

where z_0 now is a random $N\ell$-vector and where $v_0, v_1, v_2, \dots$ is a zero-mean vector-valued white noise, independent of z_0, i.e.,

$$\mathrm{E}\{v_s v_t^*\} = V_t \delta_{st}, \quad \mathrm{E}\{v_t\} = 0, \tag{8.3}$$

with $\{|V_t|\}_{t=0}^{\infty}$ being an ℓ_1 sequence, i.e.,

$$\sum_{t=0}^{\infty} |V_t| < \infty. \tag{8.4}$$

The condition (8.4) insures that $\mathrm{E}\{w_s w_t^*\}$ is bounded for all $s, t \in \mathbb{Z}_+$. We say that an optimal regulator is *universal* (for the stochastic problem) if it does not depend on z_0, G, K and $\{V_t\}_{t\in\mathbb{Z}_+}$, and it is optimal for all values of these quantities.

Theorem 8.1. *Consider the stochastic control system*

$$x_{t+1} = Ax_t + Bu_t + Ew_t \tag{8.5a}$$
$$y_t = Cx_t \tag{8.5b}$$

with w_t generated by the critically stable stochastic system (8.2). *Then any optimal universal regulator in $\mathfrak{L}$ for the deterministic problem to control* (1.1), *with w_t given by* (1.2), *so as to minimize* (1.5) *is also an optimal universal regulator in the class $\mathfrak{L}$ for the stochastic problem to control* (8.5) *so as to minimize* (8.1).

We first prove the statement of Theorem 8.1 in the special case when $v_t \equiv 0$ so that all stochastics is generated by the initial condition z_0.

Lemma 8.2. *Let $v_t \equiv 0$. Then the limit* (8.1) *exists for all (x_t, u_t) in $\mathcal{A}_{\mathfrak{L}}$ and*

$$\lim_{T\to\infty} E\{\frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t)\} = E\{\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t)\}. \tag{8.6}$$

Consequently, any optimal regulator in $\mathfrak{L}$ is an optimal regulator for the stochastic problem and does not depend on z_0.

Proof. Let us first consider the deterministic problem with $(x_t, u_t) \in \mathcal{A}_{\mathfrak{L}}$. A straightforward calculation shows that

$$\frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t) = w^*\Omega_T w + \omega_T^* w + \eta_T, \tag{8.7}$$

where w is defined as in (3.20) and Ω_T is a symmetric Hermitian matrix which depends on the choice of regulator in $\mathfrak{L}$. Since the regulator is stabilizing, $\omega_T \to 0$ and $\eta_T \to 0$ as $T \to \infty$, and, as seen from (3.20), (8.7) tends $w^*\Omega w$ for any choice of w. Consequently, $\Omega_T \to \Omega$ as $T \to \infty$. Next, let w be the stochastic vector z_0 as required in the present problem. Then

$$\mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t)\} = \mathrm{E}\{z_0^*\Omega_T z_0\} + \omega_T^*\mathrm{E}\{z_0\} + \eta_T.$$

Consequently, since $\omega_T \to 0$ as $T \to \infty$, (8.6) follows if

$$\lim_{T\to\infty} \mathrm{E}\{z_0^*\Omega_T z_0\} = \mathrm{E}\{\lim_{T\to\infty} z_0^*\Omega_T z_0\}. \tag{8.8}$$

But, $\Omega_T \to \Omega$, and therefore there is a matrix $M > \Omega$ and a $T_0 > 0$ such that $\Omega_T \leq M$ for all $T \geq T_0$ so that

$$z_0^*\Omega_T z_0 \leq z_0^* M z_0 \quad \text{for } T \geq T_0$$

and for each value of z_0. Consequently, (8.8) follows by dominated convergence; see, e.g., [12, Theorem 2.4]. □

Proof of Theorem 8.1. Let us first normalize the white noise sequence $v_0, v_1, v_2, \ldots$ by setting

$$v_t := L_t \eta_t \tag{8.9}$$

so that η_t is a zero-mean, p-dimensional, normalized white noise, i.e.,

$$\mathrm{E}\{\eta_s \eta_t^*\} = I_p \delta_{st}, \quad \mathrm{E}\{\eta_t\} = 0, \tag{8.10}$$

implying that L_t is a matrix-valued function such that $L_t L_t^* = V_t$. Then

$$w_t = \bar{w}_t + \sum_{k=1}^{p} \sum_{s=0}^{t} w_t(s,k)(\eta_s)_k, \tag{8.11}$$

where

$$w_t(s,k) = HF^t g_{sk} \quad \text{where } g_{sk} = \begin{cases} F^{-s-1} G L_s e_k & \text{for } s < t \\ K L_s e_k & \text{for } s = t \end{cases} \tag{8.12}$$

e_k being the k:th axis unit vector, and where

$$\bar{w}_t = HF^t z_0. \tag{8.13}$$

Now, any regulator in $\mathfrak{L}$ applied to (8.5) yields a closed-loop system (8.2), (8.5), (1.10), driven by the white noise η_t so that

$$x_t = \bar{x}_t + \sum_{k=1}^{p} \sum_{s=0}^{t-1} x_t(s,k)(\eta_s)_k \tag{8.14}$$

$$u_t = \bar{u}_t + \sum_{k=1}^{p} \sum_{s=0}^{t-1} u_t(s,k)(\eta_s)_k, \tag{8.15}$$

$$y_t = \bar{y}_t + \sum_{k=1}^{p} \sum_{s=0}^{t-1} y_t(s,k)(\eta_s)_k \tag{8.16}$$

where $\bar{x}_t, \bar{u}_t, \bar{y}_t$ are stochastic vector sequences generated by the initial condition z_0, and thus independent of $\{\eta_t\}$, and $x_t(s,k), u_t(s,k), y_t(s,k)$ are deterministic vector sequences. All these sequences of course depend on the particular choice of regulator. It follows from (8.14)–(8.16) that $\bar{x}_t, \bar{u}_t, \bar{y}_t$ are the conditional expected values of x_t, u_t, y_t given z_0, and therefore

$$\bar{x}_{t+1} = A\bar{x}_t + B\bar{u}_t + E\bar{w}_t \tag{8.17a}$$

$$\bar{y}_t = C\bar{x}_t \tag{8.17b}$$

Since $x_t(s,k) = \mathrm{E}\{x_t(\eta_s)_k\}$ for $t \geq k+1$, and the corresponding relations hold for $u_{t+1}(s,k)$ and $y_{t+1}(s,k)$,

$$x_{t+1}(s,k) = Ax_t(s,k) + Bu_t(s,k) + Ew_t(s,k), \;\; x_{s+1}(s,k) = Ew_s(s,k) \tag{8.18a}$$

$$y_t(s,k) = Cx_t(s,k) \tag{8.18b}$$

for $t = s+1, s+2, \dots$, it follows from (8.12) and (8.13) that $w_t(s,k)$ and $\bar{w}_t$ satisfy the autonomous system (1.7), differing only in the initial conditions, which correspond to $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ in (1.2), and therefore (8.17) and (8.18) have the same structure as the deterministic system (1.1)–(1.2).

Next, we show that the cost function can be decomposed accordingly. In fact, it is easy to check that

$$\mathrm{E}\{\Lambda(x_t, u_t)\} = \mathrm{E}\{\Lambda(\bar{x}_t, \bar{u}_t)\} + \sum_{k=1}^{p} \sum_{s=0}^{t-1} \Lambda(x_t(s,k), u_t(s,k)) \tag{8.19}$$

so, if we agree to define $x_t(s,k)$ and $u_t(s,k)$ to be zero for $k \geq t$, we have

$$\mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T} \Lambda(x_t, u_t)\} = \mathrm{E}\{\frac{1}{T}\sum_{t=0}^{T} \Lambda(\bar{x}_t, \bar{u}_t)\} + \sum_{k=1}^{p} \sum_{s=0}^{\infty} \left[\frac{1}{T}\sum_{t=s+1}^{T} \Lambda(x_t(s,k), u_t(s,k))\right].$$

By Lemma 8.2, the limit

$$\bar{\Phi} = \mathrm{E}\{\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \Lambda(\bar{x}_t, \bar{u}_t)\} \tag{8.20}$$

exists. Therefore, provided the limits

$$\Phi_{sk} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=s+1}^{T} \Lambda(x_t(s,k), u_t(s,k)) \tag{8.21}$$

exist, and provided

$$\lim_{T\to\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T}\sum_{t=s+1}^{T} \Lambda(x_t(s,k), u_t(s,k))\right] = \sum_{s=0}^{\infty} \Phi_{sk}, \tag{8.22}$$

the limit does also exist in the cost function (8.1) and

$$\Phi = \bar{\Phi} + \sum_{k=1}^{p} \sum_{s=0}^{\infty} \Phi_{sk}. \tag{8.23}$$

Under these condition, which must be verified, the stochastic control problem thus decomposes into separate decoupled control problems, all having the structure of the one considered earlier in this paper, namely the problem $\bar{P}$ of Lemma 8.2 to minimize $\bar{\Phi}$ given (8.17) and the deterministic problems P_{sk} to minimize Φ_{sk} given (8.18). The latter problems differ only in the vector amplitudes $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ in (1.2) and in the initial time (which does not affect the steady-state behavior measured by the cost function). Consequently, if there is a universal optimal regulator $Du_t = Ny_t$ in $\mathfrak{L}$ for the deterministic problem to control (1.1) so as to minimize (1.5), then

$$D(\sigma)u_t(s,k) = N(\sigma)y_t(s,k) \tag{8.24}$$

is optimal in $\mathfrak{L}$ for P_{sk}, for all $s = 0, 1, 2, \dots$ and $k = 1, 2, \dots, p$. Moreover, by Lemma 8.2,

$$D(\sigma)\bar{u}_t = N(\sigma)\bar{y}_t \tag{8.25}$$

is optimal in $\mathfrak{L}$ for the problem $\bar{P}$. Hence the cost function (8.23) for the stochastic system is minimized by these control actions. But, in view of (8.24) and (8.25), the stochastic processes u_t and y_t, given by (8.15) and (8.16), satisfy

$$D(\sigma)u_t = N(\sigma)y_t, \tag{8.26}$$

and therefore the regulator (8.26) is optimal in $\mathfrak{L}$ for the stochastic problem of Theorem 8.1. Clearly this regulator does not depend on z_0, G, K and $\{V_t\}_{t\in\mathbb{Z}_+}$, and it is optimal for all values of these quantities. Hence it is a universal regulator, as claimed.

It remains to show that the limits (8.20) and (8.21) exist and that (8.22) holds under the feedback conditions (8.25) and (8.24). It was established in Section 3 that the limits do exist under linear stabilizing feedback, so we only need to verify (8.22). To this end, recall that, for any regulator in $\mathfrak{L}$, the cost function takes the form (3.20) in the deterministic problem. Moreover, for the problems P_{sk}, it follows from (8.12) and $V_s = L_s L_s^*$ that the norm of $w = \operatorname{col}(w^{(1)}, w^{(2)}, \dots, w^{(N)})$ is bounded by $\kappa|V_s|^{\frac{1}{2}}$ for some positive constant κ which depends on the regulator. Hence (3.20) is bounded by $\kappa^2|V_s|$ so, in view of (8.4),

$$\sum_{k=0}^{\infty} \Phi_{sk} < \infty.$$

Moreover,

$$\left| \frac{1}{T} \sum_{t=s+1}^{T} \Lambda(x_t(s,k), u_t(s,k)) \right| < \kappa^2 |V_s|,$$

which is an ℓ_1 sequence. Consequently (8.22) follows from the dominated convergence theorem. □

We note that the decomposition (8.17), (8.18) and (8.23) is analogous to the one used in [15], so a natural question is whether the admissible class of regulators could be extended to include nonlinear control laws as in [15]. However, this leads to technical difficulties related to the existence of the limits (8.20) and (8.21) and the validity of (8.22).

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