Optimal Control of Linear Stochastic Systems with Applications to Time Lag Systems

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ABSTRACT

Optimal control of linear stochastic systems of general type, perturbed by a stochastic process with independent increments, is considered. The performance functional is quadratic, and various different types of observation processes, providing either complete or incomplete information about the system, are discussed. It is shown that (with the conditions imposed) the optimal control is linear in the observed data and can be determined by solving a deterministic problem with a similar dynamic structure. These results are applied to the control of linear stochastic functional differential equations, both with complete and incomplete state information. In the latter case, a separation theorem is shown to be valid: The problem is decomposed into the corresponding deterministic control problem and a problem of estimation. The optimal feedback solution of the deterministic problem is derived.

1. INTRODUCTION

In this paper we consider stochastic control problems of the following type: Let $y(t)$ be an n_1-dimensional stochastic vector process with index set $[0,T] \subset R$ defined by:

$$y(t) = y_0(t) + \int_0^t K(t,s)u(s)ds, \tag{1.1}$$

where $y_0(t)$ is a measurable[1] stochastic process with finite second-order moments, $u(t)$ is a control vector process belonging to a class $\mathcal{U}$ to be specified below, and K is an $n_1 \times n_2$-matrix function with the property:

$$\int_0^T \int_0^T |K(t,s)|^2 \, ds \, d\alpha(t) < \infty,$$
$$\int_0^T |K(t,s)|^2 \, ds < \infty \qquad \text{for all } t \in [0,T], \tag{1.2}$$

where α is a monotone increasing bounded function that is continuous on the right, thus defining a finite Borel measure, and $|\cdot|$ denotes Euclidean norm. The problem is to determine a control $u \in \mathcal{U}$ so as to minimize the performance functional:

$$E\left\{\int_0^T y'(t)\, Q_1(t) y(t)\, d\alpha(t) + \int_0^T u'(t)\, Q_2(t) u(t)\, dt\right\}, \tag{1.3}$$

where E and prime ($'$) signify expectation and transposition respectively, Q_1 and Q_2 are symmetric and bounded matrix-functions with appropriate dimensions. Q_1 is nonnegative definite and Q_2 is positive definite. Furthermore, Q_2^{-1} is bounded.

Information about the realization of the stochastic process $y(t)$ is provided through a measurable process $z(t)$ with finite second-order moments. This process is in some way related to y, and we shall call it the *observation process*. At each time t, full information about $\{z(s); 0 \leqslant s \leqslant t\}$ is available, and therefore $u(t)$ should be a nonanticipative functional of z. For this reason, it is natural to define our class $\mathcal{U}$ of admissible controls to be measurable n_2-dimensional stochastic vector processes which for each time t are $\sigma\{z(s); s \in [0,t]\}$-measurable random vectors and for which $\int_0^T E|u(t)|^2 \, dt < \infty$. ($\sigma\{\cdot\}$ denotes the σ-algebra of events generated by $\{\cdot\}$.)

To start with, in Sec. 2, this problem will be solved when:

$$y_0(t) = \bar{y}_0(t) + \int_0^t M(t,s)\, dv(s), \tag{1.4}$$

where $\bar{y}_0$ is a deterministic function such that $\int_0^T |\bar{y}_0|^2 \, d\alpha < \infty$, M is a matrix-

[1] We assume an underlying probability space $(\Omega, \mathfrak{S}, P)$ where Ω is the sample space with elements ω, $\mathfrak{S}$ is a σ-algebra of events and P is the probability measure. The stochastic process $y(t, \omega)$ is said to be measurable if it is $(\mathfrak{B} \times \mathfrak{S})$-measurable, where $\mathfrak{B}$ is the σ-algebra of Borel sets. All deterministic functions are taken to be Borel measurable. In the sequel, ω will be suppressed from notation. Further discussion of questions pertaining to measure theory will be postponed to the appendix.

function of type (1.2) and v is an n_3-dimensional measurable stochastic vector process with zero mean and *independent increments*†:

$$Ev(t) = 0; \qquad E\{v(s)v'(t)\} = I\min(s,t). \tag{1.5}$$

(I is the identity matrix). The integral in Eq. (1.4) is a stochastic integral defined in quadratic mean. Provided that M is measurable in (t,s), which we assume, the integral can be defined so that $y(t)$ is a measurable stochastic process. (See paragraph 2 in the appendix.) Furthermore, the observation process $z(t) = v(t)$. Many stochastic control problems with complete state-information can be reformulated in this way as will be demonstrated below.

In Sec. 3 we shall consider the problem obtained when $y_0(t)$ in Eq. (1.1) is an arbitrary but *Gaussian* stochastic vector process for which $E|y_0|^2$ is α-integrable and $z(t) = v(t)$ a standard *Wiener process* (Gaussian) with properties (1.5). Moreover, y_0 and v are jointly Gaussian. It will be shown that this problem can be reduced to the previous one. This is a more general problem than one might first realize. In fact, it is often possible to transform a general observation process $z(t)$ to a Wiener process $v(t)$ which, loosely speaking, contains the same information. Such a process is called an *innovation process*. We shall discuss this matter more thoroughly below.

It will be shown in Secs. 2 and 3 that these stochastic control problems can be reduced to *deterministic* control problems with a similar structure. This is an extension of the results given by Lindquist [3–5] concerning a *separation theorem* for stochastic systems with time delay. In Ref. 3 the ordinary separation theorem for linear quadratic problems (see references [6–11] and others) is modified to include delays in the performance function. In Ref. 5, an early version of this paper, time delays are introduced in the system equations and the observation process as well. The methods of these papers, which were influenced by Kailath [12–14] and Frost [13, 15], are also employed here. (The problem of Ref. 3 has also been solved in Ref. 16 by a completely different method.)

The conditions above that v have independent increments and the $\{y_0, v\}$ be Gaussian respectively, are introduced to secure that the optimal *linear* control (linear in the observation process) be optimal also in the class $\mathscr{U}$ of nonlinear controls. If we are only interested in determining the best *linear* control these conditions can be dispensed with. Thus, all results of this paper remain valid in this "wide sense" by assuming only that v have *orthogonal* increments.

So far we have tacitly assumed that the observation process z is unaffected by the control in the sense that it is constant with respect to variation of the

† *Note added in proof:* Actually it is sufficient to assume that v is a second-order *martingale*. (See Addendum.)

control function. A class $\mathscr{U}$ of admissible controls which is defined as above by means of an observation process which is unaffected by the control, will be called a *stochastic open loop* (SOL) *class*. (The word "stochastic" accentuates the fact that this is not a usual open-loop class, which only includes sure (deterministic) functions.) In the problems described above (which are solved in Sec. 2 and the beginning of Sec. 3) the optimal control is determined with respect to a SOL-class, for it is assumed that the process $v(t)$ is unaffected by the control.

On the other hand, if z is a function of u, which is usually the case when there is a functional dependence between z and y, and u is a function of z, we obtain a feedback loop. Then we must inevitably raise the question whether there really exist unique solutions to all equations defining the system. In our class $\mathscr{U}$ we can only admit controls u which are formed in the following way: $u(t) = \phi(t,z)$, where ϕ is a function such that existence and uniqueness are secured. The function ϕ which maps stochastic processes into stochastic processes,[2] should be *nonanticipative*, that is for each t $\phi(t,\cdot)$ is a function of $\{z(s); s \in [0,t]\}$ such that $\phi(t,z)$ is $\sigma\{z(s); s \in [0,t]\}$-measurable. Then $\mathscr{U}$ consists of all processes $\phi(t, z_\phi)$ where z_ϕ is the unique z-process corresponding to the admissible control law ϕ. In the sequel, we shall usually say that u is a function of z whenever there is no reason for misunderstanding.

In Secs. 3–6 we shall impose some further conditions on the control. In this way it will be possible to imbed the so-defined class of admissible controls $\mathscr{U}_0 \subset \mathscr{U}$ in a SOL-class defined by an independent increment observation process, in the sense that each $u \in \mathscr{U}_0$ belongs to this SOL-class. Then we can use the theory of Sec. 2 to determine an optimal control u^* in the SOL-class. If $u^* \in \mathscr{U}_0$ we have found an optimal control in $\mathscr{U}_0$, and it remains to determine a corresponding control law.

In Secs. 5 and 6 we consider optimal control of systems of stochastic functional differential equations—in the title of this paper loosely referred to as time-lag systems—described by the stochastic differential equation:

$$\begin{cases} dx(t) = \int_{t-h}^{t} d_s A(t,s)x(s)\,dt + B(t)u(t)\,dt + C(t)\,dv(t) & \text{for } t \geqslant 0 \\ x(t) = \xi(t) & \text{for } t \leqslant 0. \end{cases} \tag{1.6}$$

Here $x(t)$ is an n_4-dimensional stochastic vector process, $v(t)$ is the independent increment process defined by Eq. (1.5) and ξ is a measurable vector process with bounded second-order moments and mean $E\xi(t) = a(t)$, which of course is a bounded function. The $n_4 \times n_4$-matrix function A fulfils the following conditions ($h > 0$):

[2] Since we shall make no distinction between stochastically equivalent processes, we really have a mapping between equivalence classes.

$$\begin{cases} A(t,s) = 0 & \text{for } s \geqslant t \\ A(t,s) = A(t,t-h) & \text{for } s \leqslant t-h \\ s \to A(t,s) \text{ is continuous on the right} & \\ \underset{s\in[0,T]}{\text{var}} \; |A(t,s)| \leqslant m(t), & \end{cases} \tag{1.7}$$

where var means total variation and $m \in L_1(0,T)$. The first term in Eq. (1.6) is a Lebesgue-Stieltjes integral.[3] In the following we shall deal on several occasions with integrals of type $\int d_s A(t,s)\, x(s)$ where x is a measurable process for which $E|x|^2$ is bounded. Then $E|x|$ is also bounded and

$$\iint |d_s A(t,s)| E|x(s)| dt \leqslant \int m(t)\, dt \sup E|x| < \infty.$$

By applying Fubini's theorem twice ($\int |d_s A(t,s)|\, |x(s)|$ is measurable according to Ref. 29, p. 9) we then find that $\int d_s A(t,s) x(s,\omega)$ exists as an integrable function for almost all ω. Furthermore, the matrix functions B and C are square integrable. Finally, u is a measurable vector control process, for which $\int E|u|^2 dt < \infty$.

By using the transfer function Φ corresponding to the matrix function $A(t,s)$ (see Sec. 4, Eqs. (4.14)–(4.18)), we can *formally* solve Eq. (1.6) to obtain:

$$x(t) = \Phi(t,0)\,\xi(0) + \int_{-h}^{0} d_\tau \left\{ \int_0^t \Phi(t,s)\, A(s,\tau)\, ds \right\} \xi(\tau)$$

$$+ \int_0^t \Phi(t,s)\, B(s)\, u(s)\, ds + \int_0^t \Phi(t,s)\, C(s)\, dv(s). \tag{1.8}$$

Here from our previous discussion on Stieltjes integrals the second term is well defined, and so is the third term, for Φ is bounded. The last term can be regarded as a stochastic integral in quadratic mean. (We consider a measurable version.) It will be shown in Sec. 5 (Theorem 5.3) that Eq. (1.8) is a unique solution of Eq. (1.6), provided that we interpret it in the proper way [Eq. (5.25)]. Now, Schwarz's inequality can be used to see that $E|x(t)|^2$ is bounded on $[0,T]$, and therefore we are allowed to form integrals of type $\int d_s A(t,s) x(s)$.

The observation process $z(t)$ may be of various different types. We shall consider two particular cases:

A. *Complete state information:* In this case the process x itself is available for observation from $t = -h$:

$$z(t) = x(t). \tag{1.9}$$

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[3] We take the intervals of integration for Lebesgue-Stieltjes integrals to be open in the left-end and closed in the right.

Therefore at time $t = 0$, the realization of ξ is known, and for this reason we shall regard ξ as a sure function. It is then natural and in accordance with our previous discussion to define $\mathscr{U}$ as the class of measurable stochastic processes which can be expressed by a nonanticipative function of x, such that with this function inserted in Eq. (1.6) there exists a unique solution of this equation, and such that $\int_0^T E|u(t)|^2\,dt < \infty$.

B. *Incomplete state information:* The information about the x-process is provided by the n_5-dimensional vector process:

$$dz(t) = \int_0^t d_s H(t,s)\,x(s)\,dt + dw(t), \qquad z(0) = 0, \tag{1.10}$$

where the $n_5 \times n_4$ matrix function H is of type (1.7) (except that the second condition need not be fulfilled) with $m \in L_2(0,T)$, and $w(t)$ is a standard Wiener process. To simplify matters, we assume that v, w, and ξ are independent (this condition can be relaxed), and that ξ is almost surely sample continuous. Also, in order to obtain interesting results, it is necessary that v be a Wiener process and ξ is Gaussian. Then the class $\mathscr{U}$ of admissible controls should be defined in the same way as in Ex. A, except that u should now be a nonanticipative function of z such that there exist unique solutions of Eqs. (1.10) and (1.6). By making some trivial modifications below, we may include a linear nonanticipative function of u in the right member of Eq. (1.10), but we shall refrain from this since it might obscure the notations.

Now, the problem is to determine a $u \in \mathscr{U}$ (and an admissible control law to implement it) so as to minimize Eq. (1.3) when y is defined by:

$$y(t) = \int_{t-h}^{t} d_s D(t,s)\,x(s), \tag{1.11}$$

where D is a matrix function of type (1.7) such that $m(t) < \infty$ and $\int m^2\,d\alpha < \infty$.

First, to solve these problems, in Sec. 4 we consider the deterministic counterpart obtained by putting $v \equiv 0$ in Eq. (1.6), letting ξ be a sure function, and removing the expectation sign before Eq. (1.3). It is then shown that for this problem we have an optimal feedback solution:

$$u(t) = \int_{t-h}^{t} d_s L(t,s)\,x(s). \tag{1.12}$$

Now, the optimal solutions of the stochastic problems can be expressed in terms of this feedback law:

A. *Complete state information*: The optimal solution is precisely given by Eq. (1.12).

B. *Incomplete state information*: The optimal control is given by:

$$u(t) = \int_{t-h}^{t} d_s L(t,s)\,\hat{x}(s|t), \tag{1.13}$$

where $\hat{x}(s|t)$ is the least squares estimate:

$$\hat{x}(s|t) = E\{x(s)|z(\tau);\tau \in [0,t]\}. \tag{1.14}$$

This is an affine function of z given by a stochastic differential equation, which is unfortunately rather complicated. It can also be determined by solving a dual deterministic control problem as demonstrated by Lindquist [26].

In Sec. 5 the problem of delay between observation and control action is discussed in the case of complete state information. Also in this case the corresponding deterministic control law, duly reformulated with respect to the delay, is optimal. However, the arbitrariness inherent in the formulation of the deterministic control is no longer true for the stochastic case, for here the control must be adjusted to the specific information pattern.

Finally, we should point out that there are other possible applications of the results of this paper. For instance, a stochastic counterpart of the dual control problem of Ref. 26, which involves Volterra integral equations, has a dynamic structure of the same type as the one encountered in Secs. 5 and 6, and similar results hold.

2. STOCHASTIC CONTROL WITH COMPLETE INFORMATION

Let $v(t)$ be the process with independent increments defined in Sec. 1 Eq. (1.5). In the Hilbert space H of all stochastic variables with finite second-order moments with inner product $(\xi,\eta) = E\{\xi\eta\}$, define H_t to be the closed linear hull of the stochastic variables $\{v_i(s); s \in [0,t], i = 1,2\ldots n_3\}$ together with all constants. That is, H_t consists of all finite affine combinations of the components of $\{v(s); s \in [0,t]\}$ and limits in quadratic mean of such sums. Therefore, due to the definition of a stochastic integral, $\xi \in H_t$ if and only if it can be written:

$$\xi = \bar{\xi} + \int_0^t f'(s)\,dv(s), \tag{2.1}$$

where ξ is a constant and f an L_2 vector function. Now, following Doob [1], we shall call the projection of an arbitrary $\eta \in H$ onto H_t the *wide sense conditional expectation* of η relative to $\{v(s); s \in [0,t]\}$ and denote it by $\hat{E}\{\eta | v(s); s \in [0,t]\}$. It is obvious that this is the best "linear least squares estimate" of η given $\{v(s); s \in [0,t]\}$.

We can now state the following lemma (which is also true under the less restrictive condition that v has orthogonal increments):

LEMMA 2.1. *Let x be a stochastic vector such that $E|x|^2 < \infty$, and let v be as defined above. Then:*

$$\hat{E}\{x | v(s); s \in [0,t]\} = \bar{x} + \int_0^t N(s)\, dv(s), \tag{2.2}$$

where

$$\begin{cases} \bar{x} = Ex & (2.3) \\ N(s) = \dfrac{d}{ds} E\{xv'(s)\}. & (2.4) \end{cases}$$

Proof (cf. Kailath [12]). For a fixed t, the representation (2.2) follows from Eq. (2.1). Since it is immediately clear that $\bar{x} = Ex$, it only remains to determine N. (We shall write N_t to remind ourselves that it might depend on t.) To this end, denote the estimate (2.2) by $\hat{x}$ and note that the components of $x - \hat{x}$ by construction are orthogonal to H_t and in particular to the components of $v(s)$ for $s \leqslant t$:

$$E\{(x - \hat{x})\, v'(s)\} = 0.$$

But:

$$E\{\hat{x} v'(s)\} = E\left\{\int_0^t N_t(\tau)\, dv \int_0^s dv'\right\} = \int_0^s N_t(\tau)\, d\tau,$$

and therefore:

$$E\{xv'(s)\} = \int_0^s N_t(\tau)\, d\tau,$$

which is absolutely continuous and therefore Eq. (2.4) is true (for almost all s). It is then clear that $N_t(s)$ does not depend on t and the index can be dropped.

Remark. If the system $\{x, v(s); s \in [0,t]\}$ is Gaussian, the wide sense and strict sense conditional expectations coincide, and we can exchange $\hat{E}$ for E in the lemma. (See for example Ref. 2, p. 229.)

Now, let us return to the first problem posed in Sec. 1, that is to control:

$$y(t) = \bar{y}_0(t) + \int_0^t M(t,s)\, dv(s) + \int_0^t K(t,s)\, u(s)\, ds, \tag{2.5}$$

when the object is to minimize (1.3) and the class of admissible controls $\mathscr{U}$ is defined by the observation process $z(t) = v(t)$. Let us first determine the optimal control in the subclass $\mathscr{L} \subset \mathscr{U}$ of *affine* functionals of v:

$$u(t) = \bar{u}(t) + \int_0^t U(t,s)\, dv(s), \tag{2.6}$$

where $\bar{u}(t)$ is a deterministic L_2 vector function and U is an L_2 matrix-kernel, that is $\iint |U|^2 ds\, dt < \infty$. It will be shown below that the optimal control u^* in this affine class is optimal also in $\mathscr{U}$.

Inserting Eq. (2.6) in Eq. (2.5) and changing the order to integration (permitted due to a Fubini type theorem for stochastic integrals; Ref. 1, p. 431 or Ref. 2, p. 197) we have

$$y(t) = \bar{y}(t) + \int_0^t Y(t,s)\, dv(s), \tag{2.7}$$

where:

$$\begin{cases} \bar{y}(t) = \bar{y}_0(t) + \int_0^t K(t,s)\, \bar{u}(s)\, ds & (2.8) \\ Y(t,s) = M(t,s) + \int_s^t K(t,\tau)\, U(\tau,s)\, d\tau. & (2.9) \end{cases}$$

It is easily seen that:

$$E\{y' Q_1 y\} = \bar{y}' Q_1 \bar{y} + E\{(y - \bar{y})' Q_1 (y - \bar{y})\} \tag{2.10}$$

The second term in Eq. (2.10) can be written (tr is the trace-operator)[4]:

$$\operatorname{tr}\{Q_1(t)\, E\{[y(t) - \bar{y}(t)][y(t) - \bar{y}(t)]'\}\} = \operatorname{tr}\left\{Q_1(t) \int_0^t Y(t,s)\; Y'(t,s)\, ds\right\}$$

$$= \int_0^t \operatorname{tr}\{Y'(t,s)\, Q_1(t)\; Y(t,s)\} ds.$$

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[4] Remember the formal rule of calculation:

$$E\{dv(s) dv'(t)\} = \begin{cases} I\, dt & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases}$$

By changing the order of integration (permitted, for the integrand is non-negative) we have:

$$E\left\{\int_0^T [y(t)-\bar{y}(t)]'\,Q_1(t)\,[y(t)-\bar{y}(t)]d\alpha\right\} = \int_0^T\int_0^t \operatorname{tr}\{Y'(t,s)\,Q_1(t)\;Y(t,s)\}ds\,d\alpha(t)$$

$$= \int_0^T\int_s^T \operatorname{tr}\{Y'(t,s)\,Q_1(t)\;Y(t,s)\}d\alpha(t)\,ds$$

$$= \sum_{i=1}^{n_3}\int_0^T\int_s^T y_i'(t,s)\,Q_1(t)\,y_i(t,s)\,d\alpha(t)\,ds, \tag{2.11}$$

where y_i is the ith column of the matrix Y:

$$Y = (y_1, y_2, \ldots y_{n_3}). \tag{2.12}$$

In exactly the same way we derive the counterparts of Eqs. (2.10) and (2.11) for u, and then the performance functional (1.3) can be written:

$$\int_0^T \bar{y}'(t)\,Q_1(t)\,\bar{y}(t)\,d\alpha + \int_0^T \bar{u}'(t)\,Q_2(t)\,\bar{u}(t)\,dt$$

$$+\sum_i \int_0^T\left\{\int_s^T y_i'(t,s)\,Q_1(t)\,y_i(t,s)\,d\alpha + \int_s^T u_i'(t,s)\,Q_2(t)\,u_i(t,s)\,dt\right\}ds, \tag{2.13}$$

where u_i is the ith column of the matrix U:

$$U = (u_1, u_2, \ldots u_{n_3}).$$

We have now decomposed our problem, and it is clear from Eqs. (2.8), (2.9), and (2.13) that $\bar{u}$ and U should be determined so as to fulfil the requirements of the following optimality theorem:

THEOREM 2.1. *The control* $u^* \in \mathscr{L}$:

$$u^*(t) = \bar{u}^*(t) + \int_0^t U^*(t,s)\,dv(s), \tag{2.14}$$

is optimal in the class $\mathscr{L}$ *defined by Eq.* (2.6) *if and only if:*
A: *Out of all* L_2 *vector functions* $\bar{u}$, $\bar{u}^*$ *minimizes:*

$$\int_0^T y'(t)\,Q_1(t)\,\bar{y}(t)\,d\alpha(t) + \int_0^T \bar{u}'(t)\,Q_2(t)\,\bar{u}(t)\,dt,$$

when $\bar{y}$ is given by:

$$\bar{y}(t) = \bar{y}_0(t) + \int_0^t K(t,s)\,\bar{u}(s)\,ds. \tag{2.15}$$

B: *Out of all L_2 vector functions $u_i(\cdot,s)$, $u_i^*(\cdot,s)$ minimizes:*

$$\int_s^T y_i'(t,s)\,Q_1(t)\,y_i(t,s)\,d\alpha(t) + \int_s^T u_i'(t,s)\,Q_2(t)\,u_i(t,s)\,dt,$$

for all i and almost all s, when $y_i(t,s)$ is given by:

$$y_i(t,s) = m_i(t,s) + \int_s^t K(t,\tau)\,u_i(\tau,s)\,d\tau, \tag{2.16}$$

where m_i is the ith *column of the matrix M.*

Note that these problems are deterministic counterparts of our original stochastic problem and are essentially obtained by putting $v \equiv 0$ in this problem. It will be shown in Sec. 4 that the problems of Theorem 2.1 indeed have solutions constituting admissible $\bar{u}^*$ and U^* and thus there exists an optimal affine control.

THEOREM 2.2. *The optimal affine control u^* is optimal also in the class $\mathscr{U}$ of all measurable n_2-dimensional stochastic vector processes which for each time t are $\sigma\{v(s); s \in [0,t]\}$-measurable stochastic vectors and for which $\int_0^T E|u|^2\,dt < \infty$.*

Proof. Due to Fubini's theorem, every $u \in \mathscr{U}$ fulfils $E|u(t)|^2 < \infty$ for almost all t. Then for these t, the wide sense conditional expectation $\hat{u}(t) = \hat{E}\{u(t)|v(s); s \in [0,t]\}$ exists and according to Lemma 2.1:

$$\hat{u}(t) = \bar{u}(t) + \int_0^t U(t,s)\,dv(s), \tag{2.17}$$

where $\bar{u}(t) = Eu(t)$ and $U(t,s) = \partial/\partial s\{E[u(t)\,v'(s)]\}$.

The function $U(t,s)$ can be defined to be measurable in (t,s), and therefore the stochastic process $\hat{u}$ has a measurable version. (See the appendix for proofs.) Furthermore, since $\hat{u}(t)$ is a projection of $u(t)$, we have $E|\hat{u}(t)|^2 \leqslant E|u(t)|^2$ and hence $\int_0^T E|\hat{u}|^2\,dt < \infty$. The matrix function U is an L_2 kernel, that is $\int_0^T \int_0^T |U(t,s)|^2\,ds\,dt < \infty$, and $\bar{u}$ is an L_2 vector function. In fact,

$$\int_0^T \int_0^T \operatorname{tr}\{U(t,s)\,U'(t,s)\}\,ds\,dt + \int_0^T |\bar{u}(t)|^2\,dt = \int_0^T E|\hat{u}(t)|^2\,dt < \infty. \tag{2.18}$$

Now, define the process $\hat{y}$:

$$\hat{y}(t) = \bar{y}_0(t) + \int_0^t M(t,s)\,dv(s) + \int_0^t K(t,s)\,\hat{u}(s)\,ds. \tag{2.19}$$

Then inserting Eq. (2.17) in Eq. (2.19) and changing the order of integration (Ref. 1, p. 431 or Ref. 2, p. 197) shows that, for each t, $\hat{y}(t)$ belongs to H_t. Denote the residual process $y(t) - \hat{y}(t)$ by $\tilde{y}(t)$ and $u(t) - \hat{u}(t)$ by $\tilde{u}(t)$. From Eq. (2.5) and Eq. (2.19), we then have:

$$\tilde{y}(t) = \int_0^t K(t,s)\,\tilde{u}(s)\,ds. \tag{2.20}$$

For each t, $\tilde{y}(t)$ is orthogonal to H_t. In fact, let ξ, defined by Eq. (2.1), be an arbitrary element in H_t. Then for $s \leqslant t$:

$$E\{\xi\tilde{u}(s)\} = E\left\{\left[\bar{\xi} + \int_0^s f'(\tau)\,dv\right]\tilde{u}(s)\right\} + E\left\{\int_s^t f'(\tau)\,dv\,\tilde{u}(s)\right\} = 0,$$

for $\tilde{u}(s)$ is orthogonal to H_s and hence to $\bar{\xi} + \int_0^s f'(\tau)\,dv$. Moreover, v has independent increments, and therefore $\int_s^t f'(\tau)\,dv$ and $\tilde{u}(s)$ are independent, since $\tilde{u}(s)$ is $\sigma\{v(\tau); \tau \in [0,s]\}$-measurable. Therefore:

$$E\{\xi\tilde{y}(t)\} = \int_0^t K(t,s)\,E\{\xi\tilde{u}(s)\}ds = 0,$$

which establishes the orthogonality. (Clearly $E|\xi\tilde{u}(s)| \leqslant (E|\xi|^2)^{1/2}(E|\tilde{u}(s)|^2)^{1/2}$ is square integrable, and therefore Fubini's theorem applies in view of Eq. (1.2).)

Due to the orthogonality between $\hat{y}$, $\hat{u}$, and $\tilde{y}$, $\tilde{u}$ the performance functional (1.3) can be written:

$$E\left\{\int_0^T \hat{y}'\,Q_1\,\hat{y}\,d\alpha + \int_0^T \hat{u}'\,Q_2\,\hat{u}\,dt\right\} + E\left\{\int_0^T \tilde{y}'\,Q_1\,\tilde{y}\,d\alpha + \int_0^T \tilde{u}'\,Q_2\,\tilde{u}\,dt\right\}. \tag{2.21}$$

Now, the problem is to determine $\tilde{u}$ to minimize the expression on the last line of Eq. (2.21) when $\tilde{y}$ is given by Eq. (2.20), which clearly has the optimal solution $\tilde{u} = 0$, and to determine $\hat{u}$ to minimize the first part of Eq. (2.21) when $\hat{y}$ is given by Eq. (2.18), which is equivalent to the problem of Theorem 2.1. This concludes the proof.

COROLLARY 2.2. *For $\delta > 0$, let $\mathcal{U}_\delta \subset \mathcal{U}$ be the class of all controls in $\mathcal{U}$ which for each time $t \geqslant \delta$ are $\sigma\{v(s); s \in [0, t-\delta]\}$-measurable and for $t < \delta$ are sure functions. Then the optimal control in the class $\mathcal{U}_\delta$ of the problem to minimize (1.3) given Eq. (2.5) is obtained from Theorem 2.1 provided that the extra condition $U(t,s) = 0$ for $t < s + \delta$ is imposed.*

Proof. Let $v(t) = 0$ for $t \leqslant 0$. Every $u \in \mathscr{U}_\delta$ also belongs to $\mathscr{U}$, and therefore the decomposition (Eq. 2.19–2.21) is still valid, and the best performance is achieved by putting $\tilde{u} = 0$. However, for $s > t - \delta$, $E\{u(t)v'(s)\} = E\{u(t)v'(t-\delta)\}$ because $v(s) - v(t-\delta)$ is independent of $\sigma\{v(\tau); \tau \leqslant t - \delta\}$, and therefore $U(t,s) = \partial/\partial s\{E[u(t)v'(s)]\} = 0$ for $s > t - \delta$. Therefore, the optimal control is clearly given by Theorem 2.1 with the further condition that $U(t,s) = 0$ for $t < s + \delta$. The new problems in B are essentially of the same type as before and existence is shown in Sec. 4.

3. STOCHASTIC CONTROL WITH INCOMPLETE INFORMATION

Let $v(t)$ be a standard vector Wiener process (a Gaussian process with independent increments defined by Eq. (1.5)) and $y_0(t)$ a Gaussian vector process for which $E|y_0(t)|^2$ is finite and α-integrable. We shall consider the following problem: Determine a measurable control process $u(t)$, which for each t is a $\sigma\{v(s); s \in [0,t]\}$-measurable stochastic vector and for which $\int_0^T E|u(t)|^2 dt < \infty$ so as to minimize the performance functional (1.3) when the process $y(t)$ is defined by:

$$y(t) = y_0(t) + \int_0^t K(t,s)\, u(s)\, ds. \tag{3.1}$$

The two underlying stochastic processes y_0 and v, which are jointly Gaussian, are not affected by the choice of control function.

Since $\{y_0, v\}$ are Gaussian, the wide sense and strict sense conditional expectations coincide, and therefore Lemma 2.1 implies:

$$\hat{y}_0(t|t) = E\{y_0(t)|v(s); s \in [0,t]\} = \bar{y}_0(t) + \int_0^t M(t,s)\, dv(s), \tag{3.2}$$

where:

$$\begin{cases} \bar{y}_0(t) = E y_0(t) & (3.3) \\ M(t,s) = \dfrac{\partial}{\partial s} E\{y_0(t)\, v'(s)\}. & (3.4) \end{cases}$$

Furthermore, since the last term in Eq. (3.1) is $\sigma\{v(s); s \in [0,t]\}$-measurable, $\hat{y}(t|t) = E\{y(t)|v(s); s \in [0,t]\}$ is given by:

$$\hat{y}(t|t) = \bar{y}_0(t) + \int_0^t M(t,s)\, dv(s) + \int_0^t K(t,s)\, u(s)\, ds, \tag{3.5}$$

which is a process of the same type as Eq. (2.5). (By a similar method as the one used to derive Eq. (2.18), it can be shown that M indeed fulfils condition (1.2), and that $\int_0^T |\bar{y}_0(t)|^2 d\alpha < \infty$.) The residual process $\tilde{y}(t|t) = y(t) - \hat{y}(t|t)$, which is orthogonal to $\hat{y}(t|t)$, is given by:

$$\tilde{y}(t|t) = y_0(t) - \bar{y}_0(t) - \int_0^t M(t,s)\, dv(s), \tag{3.6}$$

which is not affected by the control.

Due to the orthogonality, we obtain:

$$E\{y'(t)\, Q_1(t)\, y(t)\} = E\{\hat{y}'(t|t)\, Q_1(t)\, \hat{y}(t|t)\} + E\{\tilde{y}'(t|t)\, Q_1(t)\, \tilde{y}(t|t)\}.$$

Since the last term is unaffected by the control, the performance functional (1.3) can be exchanged for:

$$E\left\{\int_0^T \hat{y}'(t|t)\, Q_1(t)\, \hat{y}(t|t)\, d\alpha + \int_0^T u'(t)\, Q_2(t)\, u(t)\, dt\right\}, \tag{3.7}$$

which together with Eq. (3.5) forms a problem of the type discussed in Sec. 2.

THEOREM 3.1. *The optimal solution u^* of the problem posed above is given by:*

$$u^*(t) = \bar{u}^*(t) + \int_0^t U^*(t,s)\, dv(s), \tag{3.8}$$

where $\bar{u}^*$ and U^* are the optimal solutions of the deterministic problems of Theorem 2.1 *where $\bar{y}_0$ and M are defined by Eq.* (3.3) *and Eq.* (3.4) *respectively.*

We shall now consider a problem with a more general observation process $z(t)$ defined by the stochastic differential equation:

$$dz(t) = q(t)\, dt + dw(t),$$
$$z(0) = 0, \tag{3.9}$$

where the measurement noise $w(t)$ is a standard vector Wiener process, and $q(t)$ is a vector process which is causally dependent on $\{y(s); s \in [0,t]\}$ (y is defined by Eq. (3.1)) in a way more carefully specified below. It is assumed that the dependence of the control thus introduced into q, enters linearly[5]:

$$q(t) = q_0(t) + \int_0^t N(t,s)\, u(s)\, ds, \tag{3.10}$$

[5] We may exchange the last term in Eq. (3.10) for $\int_0^t d_s N(t,s) u(s)$ if N is defined so that the conclusions concerning Eq. (3.14) are unaltered.

where $q_0(t)$ is a measurable vector process for which $E|q_0(t)|^2$ is integrable and N is an L_2 matrix kernel. We further assume that the joint finite-dimensional distributions of $\{y_0(t), q_0(t), \int_0^t q_0(s)\,ds; t \in [0,T]\}$ are Gaussian (see paragraph 4 in the appendix), and to simplify matters, that w is independent of these processes. (As we have done above, we shall put a zero index on the uncontrolled counterparts of processes normally affected by the control.) Then if we define the uncontrolled observation process:

$$dz_0(t) = q_0(t)\,dt + dw(t),$$
$$z_0(0) = 0, \tag{3.11}$$

we can write:

$$dz(t) = dz_0(t) + \int_0^t N(t,s)\,u(s)\,ds\,dt. \tag{3.12}$$

When $u(t)$ is a function of $\{z(s); s \in [0,t]\}$, the feedback loop described by Eq. (3.12) may introduce some rather intricate problems of existence. Whether or not there exists a unique solution to Eq. (3.9) of course depends on the control function used. In studying this problem we shall begin with a class $\mathscr{L}$ of *affine* controls:

$$u(t) = f(t) + \int_0^t F(t,s)\,dz(s) \tag{3.13}$$

where f is an L_2 vector function, F is an L_2 matrix kernel, and the integral is defined in quadratic means (q.m.).

Inserting Eq. (3.13) into (3.12) and changing the order of integration[6], we obtain an expression of the following type:

$$dz(t) = dz_0(t) + \int_0^t \Gamma(t,s)\,dz(s)\,dt + h(t)\,dt \tag{3.14}$$

where Γ is an L_2 matrix kernel and h is an L_2 vector function.

Now, let the L_2 kernel $\Lambda(t,s)$ be defined by the Volterra resolvent equation:

$$\Gamma(t,s) - \Lambda(t,s) = -\int_s^t \Lambda(t,\tau)\,\Gamma(\tau,s)\,d\tau. \tag{3.15}$$

[6] Since $dz = q\,dt + dw$, the stochastic double integral consists of two terms. In the first term, which also exists as an ordinary (Lebesgue) integral changing the order of integration is permitted (almost surely) according to Fubini's theorem. ($\int |q|^2 dt < \infty$ a.s.) For the second term, we find the relevant theorem in Ref. 1, p. 431 or in Ref. 2, p. 197.

By changing the order of integration, we obtain from Eq. (3.15) and Eq. (3.14):

$$\int_0^t \Gamma(t,s)\,dz(s) = \int_0^t \Lambda(t,s)\,dz(s) - \int_0^t \int_s^t \Lambda(t,\tau)\,\Gamma(\tau,s)\,d\tau\,dz(s)$$

$$= \int_0^t \Lambda(t,s)\,[dz(s) - \int_0^s \Gamma(s,\tau)\,dz(\tau)\,ds]$$

$$= \int_0^t \Lambda(t,s)\,dz_0(s) + \int_0^t \Lambda(t,s)\,h(s)\,ds,$$

$$\therefore\ dz(t) = dz_0(t) + \left[\int_0^t \Lambda(t,s)\,dz_0(s) + \int_0^t \Lambda(t,s)\,h(s)\,ds + h(t)\right]dt. \quad (3.16)$$

Then, from Eqs. (3.14) and (3.16) it is clear that z and z_0 can be obtained from each other by nonanticipative affine transformations, and therefore Eq. (3.9) has a unique solution[7]. We shall now impose the further *condition on the functional relationship between q and y* that there exists a unique solution to Eq. (3.1) for each $u \in \mathscr{L}$. (This condition is fulfilled in most problems of any practical interest.)

Now, if $\mathscr{U}$ is the class of measurable control processes u that are non-anticipative (not necessarily affine) functions of z such that for each u there exist unique solutions to Eqs. (3.9) and (3.1), and for which $\int_0^T E|u(t)|^2\,dt < \infty$, then *define* $\mathscr{U}_0$ as the subclass of $\mathscr{U}$ characterized by:

$$\sigma\{z(s); s \in [0,t]\} = \sigma\{z_0(s); s \in [0,t]\}, \quad (3.17)$$

for every $t \in [0,T]$, that is $\mathscr{U}_0$ is the largest subclass of $\mathscr{U}$, for which the σ-algebras generated by the observation process are constant with respect to variations of the control.

We have already found an important subclass of $\mathscr{U}_0$, namely the class $\mathscr{L}$ of affine controls (3.13):

THEOREM 3.2. $\mathscr{L} \subset \mathscr{U}_0$.

In order to be able to apply Theorem 3.1, we must transform $z(t)$ to a Wiener process. To this end, we define the *innovation process*

$$dv(t) = dz(t) - E\{q(t)|z(s); s \in [0,t]\}dt,$$
$$v(0) = 0. \quad (3.18)$$

[7] As a solution we usually accept any stochastic process (or really equivalence class of processes) for which both members of the equation constitute well-defined and equivalent stochastic processes. However, whenever there is a sample continuous version, as in this case, the equivalence class will be represented by that process. (See Sec. 5, remark 3.)

In view of Eq. (3.17) and the fact that the last term in Eq. (3.10) is $\sigma\{z(s); s \in [0,t]\}$-measurable, for each $u \in \mathscr{U}_0$ v takes the form:

$$dv(t) = dz_0(t) - \hat{q}_0(t|t)\,dt, \tag{3.19}$$

where:

$$\hat{q}_0(t|t) = E\{q_0(t)|z_0(s); s \in [0,t]\}.$$

To ensure that this expression is well-defined (that $\hat{q}_0(t|t)$ is measurable and a.s. integrable) we need the following lemma:

LEMMA 3.1. *The stochastic process $\hat{q}_0(t|t)$ has a measurable version, such that $\int E|\hat{q}_0(t|t)|^2\,dt < \infty$.*

Proof. It is no limitation to consider the case that $Eq_0(t) = 0$. Since (for almost all t) $E|q_0(t)|^2 < \infty$ and the system of random variables $\{q_0(t), z_0(s); s \in [0,t]\}$ is Gaussian (see paragraph 4 in the appendix), $\hat{q}_0(t|t)$ can be represented in the form:

$$\hat{q}_0(t|t) = \int_0^t \Pi(t,s)\,dz_0(s), \tag{3.20}$$

where $s \to \Pi(t,s)$ is an L_2 function (see Ref. 2, pp. 228–229 and paragraph 5 in the appendix). Since $\tilde{q}_0(t|t) = q_0(t) - \hat{q}_0(t|t)$ is orthogonal to $z_0(s)$ for $s \leqslant t$, and q_0 and w are independent, Π should be determined to satisfy:

$$\begin{aligned} E\{\hat{q}_0(t|t)\,z_0'(s)\} &= E\{q_0(t)\,z_0'(s)\} \\ &= \int_0^s E\{q_0(t)q_0'(\tau)\}d\tau = \int_0^s Q(t,\tau)d\tau \end{aligned}$$

On the other hand, using Eq. (3.20) we have:

$$E\{\hat{q}_0(t|t)\,z_0'(s)\} = \int_0^s \left[\int_0^t \Pi(t,\tau)\,Q(\tau,\sigma)\,d\tau + \Pi(t,\sigma)\right]d\sigma.$$

Therefore, we can define Π as the L_2 matrix kernel solution of the family of Fredholm integral equations (cf. Eq. (4.32) and following):

$$\Pi(t,s) + \int_0^t \Pi(t,\tau)\,Q(\tau,s)\,d\tau = Q(t,s),$$

which clearly delivers a measurable function Π.

Then by Fubini's theorem and paragraph 2 in the appendix, $\hat{q}_0(t|t)$ has a measurable version. The last assertion follows from the fact that $E|\hat{q}_0|^2 \leqslant E|q_0|^2$, for $\hat{q}_0$ is a projection. This concludes the proof.

LEMMA 3.2. *The innovation process as defined by Eq.* (3.19) *is a standard Wiener process* (*such that* ν *and* y_0 *are jointly Gaussian*) *with the property*:

$$\sigma\{\nu(s); s \in [0,t]\} = \sigma\{z_0(s); s \in [0,t]\}. \tag{3.21}$$

Proof. (Cf. Kailath [12, 14, 17] and Frost [15]. Also see Kushner [6], p. 138): A proof that ν is a Wiener process can be found in Ref. 14. To prove Eq. (3.21), remember that Eq. (3.19) can be written:

$$d\nu(t) = dz_0(t) - \int_0^t \Pi(t,s)\, dz_0(s)\, dt - g(t)\, dt, \tag{3.22}$$

where Π is an L_2 kernel and g an L_2 vector function. Then proceed as in the proof of Theorem 3.2 to see that ν and z_0 can be obtained from each other by nonanticipative affine transformations. (A heuristic proof along these lines can be found in Ref. 12.) The joint distribution property is clear from Eq. (3.22) (see paragraph 4 in the appendix).

THEOREM 3.3. *The problem to determine an optimal control* $u \in \mathscr{U}_0$ *so as to minimize* (1.3) *when* $y(t)$ *is given by Eq.* (3.1), *has the optimal solution described by Theorem* 3.1, *where* v *should be exchanged for* ν *as defined by Eq.* (3.18). *If* y_0 *and* w *are independent*, M *is given by*:

$$M(t,s) = E\{y_0(t)\tilde{q}_0'(s|s)\}. \tag{3.23}$$

Proof. The innovation process for $u \in \mathscr{U}_0$, described by Eq. (3.19) is clearly unaffected by the control. Then Theorem 3.1 applies with $v = \nu$, and yields an optimal solution:

$$u^*(t) = \bar{u}^*(t) + \int_0^t U^*(t,s)\, d\nu(s). \tag{3.24}$$

Since z_0, by inversion of Eq. (3.22) = Eq. (3.19), can be expressed as a nonanticipative affine function of ν, the same is true for the z-process corresponding to u^*, called z^*. This can be seen by inserting Eq. (3.24) in Eq. (3.12) and changing the order to integration, thus obtaining z^* as an affine nonanticipative function of ν. Inversion of this function, in the way demonstrated above, establishes u^* as a function of type (3.13), that is $u^* \in \mathscr{L} \subset \mathscr{U}_0$. Then u^* is obviously optimal in $\mathscr{U}_0$, since for every $u \in \mathscr{U}_0$ $u(t)$ is $\sigma\{\nu(s); s \in [0,t]\}$-measurable for each t (see Eqs. (3.17) and (3.21)).

If y_0 and w are independent, inserting $\nu(s) = \int_0^s \tilde{q}_0(\tau|\tau)d\tau + w(s)$ into Eq. (3.4) gives:

$$M(t,s) = \frac{\partial}{\partial s} E\{y_0(t)\nu'(s)\} = \frac{\partial}{\partial s} \int_0^s E[y_0(t)\tilde{q}_0'(\tau|\tau)]d\tau,$$

which concludes the proof.

Theorem 3.1 may also be applied to problems with other types of z-processes. The only major condition is that there can be established a non-anticipative and invertible transformation between z and a Wiener process that is unaffected by the control. For an indication of how this may be done for colored w-noise, see for instance Ref. 7, p. 87 and Ref. 18.

4. THE DETERMINISTIC PROBLEM

To start with, consider the problem to determine an n_2-dimensional L_2 vector function u so as to minimize

$$\int_s^T y'(t)\, Q_1(t)\, y(t)\, d\alpha(t) + \int_s^T u'(t)\, Q_2(t)\, u(t)\, dt, \tag{4.1}$$

when y is an n_1-dimensional vector function defined by

$$y(t) = y_0(t) + \int_s^t K(t,\tau)\, u(\tau)\, d\tau, \tag{4.2}$$

where y_0 is a vector function for which $\int |y_0|^2\, d\alpha < \infty$. All other functions are defined in Sec. 1.

This is precisely the type of problem encountered in Theorem 2.1. Since similar problems have been studied quite extensively in the literature (see for instance Ref. 19), we shall only give a brief outline of the general problem so as to get over to feedback solutions of time lag systems as quickly as possible.

Introducing the Hilbert space H_1 of functions: $(s,T] \to R^{n_1}$ with the inner product $\langle x,y\rangle = \int_s^T x'(t) y(t)\, d\alpha$ and the Hilbert space H_2 of functions: $[s,T] \to R^{n_2}$ with the inner product $[u,v] = \int_s^T u'(t) v(t)\, dt$, our problem can be written: minimize

$$\langle y, Q_1 y\rangle + [u, Q_2 u], \tag{4.3}$$

when:

$$y = y_0 + Ku. \tag{4.4}$$

The meaning of the linear operators $Q_1: H_1 \to H_1$, $Q_2: H_2 \to H_2$ and $K: H_2 \to H_1$ is clear from the context. It is easily seen that the adjoint operator $K^*: H_1 \to H_2$ is given by $(K^*y)(t) = \int_t^T K'(\tau,t) y(\tau)\, d\alpha(\tau)$ and that $Q_i^* = Q_i$ $i = 1,2$ [for $Q_i'(t) = Q_i(t)$].

Then, the performance functional (4.3) can be written:

$$\langle y_0 + Ku, Q_1(y_0 + Ku)\rangle + [u, Q_2 u]$$
$$= [u, \Lambda u] + [u, K^* Q_1 y_0] + [K^* Q_1 y_0, u] + \langle y_0, Q_1 y_0\rangle,$$

where $\Lambda = Q_2 + K^* Q_1 K$ is a positive operator with a bounded symmetric inverse {for $Q_2(t)$ is positive definite and has an inverse that is bounded on $[0, T]$}.

We then have:

$$[u + \Lambda^{-1} K^* Q_1 y_0, \Lambda(u + \Lambda^{-1} K^* Q_1 y_0)] + \langle y_0, (Q_1 - Q_1 K \Lambda^{-1} K^* Q_1) y_0 \rangle,$$

which has a unique minimum for $u = -\Lambda^{-1} K^* Q_1 y_0$ that is:

$$Q_2 u + K^* Q_1 K u + K^* Q_1 y_0 = 0, \tag{4.5}$$

which is a Fredholm integral equation:

$$u(t) + \int_s^T P(t, \tau) u(\tau) \, d\tau = f(t), \tag{4.6}$$

where:

$$\begin{cases} P(t,\tau) = Q_2^{-1}(t) \displaystyle\int_{\max(t,\tau)}^{T} K'(\sigma, t) Q_1(\sigma) K(\sigma, \tau) \, d\alpha(\sigma) & (4.7) \\ f(t) = -Q_2^{-1}(t) \displaystyle\int_t^T K'(\tau, t) Q_1(\tau) y_0(\tau) \, d\alpha(\tau), & (4.8) \end{cases}$$

having a unique L_2 solution:

$$u(t) = f(t) - \int_s^T R(t, \tau, s) f(\tau) \, d\tau. \tag{4.9}$$

Here the resolvent L_2 kernel $R(\cdot, \cdot, s)$ satisfies:

$$R(t, \tau, s) - P(t, \tau) = -\int_s^T P(t, \sigma) R(\sigma, \tau, s) \, d\sigma \tag{4.10$'$}$$

$$= -\int_s^T R(t, \sigma, s) P(\sigma, \tau) \, d\sigma. \tag{4.10$''$}$$

For a discussion of the properties of R the reader is referred to a recent book by Bellman [20], pp. 281–284 where also a suggestion how to solve R is given.

Then, the solutions $\bar{u}^*$ and U^* of the deterministic problems of Theorem 2.1 are given by Eq. (4.9), where in A $y_0 = \bar{y}_0$ and $s = 0$, and in B $y_0(\tau)$ should be exchanged for $m_t(\tau, s)$ thus providing a $U^*(t, s)$ which clearly possesses the necessary properties for Eq. (2.13) to exist as a measurable stochastic process.

Now, let us turn to a problem with a more specific dynamic structure.

We shall consider the problem to control a system of functional differential equations:

$$\begin{cases} \dot{x}(t) = \int_{t-h}^{t} d_s A(t,s)\, x(s) + B(t)\, u(t) & \text{for } t > 0 \\ x(t) = \xi(t) & \text{for } t \leqslant 0 \end{cases} \tag{4.11}$$

$$y(t) = \int_{t-h}^{t} d_s D(t,s)\, x(s), \tag{4.12}$$

(where as usual u is the control function) so as to minimize:

$$\int_0^T y'(t)\, Q_1(t)\, y(t)\, d\alpha(t) + \int_0^T u'(t)\, Q_2(t)\, u(t)\, dt. \tag{4.13}$$

This is the deterministic counterpart of the problem described by Eqs. (1.3) (1.6) and A, B, and D are defined as in Sec. 1. The initial function ξ is bounded on $[-h,0]$. (However, see Remark 2 in Sec. 5.)

Optimal feedback solutions of problems of this type (in somewhat less general formulations) have been given by Manitius [22] who used a method similar to the one presented in this paper, and Kushner and Barnea [23].

For $t \geqslant s \geqslant 0$ the unique solution of Eq. (4.11) can be written:

$$x(t) = \Phi(t,s)\, x(s) + \int_{s-h}^{s} d_\tau \left\{ \int_s^t \Phi(t,\sigma)\, A(\sigma,\tau)\, d\sigma \right\} x(\tau)$$

$$+ \int_s^t \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau, \tag{4.14}$$

where the transfer matrix Φ is defined by:

$$\Phi(t,s) + \int_s^t \Phi(t,\tau)\, A(\tau,s)\, d\tau = I, \tag{4.15}$$

for $t \geqslant s$, and

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(t,s) = \int_s^t d_\tau A(t,\tau)\, \Phi(\tau,s) & \text{for } t > s \quad (4.16) \\ \Phi(s,s) = I; & (4.17) \\ \Phi(t,s) = 0 & \text{for } t < s. \quad (4.18) \end{cases}$$

It can be shown that Φ is bounded (use Gronwall's lemma in Eq. (4.15)), $t \to \Phi(t,s)$ is absolutely continuous, and $s \to \Phi(t,s)$ is of bounded variation. (See Refs. 24–27.)

For all $t \geqslant s$, we have $[D(t,s) \equiv D(t,t-h)$ for $s \leqslant t-h]$:

$$y(t) = \int_{s-h}^{s} d_\tau D(t,\tau)x(\tau) + \int_{s}^{t} d_\tau D(t,\tau)x(\tau)$$

$$= \int_{s-h}^{s} d_\tau D(t,\tau)x(\tau) + \int_{s}^{t} d_\tau D(t,\tau)\Phi(\tau,s)x(s)$$

$$+ \int_{s}^{t} d_\beta D(t,\beta) \int_{s-h}^{s} d_\tau \left\{ \int_{s}^{\beta} \Phi(\beta,\sigma)A(\sigma,\tau)\,d\sigma \right\} x(\tau)$$

$$+ \int_{s}^{t} d_\sigma D(t,\sigma) \int_{s}^{\sigma} \Phi(\sigma,\tau)B(\tau)u(\tau)\,d\tau.$$

The properties of the functions as defined above and in Sec. 1 allow us to change the order to integration in the last two terms, to obtain:

$$\int_{s-h}^{s} d_\tau \left\{ \int_{s}^{t} \left[\int_{\sigma}^{t} d_\beta D(t,\beta)\Phi(\beta,\sigma) \right] A(\sigma,\tau)\,d\sigma \right\} x(\tau),$$

where the unsymmetric Fubini theorem of Cameron and Martin [28] and the usual Fubini theorem have been employed, and for the last term:

$$\int_{s}^{t} \left[\int_{\tau}^{t} d_\sigma D(t,\sigma)\Phi(\sigma,\tau) \right] B(\tau)u(\tau)\,d\tau.$$

Therefore, defining the matrix functions Ψ and Γ to be:

$$\Psi(t,s) = \int_{s}^{t} d_\tau D(t,\tau)\Phi(\tau,s), \tag{4.19}$$

$$\Gamma(t,s,\tau) = D(t,\tau) - \Psi(t,s)\theta(s-\tau) + \int_{s}^{t} \Psi(t,\sigma)A(\sigma,\tau)\,d\sigma, \tag{4.20}$$

where θ is the step function:

$$\theta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{4.21}$$

we obtain the following expression for y:

$$y(t) = y_0(t,s,x_s) + \int_{s}^{t} K(t,\tau)u(\tau)\,d\tau \qquad \text{for } t > s \tag{4.22}$$

where:

$$\begin{cases} y_0(t,s,x_s) = \int\limits_{s-h}^{s} d_\tau \Gamma(t,s,\tau)\, x(\tau) & (4.23) \\ K(t,s) = \Psi(t,s)\, B(s). & (4.24) \end{cases}$$

Here $x_s \to y_0(t,s,x_s)$ is a function from the space of bounded funtions on $(-h,0]$ to R^{n_1}, and $x_s(\tau) = x(s+\tau)$. It is easily seen that $t \to y_0(t,s,x_s)$ and K are functions of the same type as those in Eq. (4.2), and that $\mathrm{var}_{\tau\in[0,T]}|\Gamma(t,s,\tau)|$ is dominated by an α-L_2 function $g(t)$. For the special case $y = x$, we have $D(t,\tau) = -\mathrm{I}\theta(t-\tau)$, that is:

$$\Psi(t,s) = \Phi(t,s),$$

$$\Gamma(t,s,\tau) = -\Phi(t,s)\,\theta(s-\tau) + \int\limits_{s}^{t} \Phi(t,\sigma)\, A(\sigma,\tau)\, d\sigma.$$

Now, the problem to minimize (4.13) when y is given by Eq. (4.22):

$$y(t) = y_0(t,0,\xi_0) + \int\limits_{0}^{t} K(t,\tau)\, u(\tau)\, d\tau, \tag{4.25}$$

is, putting $s = 0$, precisely of the type discussed in the beginning of this section. The optimal L_2 solution, which we shall denote $u^*(t)$, is provided by Eq. (4.9). However, we are not primarily interested in an "open loop control function" but in a *feedback* solution, that is a function $u^*(t,x_t{}^*)$ where $x_t{}^*$ is the x_t corresponding to u^*. For this reason, we shall return to Eq. (4.5), from which we have (the elements of H_1 and H_2 are now functions on $(0,T]$ and $[0,T]$ respectively):

$$u = -Q_2^{-1} K^* Q_1(y_0 + Ku), \tag{4.26}$$

which together with Eq. (4.4) defines the optimal control u^* on $[0,T]$ as a function of y^* (which is obtained by inserting u^* in Eq. (4.4)):

$$u^*(t) = -Q_2^{-1}(t) \int\limits_{t}^{T} K'(\tau,t)\, Q_1(\tau)\, y^*(\tau)\, d\alpha(\tau) \tag{4.27}$$

We now exploit the fact that Eq. (4.26) only determines u^* a.e., to *define* u^* so that Eq. (4.27) is valid for all $t \in [0,T]$. (The change of u on a null set does not affect y and hence does not effect the right member of Eq. (4.27) neither.) Since Eq. (4.22) is valid for all (t,s) such that $0 \leqslant s < t \leqslant T$, we obtain along the optimal trajectory:

$$y^*(t) = y_0(t,s,x_s{}^*) + \int\limits_{s}^{t} K(t,\tau)\, u^*(\tau)\, d\tau \qquad \text{for } t > s. \tag{4.28}$$

Now, inserting Eq. (4.28) into Eq. (4.27) and changing the order to integration (which is safely done due to Eq. (1.2)), we obtain a family of Fredholm integral equations of the same type as previously encountered:

$$u^*(t) + \int_s^T P(t,\tau)u^*(\tau)\,d\tau = f(t,s), \tag{4.29}$$

where P is defined as in Eq. (4.7) and $f(t,s)$ is given by:

$$f(t,s) = \int_{s-h}^{s} d_\tau \Lambda(t,s,\tau)\,x^*(\tau), \tag{4.30}$$

where:

$$\Lambda(t,s,\tau) = -Q_2^{-1}(t)\int_t^T K'(\sigma,t)\,Q_1(\sigma)\,\Gamma(\sigma,s,\tau)\,d\alpha(\sigma), \tag{4.31}$$

which is obtained from Eq. (4.23) by using the unsymmetric Fubini theorem of Cameron and Martin [28].

Equation (4.29) is valid for *all* t and s such that $0 \leqslant s \leqslant t \leqslant T$. This introduces a certain arbitrariness in the formulation of the optimal control (which is no longer the case in stochastic control). In fact, prescribing that Eq. (4.10) be valid for all t, the solution of Eq. (4.29) is given by (see Eq. (4.9)):

$$u^*(t) = f(t,s) - \int_s^T R(t,\tau,s)f(\tau,s)\,d\tau, \tag{4.32}$$

for *all* t and s such that $s \leqslant t$. (Cf. Ref. 21 in which the theory of integral equations is presented in terms of "everywhere" results. The extra conditions imposed on the L_2 kernels, prescribing that the single integrals be finite, is clearly fulfilled in our case.)

From Eq. (4.10)′ it can be seen that $(t,\tau) \to R(t,\tau,t)$ is an L_2 kernel. (First note that $\|R(\cdot,\cdot,s)\|^2 = \int\int |R(t,\tau,s)|^2\,d\tau\,dt$ is continuous in s and hence bounded on $[0,T]$. In fact, $R(\cdot,\cdot,s+h) - R(\cdot,\cdot,s)$ is a solution of an equation of type (4.6) where f is an L_2 matrix function tending to zero in norm as $h \to 0$. Then use Schwarz's inequality.) Then, putting $s = t$ in Eq. (4.32) and changing the order of integration (the unsymmetric Fubini theorem), we have:

$$u^*(t) = \int_{t-h}^{t} d_s L(t,s)\,x^*(s), \tag{4.33}$$

where we have used Eq. (4.30) to obtain:

$$L(t,s) = \Lambda(t,t,s) - \int_t^T R(t,\tau,t)\,\Lambda(\tau,t,s)\,d\tau. \tag{4.34}$$

Equation (4.33) is a function of type $u^*(t, x_t^*)$, which was requested, but it remains to prove that it is feasible in the sense that there exists a unique solution to Eq. (4.11) when u is given by this function. But this is clearly the case, for it is not hard to see that L is a function of type (1.7) with $m \in L_2(0,T)$ (use the dominated convergence theorem to prove the continuity on the right), and thus $B(t)L(t,s)$ is a function of the same type as $A(t,s)$. Therefore x^* is the solution of a homogeneous linear functional differential equation.

THEOREM 4.1. *There exists an optimal feedback solution of the problem to control Eq.* (4.11) *and Eq.* (4.12) *so as to minimize* (4.13) *and it is given by Eq.* (4.33).

In some practical problems there may be a delay in the observation of the process. For this reason we may need a control law $u^*(t, x_{t-\delta}^*)$, where $\delta > 0$. Such a function is easily obtained from Eq. (4.32) for $t \geq \delta$. In fact, put $s = t - \delta$ and proceed in same way as above:

$$u^*(t) = \int_{t-h-\delta}^{t-\delta} d_s L(t,s,\delta)\, x^*(s), \tag{4.35}$$

where:

$$L(t,s,\delta) = \Lambda(t,t-\delta,s) - \int_{t-\delta}^{T} R(t,\tau,t-\delta)\, \Lambda(\tau,t-\delta,s)\, d\tau. \tag{4.36}$$

Of course, Eqs. (4.33) and (4.35) represent the same function of t $u^*(t)$, but the relevance of Eq. (4.35) will be revealed in Sec. 5. For in the stochastic case, the two situations leading to Eq. (4.33) and Eq. (4.35) represent different information patterns.

Finally, it should be pointed out that, in the feedback solutions (4.33) and (4.35), the dependence on the initial function ξ and the initial time $t = 0$ is accumulated in x. The "feedback gain" L does not depend on these data, as should be clear from the derivation above, and *can be used for any s in the family of performance indices* (4.1) *and for all* x_s.

5. CONTROL OF LINEAR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH COMPLETE STATE INFORMATION

Consider the problem to control the system:

$$\begin{cases} dx(t) = \displaystyle\int_{t-h}^{t} d_s A(t,s)\, x(s)\, dt + B(t)\, u(t)\, dt + C(t)\, dv(t) & \text{for } t \geq 0, \\ x(t) = \xi(t) & \text{for } t \leq 0 \end{cases} \tag{5.1}$$

where $v(t)$ is the independent increment process (1.5), ξ is a deterministic function, and all other functions are defined as in Sec. 1.

Let $\mathcal{U}_0$ be the class of all measurable control processes $u(t)$, which are nonanticipative functions of x such that there exists a unique solution of Eq. (5.1) with the property that $x(t)$ is $\sigma\{v(s); s \in [0,t]\}$-measurable[8]. Furthermore, $\int_0^T E|u(t)|^2 dt < \infty$ should hold.

This is a subclass of $\mathcal{U}$ defined in section 1, and we can immediately give an important class of controls belonging to $\mathcal{U}_0$:

LEMMA 5.1. *Let $\mathcal{L}$ be the affine class of controls*:

$$u(t) = \int_{-h}^{t} d_s \Pi(t,s)\, x(s) + f(t), \tag{5.2}$$

where Π is a function of type (1.7) *with $m \in L_2(0,T)$, except that the second condition need not be fulfilled, and f is an L_2 vector function. Then $\mathcal{L} \subset \mathcal{U}_0$.*

Proof. Inserting Eq. (5.2) in Eq. (5.1) we obtain:

$$dx(t) = \int_{-h}^{t} d_s[A(t,s) + B(t)\Pi(t,s)]x(s)\, dt + B(t)f(t)\, dt + C(t)\, dv(t),$$

for $t \geqslant 0$ and $x(t) = \xi(t)$ for $t \leqslant 0$, which has the solution:

$$x(t) = \bar{x}(t) + \int_{0}^{t} \Psi(t,s)\, C(s)\, dv(s), \tag{5.3}$$

where $\bar{x}(t) = Ex(t)$ and Ψ is the (bounded) transfer function (4.15) corresponding to $A(t,s) + B(t)\Pi(t,s)$, which is a function of type (1.7) (except for the second condition, see remark 1 in the end of this section). Equation (5.3) can immediately be inferred from Eq. (1.8) putting $u = f$ and remembering that ξ is deterministic. Since $E|x(t)|^2$ is bounded, Schwarz's inequality can be used to show that $E|u(t)|^2$ is integrable. This concludes the proof.

The problem is to determine a control $u \in \mathcal{U}_0$ so as to minimize:

$$E\left\{\int_0^T y'(t)\, Q_1(t) y(t)\, d\alpha + \int_0^T u'(t)\, Q_2(t)\, u(t)\, dt\right\}, \tag{5.4}$$

when y is given by:

$$y(t) = \int_{t-h}^{t} d_s D(t,s)\, x(s), \tag{5.5}$$

where Q_1, Q_2 and D are defined in Sec. 1.

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[8] Then $\sigma\{x(s); s \in [0,t]\} \subset \sigma\{v(s); s \in [0,t]\}$ and therefore $u(t)$ is $\sigma\{v(s); s \in [0,t]\}$-measurable.

Denote the x-process obtained when $u \equiv 0$ by x_0, and define $\bar{x}_0(t) = Ex_0(t)$, that is from Eq. (1.8):

$$\bar{x}_0(t) = \Phi(t,0)\,\xi(0) + \int_{-h}^{0} d_\tau \left\{ \int_0^t \Phi(t,s)\,A(s,\tau)\,ds \right\} \xi(\tau), \tag{5.6}$$

and hence Eq. (1.8) can be written:

$$x(t) = \bar{x}_0(t) + \int_0^t \Phi(t,s)\,C(s)\,dv(s) + \int_0^t \Phi(t,s)\,B(s)\,u(s)\,ds. \tag{5.7}$$

Now, inserting Eq. (5.7) into Eq. (5.5) and changing the order of integration (for the last term this is permitted due to Fubini's theorem, for the stochastic integral we need a trivially modified version of the stochastic Fubini theorem previously encountered, Ref. 1, p. 431 or Ref. 2, p. 197), we obtain (use Eq. (4.18)):

$$y(t) = \bar{y}_0(t) + \int_0^t M(t,s)\,dv(s) + \int_0^t K(t,s)\,u(s)\,ds, \tag{5.8}$$

where:

$$\begin{cases} \bar{y}_0(t) = \displaystyle\int_{t-h}^{t} d_\tau\, D(t,\tau)\,\bar{x}_0(\tau) & (5.9) \\ M(t,s) = \displaystyle\int_{t-h}^{t} d_\tau\, D(t,\tau)\,\Phi(\tau,s)\,C(s) & (5.10) \\ K(t,s) = \displaystyle\int_{t-h}^{t} d_\tau\, D(t,\tau)\,\Phi(\tau,s)\,B(s). & (5.11) \end{cases}$$

Then $y(t)$ is precisely the type of process described by Eq. (2.5), for y_0, M and K clearly have the properties prescribed in Sec. 2. In fact, by assumption $\text{var}_{s \in [0,t]}|D(t,s)|$ is dominated by a finite α-L_2 function, $|B|$ and $|C|$ are L_2 functions, and Φ and $\bar{x}_0$ are bounded.

Therefore, for the problem to minimize (5.4) given Eq. (5.8), we can apply the results of Sec. 2 to obtain an optimal control u^* in the SOL-class defined by the observation process $v(t)$. According to Theorem 2.2, this control should be found in the affine class (2.6). Then inserting Eq. (2.6) into Eq. (5.7) and changing the order of integration (as we have previously pointed out, this is permitted), we have:

$$x(t) = \bar{x}(t) + \int_0^t X(t,s)\,dv(s), \tag{5.12}$$

where:

$$\begin{cases} \bar{x}(t) = \bar{x}_0(t) + \int\limits_0^t \Phi(t,s)\, B(s)\, \bar{u}(s)\, ds & (5.13) \\ X(t,s) = \Phi(t,s)\, C(s) + \int\limits_s^t \Phi(t,\tau)\, B(\tau)\, U(\tau,s)\, d\tau. & (5.14) \end{cases}$$

Taking Eq. (5.6) into account, it is clear from Eq. (4.14) with $s = 0$ (which is the solution of Eq. (4.11)) that $\bar{x}$ is given by:

$$\begin{cases} \dot{\bar{x}}(t) = \int\limits_{t-h}^t d_s A(t,s)\, \bar{x}(s) + B(t)\, \bar{u}(t) & \text{for } t > 0 \\ x(t) = \xi(t) & \text{for } t \leqslant 0. \end{cases} \quad (5.15)$$

Unfortunately, Eq. (5.14) is not expressed in the form (4.14). However, since $s \to \Phi(t,s)$ is of bounded variation (see Sec. 4 for references), it has at most a countable number of discontinuities. Then exchanging $\Phi(t,s)C(s)$ for $\Phi(t,s-)C(s)$ in Eqs. (5.10) and (5.14) only involves a null set, and therefore $x(t)$ and $y(t)$ are not affected. But from Eq. (4.15) we have:

$$\Phi(t,s-) = \Phi(t,s) + \int\limits_s^t \Phi(t,\tau)\, [A(\tau,s) - A(\tau,s-)]\, d\tau.$$

Therefore, the *redefined* X-function can be written:

$$X(t,s) = \Phi(t,s)\, C(s) + \int\limits_{s-}^{s} d_\sigma \left\{ \int\limits_s^t \Phi(t,\tau)\, A(\tau,\sigma)\, d\tau \right\} C(\sigma)$$

$$+ \int\limits_s^t \Phi(t,\tau)\, B(\tau)\, U(\tau,s)\, d\tau, \quad (5.16)$$

which is an expression of type (4.14). Then the column vector functions $x_i(t,s)$ clearly satisfy:

$$\begin{cases} \dfrac{\partial x_i}{\partial t}(t,s) = \int\limits_{t-h}^t d_\tau A(t,\tau)\, x_i(\tau,s) + B(t)\, u_i(t,s) & \text{for } t > s \\ x_i(s,s) = c_i & \\ x_i(t,s) = 0 & \text{for } t < s, \end{cases} \quad (5.17)$$

where c_i and u_i are the column vectors corresponding to C and U.

Now we can determine u^* from Theorem 2.1. By inserting Eqs. (5.9)–(5.11) into Eqs. (2.15) and (2.16) and changing the order of integration we obtain:

$$\begin{cases} \bar{y}(t) = \int_{t-h}^{t} d_\tau D(t,\tau)\,\bar{x}(\tau) & (5.18) \\ y_i(t,s) = \int_{t-h}^{t} d_\tau D(t,\tau)\,x_i(\tau,s), & (5.19) \end{cases}$$

where we have used property (4.18). (Really, in Eq. (5.19) we have the x_i-function as defined by Eq. (5.14), but our redefinition does not affect y.) Then the deterministic problems of Theorem 2.1 belong to the class defined by Eqs. (4.11)–(4.13), and according to Theorem 4.1, we have:

$$\begin{cases} \bar{u}^*(t) = \int_{t-h}^{t} d_\tau L(t,\tau)\,\bar{x}^*(\tau) & (5.20) \\ u_i^*(t,s) = \int_{t-h}^{t} d_\tau L(t,\tau)\,x_i^*(\tau,s). & (5.21) \end{cases}$$

The optimal stochastic control is then given by Eq. (2.14):

$$\begin{aligned} u^*(t) &= \bar{u}^*(t) + \int_0^t U^*(t,s)\,dv(s) \\ &= \int_{t-h}^{t} d_\tau L(t,\tau)\left[\bar{x}^*(\tau) + \int_0^t X^*(\tau,s)\,dv(s)\right] \\ &= \int_{t-h}^{t} d_\tau L(t,\tau)\,x^*(\tau), \end{aligned} \tag{5.22}$$

where we have used Eq. (5.12) and the fact that $X(\tau,s) = 0$ for $\tau < s$ (see Eq. (5.17)). Furthermore, we have changed the order of integration in a stochastic integral. A justification of this is given above.

THEOREM 5.1. *The problem to determine an optimal control* $u \in \mathcal{U}_0$ *so as to minimize* (5.4) *when* y *is given by Eqs.* (5.5) *and* (5.1) *has the following solution*:

$$u^*(t) = \int_{t-h}^{t} d_s L(t,s)\,x^*(s), \tag{5.23}$$

where L *is given by Eq.* (4.34).

Proof. Since L is a function of type (1.7) with $m \in L_2(0,T)$ (see Sec. 4), $u^* \in \mathscr{L}$. Then by Lemma 5.1, $u^* \in \mathscr{U}_0$ which is included in the SOL-class employed above, and therefore the proposition follows.

The optimal control (5.23) is clearly identical to the optimal deterministic feedback control (4.33). However, it is clear from the derivation of Eq. (5.23) that we have no longer the arbitrariness in the formulation of the control demonstrated for deterministic controls in Sec. 4. Indeed the formulation of the stochastic control has to be adapted to the information provided by the observation. So if we modify the class of admissible controls of the problem posed above to include only those controls in $\mathscr{U}_0$ which for $t \geqslant \delta$ are non-anticipative functions of $x(t-\delta)$, and for $t < \delta$ are sure functions, we can imbed this new class $\mathscr{U}_\delta$ in the SOL-class of Corollary 2.2. Then all results of this section up to Eq. (5.20) are still valid with the extra condition that $U(t,s) = 0$ for $s > t - \delta$. Equation (5.21) is still valid for $t \geqslant s + \delta$ (for problem B of Theorem 2.1 is essentially modified so that the lower limit of integration s is changed for $s + \delta$), but Eqs. (5.20) and (5.21) cannot be used to synthesize a control in $\mathscr{U}_\delta$. However, for $t \geqslant \delta$ we can use the equivalent deterministic control (4.35) to obtain:

$$\begin{cases} \bar{u}^*(t) = \int_{t-h-\delta}^{t-\delta} d_\tau L(t,\tau,\delta)\bar{x}^*(\tau) \\ u_t^*(t,s) = \int_{t-h-\delta}^{t-\delta} d_\tau L(t,\tau,\delta)\, x_t^*(\tau,s). \end{cases}$$

In fact, proceeding in the same way as above we obtain:

$$\begin{aligned} u^*(t) &= \bar{u}^*(t) + \int_0^{t-\delta} U^*(t,s)\, dv(s) \\ &= \int_{t-h-\delta}^{t-\delta} d_\tau L(t,\tau,\delta)\left[\bar{x}^*(\tau) + \int_0^{t-\delta} X^*(\tau,s)\, dv(s)\right] \\ &= \int_{t-h-\delta}^{t-\delta} d_\tau L(t,\tau,\delta)\, x^*(\tau) \qquad \text{for } t \geqslant \delta \end{aligned} \tag{5.24}$$

THEOREM 5.2. *The problem to determine a* $u \in \mathscr{U}_\delta$ *so as to minimize* (5.4) *given Eqs.* (5.5) *and* (5.1) *has the optimal solution* (5.24), *for* $t \geqslant \delta$. *For* $t < \delta$, $u^*(t) = \bar{u}^*(t)$.

Proof. This follows immediately from the fact that $u^* \in \mathscr{L} \cap \mathscr{U}_\delta \subset \mathscr{U}_\delta$ which is included in the SOL-class of Corollary 2.2. For $t < \delta$, the last term in Eq. (2.15) is zero.

The problem of Theorem 5.2 clearly corresponds to an important practical situation, namely the one obtained when there is a delay in the reception of the observation.

Finally, we shall settle the question of existence and uniqueness for the *open-loop* system discussed above (ξ is now stochastic): Equation (1.6) should be interpreted in the following way:

$$\begin{cases} x(t) = \xi(0) + \int_0^t \int_{\tau-h}^{\tau} d_s A(\tau,s)\, x(s)\, d\tau + \int_0^t B(s)\, u(s)\, ds \\ \qquad + \int_0^t C(s)\, dv(s) \qquad \text{for } t \geqslant 0 \\ x(t) = \xi(t) \qquad \text{for } t \leqslant 0, \end{cases} \tag{5.25}$$

where the last integral is defined in q.m. (A measurable version is considered.) A *solution* of Eq. (1.6) is a measurable stochastic process such that the integrals of Eq. (5.25) exist for almost all sample functions and such that for each $t \in [-h,T]$ Eq. (5.25) holds a.s. That is, the left side and the right side of Eq. (5.25) constitute equivalent stochastic processes. We shall make no distinction between equivalent solutions. As we have pointed out above, we are interested in solutions with bounded second-order moments.

LEMMA 5.2. *There is a unique (stochastic) solution of*:

$$\begin{cases} x(t) = \int_0^t \int_{\tau-h}^{\tau} d_s A(\tau,s)\, x(s)\, d\tau \qquad \text{for } t \geqslant 0 \\ x(t) = 0 \qquad \text{for } t \leqslant 0, \end{cases} \tag{5.26}$$

with bounded second-order moments, and it is $y(t) = 0$.

Proof. Let $y(t)$ be an arbitrary solution of Eq. (5.26) in the sense described above, and for each sample function for which the integrals are defined (almost all; see Sec. 1), define $\eta(t,\omega)$ and $\bar{\eta}(t,\omega)$ to be:

$$\eta(t,\omega) = \int_0^t \int_{\tau-h}^{\tau} d_s A(\tau,s)\, y(s,\omega)\, d\tau, \tag{5.27}$$

$$\bar{\eta}(t,\omega) = \int_0^t \int_{\tau-h}^{\tau} d_s A(\tau,s)\, \eta(s,\omega)\, d\tau, \tag{5.28}$$

for $t \geqslant 0$ and identically zero for $t \leqslant 0$. Then, since y is a solution of Eq. (5.26), y and η are equivalent, and so are η and $\bar{\eta}$ for:

$$E|\eta(t) - \bar{\eta}(t)| \leqslant \int_0^t \int_{\tau-h}^{\tau} |d_s A(\tau, s)| E|y(s) - \eta(s)| d\tau = 0.$$

But, according to Eqs. (5.27) and (5.28), η and $\bar{\eta}$ have continuous sample functions and therefore they must be identical. In fact, due to the equivalence, $\eta(t, \omega) = \bar{\eta}(t, \omega)$ for all rational t except for a countable union of ω-null sets. Then, except for this null set, by the continuity, $\eta(t, \omega) = \bar{\eta}(t, \omega)$ for all other t as well. Therefore for almost all ω, $t \to \eta(t, \omega)$ is a "deterministic" solution of Eq. (5.26) and must be identically zero (see Sec. 4). The solution $y(t)$ is therefore equivalent to the zero function. This concludes the proof of the lemma.

THEOREM 5.3. *With u and ξ defined as in Sec.* 1, *there exists a unique solution with bounded second-order moments of the linear stochastic functional differential equation* (1.6), *and this solution is provided by Eq.* (1.8).

Proof. If there exist two solutions of Eq. (5.25) with bounded second-order moments, the difference between them must be a solution of Eq. (5.26) and is therefore, by Lemma 5.2 (equivalent to) zero. Therefore, a possible solution must be unique.

It remains to be shown that:

$$\begin{cases} x(t) = \Phi(t,0)\,\xi(0) + \displaystyle\int_0^t \Phi(t,s) \int_{-h}^0 d_\beta A(s,\beta)\,\xi(\beta)\,ds \\ \qquad + \displaystyle\int_0^t \Phi(t,s)B(s)\,u(s)\,ds + \int_0^t \Phi(t,s)\,C(s)\,dv(s) \qquad \text{for } t \geqslant 0 \\ x(t) = \xi(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \leqslant 0, \end{cases} \tag{5.29}$$

satisfies Eq. (1.6) = Eq. (5.25) in the sense described above. Equation (5.29) is precisely Eq. (1.8) where we have changed the order of integration in the second term. This is permitted according to the unsymmetric Fubini theorem [28] provided that:

$$I(\omega) = \int_0^t |\Phi(t,s)| \int_{-h}^0 |d_\tau A(s,\tau)| |\xi(\tau,\omega)| ds < \infty. \tag{5.30}$$

Since $E|\xi(\tau)|$ is bounded (for $E|\xi(\tau)|^2$ is bounded), Eq. (5.30) is true with $|\xi|$ exchanged for $E|\xi|$. Then, by Fubini's theorem, $EI < \infty$ and therefore

Eq. (5.30) is valid almost surely. Now, with x given by Eq. (5.29) we have a.s.:

$$\int_0^t d_\sigma A(t,\sigma)\, x(\sigma) = \int_0^t d_\sigma A(t,\sigma)\, \Phi(\sigma,0)\, \xi(0)$$

$$+ \int_0^t \int_s^t d_\sigma A(t,\sigma)\, \Phi(\sigma,s) \int_{-h}^0 d_\beta A(s,\beta)\, \xi(\beta)\, ds$$

$$+ \int_0^t \int_s^t d_\sigma A(t,\sigma)\, \Phi(\sigma,s)\, B(s)\, u(s)\, ds$$

$$+ \int_0^t \int_s^t d_\sigma A(t,\sigma)\, \Phi(\sigma,s)\, C(s)\, dv(s).$$

Here we have used the usual Fubini theorem and, in the last term, a modified version of the stochastic Fubini theorem. (The theorems of Ref. 1, p. 431 and Ref. 2, p. 197, which give almost sure results, are easily seen to be valid when the Lebesgue integral is exchanged for a Stieltjes integral.) Again applying these theorems and Eq. (4.16) we have a.s.:

$$\xi(0) + \int_0^t \int_{-h}^{\tau} d_\sigma A(\tau,\sigma)\, x(\sigma)\, d\tau + \int_0^t B(s)\, u(s)\, ds + \int_0^t C(s)\, dv(s)$$

$$= \left[I + \int_0^t \frac{\partial \Phi}{\partial \tau}(\tau,0)\, d\tau \right] \xi(0) + \int_0^t \left[I + \int_s^t \frac{\partial \Phi}{\partial \tau}(\tau,s)\, d\tau \right] \int_{-h}^0 d_\beta A(s,\beta)\, \xi(\beta)\, ds$$

$$+ \int_0^t \left[I + \int_s^t \frac{\partial \Phi}{\partial \tau}(\tau,s)\, d\tau \right] B(s)\, u(s)\, ds + \int_0^t \left[I + \int_s^t \frac{\partial \Phi}{\partial \tau}(\tau,s)\, d\tau \right] C(s)\, dv(s),$$

which is equal to Eq. (5.29) because of:

$$\Phi(t,s) = I + \int_s^t \frac{\partial \Phi}{\partial \tau}(\tau,s)\, d\tau.$$

Here we have used the fact that indefinite integrals of equivalent processes form equivalent processes, which can immediately be inferred from Fubini's theorem. Since $E|x(t)|^2$ is bounded (to see this, use Schwarz's inequality as indicated in Sec. 1), Eq. (5.29) clearly constitutes the unique solution of Eq. (1.6). This concludes the proof.

Remark 1. It should be noted that the second condition in Eq. (1.7) is in no way crucial. Indeed, we can put $h > T$ to see that the theorem holds for intervals of integration like $(0, t]$ in the Stieltjes integral of Eq. (1.6).

Remark 2. Regarding ξ and u as deterministic functions and putting $v = 0$ in the proof of Theorem 5.3, it is clear that ξ need not be bounded in order that Eq. (5.13) be the solution, of Eq. (5.15). It suffices that the unsymmetric Fubini theorem applies to the last term in Eq. (5.6). This is obviously the case for almost all sample functions of $\xi(t, \omega)$.

Remark 3. Really, Theorem 5.3 states that there is a unique equivalence class of solutions of Eq. (5.25), for it is clear from the proof of Lemma 5.2 that if $x(t)$ is a solution, any (measurable) process equivalent to $x(t)$ is also a solution. We have been forced to consider equivalence classes rather than individual processes (they have the same finite-dimensional distributions) because of the intrinsic lack of uniqueness in the definition of the stochastic integrals. However, if $v(t)$ is a Wiener process, these integrals have a.s. sample continuous versions, and there is a unique sample continuous solution of Eq. (5.25) such that the two members of Eq. (5.25) are equal (not merely equivalent), for two equivalent sample continuous processes are equal a.s. Therefore, in Secs. 3 and 6 we consider this kind of solutions. If in addition, as is the case in Sec. 6, $\xi(t)$ is Gaussian, independent of v, and almost surely sample continuous, the process $x_0(t)$, obtained by putting $u \equiv 0$ in Eq. (1.8), is Gaussian and a.s. sample continuous. (See paragraph 4 in the appendix.)

6. CONTROL OF LINEAR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INCOMPLETE STATE INFORMATION

Consider the systems of stochastic functional differential equations defined in Sec. 1:

$$\begin{cases} dx(t) = \displaystyle\int_{t-h}^{t} d_s A(t,s)\, x(s)\, dt + B(t)\, u(t)\, dt + C(t)\, dv(t) & \text{for } t \geqslant 0 \qquad (6.1) \\ x(t) = \xi(t) & t \leqslant 0 \\ dz(t) = \displaystyle\int_{0}^{t} d_s H(t,s)\, x(s)\, dt + dw(t) & \text{for } t \geqslant 0 \qquad (6.2) \\ z(0) = 0. \end{cases}$$

Here ξ is Gaussian, $v(t)$ and $w(t)$ are standard Wiener processes, and moreover these processes have the properties prescribed in point B Sec. 1, where also a class $\mathscr{U}$ of admissible controls based on the observation process z was defined. The problem is to determine a $u \in \mathscr{U}$ so as to minimize:

$$E\left\{\int_0^T y'(t)\, Q_1(t)\, y(t)\, d\alpha(t) + \int_0^T u'(t)\, Q_2(t)\, u(t)\, dt\right\}, \tag{6.3}$$

when y is given by:

$$y(t) = \int_{t-h}^{t} d_s\, D(t,s)\, x(s). \tag{6.4}$$

A more general problem of this type was described in Sec. 3, and we shall make use of the results obtained there. To this end, we shall first assure ourselves that the conditions imposed in Sec. 3 are fulfilled in the present problem. As in Sec. 3, uncontrolled versions ($u = 0$) of processes normally affected by the control will be indexed with a zero. The solution of Eq. (6.1) is given by Eq. (1.8), that is:

$$x(t) = x_0(t) + \int_0^t \Phi(t,s)\, B(s)\, u(s)\, ds, \tag{6.5}$$

where:

$$x_0(t) = \Phi(t,0)\, \xi(0) + \int_{-h}^{0} d_\tau\left\{\int_0^t \Phi(t,s)\, A(s,\tau)\, ds\right\}\xi(\tau) + \int_0^t \Phi(t,s)\, C(s)\, dv(s), \tag{6.6}$$

which is a Gaussian process for which almost all sample functions are continuous (see Remark 3 in Sec. 5). By point 4 in the appendix, all zero-indexed processes in the sequel will be jointly Gaussian. The controlled versions are *a priori* non-Gaussian, for the control law might be a nonlinear function. (However, it so happens that this is not the case for the optimal solution.) Then inserting Eq. (6.5) into Eq. (6.4) and changing the order of integration (Fubini), we have:

$$y(t) = y_0(t) + \int_0^t K(t,s)\, u(s)\, ds, \tag{6.7}$$

where:

$$\begin{cases} y_0(t) = \displaystyle\int_{t-h}^{t} d_\tau\, D(t,\tau)\, x_0(\tau) & (6.8) \\[2ex] K(t,s) = \displaystyle\int_{t-h}^{t} d_\tau\, D(t,\tau)\, \Phi(\tau,s)\, B(s), & (6.9) \end{cases}$$

which is precisely an expression of type (3.1), and it is readily seen that y_0 and K fulfil the proper conditions. (For y_0, note that $E|x_0(t)|^2$ is bounded. Then, use Schwarz's inequality and the properties of D. For K, see Sec. 5.)

The process q of Eq. (3.9) is:

$$q(t) = \int_0^t d_s H(t,s)\, x(s), \tag{6.10}$$

and by inserting Eq. (6.5) in this expression and changing the order of integration we obtain:

$$q(t) = q_0(t) + \int_0^t \left[\int_s^t d_\tau H(t,\tau)\, \Phi(\tau,s) \right] B(s)\, u(s) ds, \tag{6.11}$$

which is the prescribed linear feedback loop (Eq. 3.10). That the integrability conditions of q and q_0 are fulfilled can be seen in exactly the same way as for Eq. (6.8). Finally, to ascertain that $\mathscr{L} \subset \mathscr{U}_0$, we must check whether the condition on the functional relationship between q and y imposed in Sec. 3 is fulfilled. By inserting Eq. (6.2) into Eq. (3.13) and using the unsymmetric Fubini theorem [28] to change the order of integration, we obtain:

$$u(t) = f(t) + \int_0^t F(t,s) \int_0^s d_\tau H(s,\tau)\, x(\tau)\, ds + \int_0^t F(t,s)\, dw(s)$$

$$= f(t) + \int_0^t d_\tau \left\{ \int_\tau^t F(t,s)\, H(s,\tau)\, ds \right\} x(\tau) + \int_0^t F(t,s)\, dw(s). \tag{6.12}$$

Now, since $B(t) \int_\tau^t F(t,s) H(s,\tau) ds$ is a function of type (1.7) (except for the second condition) with $m \in L_1(0,T)$ (the third condition is obtained with the aid of the dominated convergence theorem and the last condition directly from the definition), it is clear from Theorem 5.3 (slightly modified) that Eq. (6.1) with Eq. (6.12) inserted has a unique solution.

In Sec. 3 we defined the class $\mathscr{U}_0 \subset \mathscr{U}$ of all measurable control processes u for which $\int E|u|^2 dt < \infty$ and which are nonanticipative functions of z such that $u(t)$ is $\sigma\{z_0(s); s \in [0,t]\}$-measurable for each t. This was done to enable the imbedding of the class of admissible controls in the SOL-class of the innovation process, which in the present problem is given by:

$$d\nu(t) = dz(t) - \int_0^t d_s H(t,s)\, \hat{x}(s\,|\,t)\, dt, \tag{6.13}$$

where:

$$\hat{x}(s\,|\,t) = E\{x(s)|z(\tau); \tau \in [0,t]\}, \tag{6.14}$$

and which is the same for all $u \in \mathscr{U}_0$. For these u it generates the same family of σ-algebras as z (Lemma 3.2), and for this reason we can equally well condition with respect to ν when determining expressions of type (6.14), which is an advantage since ν is a Wiener process. Since x_0 and ν are jointly Gaussian and $u(t)$ is $\sigma\{\nu(\tau); \tau \in [0,t]\}$-measurable we obtain:

$$\hat{x}(s\,|\,t) = \bar{x}_0(s) + \int_0^t G(s,\tau)\,d\nu(\tau) + \int_0^s \Phi(s,\tau)\,B(\tau)\,u(\tau)\,d\tau, \tag{6.15}$$

where, according to Lemma 2.1, $\bar{x}_0(s) = Ex_0(s)$ and :

$$G(s,\tau) = \frac{\partial}{\partial\tau} E\{x_0(s)\nu'(\tau)\} = E\{x_0(s)\tilde{q}_0{}'(\tau|\tau)\}. \tag{6.16}$$

Here we have made use of the independence between x_0 and w (due to independence between ξ, v and w) in the same way as in Theorem. 3.3. ($\tilde{q}_0$ is defined by (3.24).)

In order to determine $G(t,s)$ we shall need the following lemma, which is an immediate consequence of Theorem 5.3:

LEMMA 6.1. *For all t and s such that $0 \leqslant s \leqslant t$, we have*:

$$x_0(t) = \Phi(t,s)\,x_0(s) + \int_{s-h}^{s} d_\tau\left\{\int_s^t \Phi(t,\sigma)\,A(\sigma,\tau)\,d\sigma\right\}x_0(\tau)$$

$$+ \int_s^t \Phi(t,\tau)\,C(\tau)\,dv(\tau). \tag{6.17}$$

Since $\tilde{q}_0(s|s)$ is an affine function of ξ, w and $\{v(\tau); \tau \in [0,s]\}$ (for all estimates are linear when $u \in \mathscr{U}_0$), $\tilde{q}_0(s|s)$ and the last term of Eq. (6.17) are independent, and therefore for $t \geqslant s$ we have the following equation for $G(t,s) = E\{x_0(t)\tilde{q}_0{}'(s\,|\,s)\}$:

$$G(t,s) = \Phi(t,s)\,G(s,s) + \int_{s-h}^{s} d_\tau\left\{\int_s^t \Phi(t,\sigma)\,A(\sigma,\tau)\,d\sigma\right\}G(\tau,s). \tag{6.18}$$

Now, according to Theorem 3.3, the optimal control in $\mathscr{U}_0$ should be found in the affine class:

$$u(t) = \bar{u}(t) + \int_0^t U(t,s)\,d\nu(s) \tag{6.19}$$

Inserting Eq. (6.19) into Eq. (6.15) and changing the order of integration (permitted according to Ref. 1, p. 431 or Ref. 1, p. 197), we obtain:

$$\hat{x}(s\,|\,t) = \bar{x}(s) + \int_0^t X(s,\tau)\,d\nu(\tau), \tag{6.20}$$

where:

$$\begin{cases} \bar{x}(s) = \bar{x}_0(s) + \int_0^s \Phi(s,\tau)\, B(\tau)\, \bar{u}(\tau)\, d\tau & & (6.21) \\ X(s,\tau) = G(s,\tau) + \int_\tau^s \Phi(s,\sigma)\, B(\sigma)\, U(\sigma,\tau)\, d\sigma & \text{for } s \geqslant \tau & (6.22) \\ X(s,\tau) = G(s,\tau) & \text{for } s \leqslant \tau. & (6.23) \end{cases}$$

By inserting Eq. (6.18) into Eq. (6.22) we obtain an expression of type (4.14), and it is then clear that the column vectors x_i of X satisfy:

$$\begin{cases} \frac{\partial x_i}{\partial t}(t,s) = \int_{t-h}^t d_\tau A(t,\tau)\, x_i(\tau,s) + B(t)\, u_i(t,s) & \text{for } t > s \qquad (6.24) \\ x_i(t,s) = g_i(t,s) & \text{for } t \leqslant s, \end{cases}$$

where g_i are the column vectors of G. In the same way:

$$\begin{cases} \dot{\bar{x}}(t) = \int_{t-h}^t d_s A(t,s)\, \bar{x}(s) + B(t)\, \bar{u}(t) & \text{for } t > 0 \\ \bar{x}(t) = a(t) & \text{for } t \leqslant 0, \qquad (6.25) \end{cases}$$

which is obtained from Eqs. (6.21) and (6.6) ($\bar{x}_0 = Ex_0$).

The optimal control of Theorem 3.3 and Theorem 3.1 can now be determined from Theorem 2.1. To this end, first note that the column vectors m_i of M are given by:

$$m_i(t,s) = \int_{t-h}^t d_\tau D(t,\tau)\, g_i(\tau,s). \tag{6.26}$$

This is an immediate consequence of Eqs. (6.8), (3.23), and (6.16). In the same way:

$$\bar{y}_0(t) = \int_{t-h}^t d_\tau\, D(t,\tau)\, \bar{x}_0(\tau). \tag{6.27}$$

Now, insertion of Eqs. (6.26), (6.27), and (6.9) into Eqs. (2.15) and (2.16) yields after changing the order of integration:

$$\begin{cases} \bar{y}(t) = \int_{t-h}^t d_\tau\, D(t,\tau)\, \bar{x}(\tau) & & (6.28) \\ y_i(t,s) = \int_{t-h}^t d_\tau\, D(t,\tau)\, x_i(\tau,s) & \text{for } t \geqslant s, & (6.29) \end{cases}$$

where we have used Eqs. (6.21) and (6.22) and also the property (4.18). Therefore the deterministic problems of Theorem 2.1 are of the type described in Theorem 4.1 and have the optimal solutions:

$$\begin{cases} \bar{u}^*(t) = \int_{t-h}^{t} d_\tau L(t,\tau)\,\bar{x}^*(\tau) & (6.30) \\ u_i^*(t,s) = \int_{t-h}^{t} d_\tau L(t,\tau)\,x_i^*(\tau,s). & (6.31) \end{cases}$$

The optimal stochastic control is then given by Eq. (2.14):

$$\begin{aligned} u^*(t) &= \bar{u}^*(t) + \int_0^t U^*(t,s)\,d\nu(s) \\ &= \int_{t-h}^{t} d_\tau L(t,\tau)\,[\bar{x}^*(\tau) + \int_0^t X^*(\tau,s)\,d\nu(s)] \\ &= \int_{t-h}^{t} d_\tau L(t,\tau)\,\hat{x}^*(\tau\,|\,t) \end{aligned} \tag{6.32}$$

where Eq. (6.20) has been used.

Finally, we shall only briefly discuss the calculation of $\hat{x}(s\,|\,t)$. For $s \leqslant t$, Eq. (6.20) yields the smoothing estimate:

$$\begin{aligned} \hat{x}(s\,|\,t) &= \bar{x}(s) + \int_0^s X(s,\tau)\,d\nu(\tau) + \int_s^t X(s,\tau)\,d\nu(\tau) \\ &= \hat{x}(s\,|\,s) + \int_s^t G(s,\tau)\,d\nu(\tau). \end{aligned} \tag{6.33}$$

This equation is valid also for $s < 0$ provided that we define $G(s,\tau)$ for $\tau < 0$ to be zero and define $\hat{x}(s\,|\,s) = \bar{x}(s) = a(s)$ for $s \leqslant 0$. A stochastic differential equation for the filtering estimate can be determined by means of Eqs. (6.20) and (6.24):

$$\begin{aligned} \hat{x}(t\,|\,t) - \bar{x}(t) &= \int_0^t [X(s,s) + \int_s^t \frac{\partial X}{\partial \tau}(\tau,s)\,d\tau]d\nu(s) \\ &= \int_0^t G(s,s)\,d\nu(s) + \int_0^t \int_0^\tau \frac{\partial X}{\partial \tau}(\tau,s)\,d\nu(s)\,d\tau. \end{aligned} \tag{6.34}$$

The last term in Eq. (6.34) can be written (using Eq. (6.24)):

$$\int_0^t \left[\int_{\tau-h}^{\tau} d_\sigma A(\tau,\sigma) \int_0^{\tau} X(\sigma,s)\, d\nu(s) + B(\tau) \int_0^{\tau} U(\tau,s)\, d\nu(s) \right] d\tau$$

$$= \int_0^t \left\{ \int_{\tau-h}^{\tau} d_\sigma A(\tau,\sigma) \left[\hat{x}(\sigma|\tau) - \bar{x}(\sigma)\right] + B(\tau) \left[u(\tau) - \bar{u}(\tau)\right] \right\} d\tau. \tag{6.35}$$

Therefore Eqs. (6.33)–(6.35) together with Eq. (6.25) yield the following equations (with obvious interpretations) of the filtering and smoothing estimates:

$$\begin{cases} d\hat{x}(t|t) = \int_{t-h}^{t} d_\sigma A(t,\sigma)\, \hat{x}(\sigma|t)\, dt + B(t) u(t)\, dt + G(t,t)\, d\nu(t) & \text{for } t \geqslant 0 \quad (6.36) \\ d_t \hat{x}(s|t) = G(s,t)\, d\nu(t) & \text{for } s \leqslant t. \quad (6.37) \end{cases}$$

Note that we have changed the order of integration in stochastic integrals on several occasions without saying so. Justifications for this are given in Sec. 5.

It is easily seen from Eq. (6.16) and the fact that $\tilde{x}_0(s|t) = x_0(s) - \hat{x}_0(s|t)$ is orthogonal to $\hat{x}(s|t)$ that the gain matrix G is given by:

$$G(t,s) = \int_0^s Q(t,\tau,s)\, d_\tau H'(s,\tau) \qquad \text{for } s \geqslant 0$$

$$G(t,s) = 0 \qquad \text{for } s \leqslant 0, \tag{6.38}$$

where Q is the "error covariance matrix":

$$Q(t,\tau,s) = E\{\tilde{x}_0(t|s)\, \tilde{x}_0{}'(\tau|s)\}. \tag{6.39}$$

THEOREM 6.1 (*Separation Theorem*). *The problem to determine* $u \in \mathcal{U}_0$ *so as to minimize* (6.3) *when* y *is given by Eqs.* (6.4) *and* (6.1) *has the following solution:*

$$u^*(t) = \int_{t-h}^{t} d_s L(t,s)\, \hat{x}^*(s|t),$$

where L *is the feedback gain matrix of the corresponding deterministic problem* (*Sec.* 4), *and* $\hat{x}(s|t)$ *is given by Eqs.* (6.36) *and* (6.37). *The estimation problem and the deterministic control problem can be solved independently.*

An alternative approach to the estimation problem based on the solution of a dual control problem is described in Ref. 26.

APPENDIX

1. We assume an underlying complete probability space $(\Omega, \mathfrak{S}, P)$, where Ω is the sample space with elements ω, $\mathfrak{S}$ is a σ-algebra of events with respect to which all random variables in our paper are measurable, and P is the probability measure. Let $\mathfrak{B}$ be the σ-algebra of Borel sets on $[0,T]$ and let $\mathfrak{B} \times \mathfrak{S}$ denote the σ-algebra generated by the class of all sets of the form $A \times B$ where $A \in \mathfrak{B}$ and $B \in \mathfrak{S}$. The stochastic process $x(t)$ is *measurable* if the function $x(t,\omega)$ is $\mathfrak{B} \times \mathfrak{S}$-measurable. All deterministic functions defined in this paper are $\mathfrak{B}$- or $\mathfrak{B} \times \mathfrak{B}$-measurable functions. Therefore, by Fubini's theorem, a stochastic process of the form $\int K(t,s)x(s,\omega)ds$ is measurable, and the same is true for processes of the type $\int d_s A(t,s)x(s,\omega)$ appearing in this paper (see Lemma 1.7 in Ref. 29, p. 9).

2. In the following lemma we modify the results given in Ref. 1, p. 430 and Ref. 2, p. 196 to our concept of measurability:

LEMMA: Let $v(t)$ be a process with orthogonal increments described by Eq. (1.5), and let $M(t,s)$ be a $\mathfrak{B} \times \mathfrak{B}$-measurable function such that $\int |M(t,s)|^2 ds < \infty$ for almost all t. Then the stochastic integral:

$$y(t) = \int M(t,s)\, dv(s),$$

can be defined so that $y(t)$ is a measurable process.

Proof: Let $M_n(t,s)$ be a sequence of functions such that:

$$M_n(t,s) = \sum_{i=1}^{n} \Psi_i(t)\, \Phi_i(s),$$

where Ψ_i is $\mathfrak{B}$-measurable, $\int |\Phi_i(s)|^2 ds < \infty$, and:

$$\int |M_n(t,s) - M(t,s)|^2 ds \to 0, \tag{A.1}$$

for almost all t when $n \to \infty$. [Such a sequence can be constructed by expanding M as a Fourier series $\sum \langle M(t,\cdot), \Phi_i \rangle \Phi_i(s)$.] Then, the processes:

$$y_n(t) = \int M_n(t,s)\, dv(s) = \sum_i \Psi_i(t) \int \Phi_i(s)\, dv(s),$$

are $(\mathfrak{B} \times \mathfrak{S})$-measurable and, in view of Eq. (A.1):

$$E|y_n(t) - y(t)|^2 \to 0.$$

Therefore, for each t, there is subsequence such that $y(t) = \lim y_{n_k}(t)$ almost surely. Hence $\tilde{y}(t) = \overline{\lim}\, y_{n_k}(t)$, which is measurable (see for instance Ref. 30, p. 153), is equivalent to $y(t)$. This concludes the proof.

3. We will show that:

$$U(t,s) = \frac{\partial}{\partial s} E\{u(t)\, v'(s)\},$$

which for each t (almost) exists for almost all s, can be redefined on this exceptional set to be $\mathfrak{B} \times \mathfrak{B}$-measurable. Since $E\{u(t) v'(s)\}$ is $\mathfrak{B} \times \mathfrak{B}$-measurable (Fubini):

$$U_n(t,s) = n\left[E\left\{u(t)\, v'\left(s + \frac{1}{n}\right)\right\} - E\{u(t)\, v'(s)\}\right],$$

are $\mathfrak{B} \times \mathfrak{B}$-measurable, and this sequence tends pointwise to $U(t,s)$ except on the set described above. Then, define $U(t,s)$ to be $\overline{\lim}\ U_n(t,s)$.

4. Consider an indefinite integral $\int_0^t q(s) ds$ of a Gaussian process $q(t)$. If this integral can be defined as an almost sure limit or a limit in q.m. of a Riemann approximation sum, it constitutes a Gaussian process. (However, for more general conditions, see Ref. 31). In fact, a Riemann sum is a finite linear combination of jointly distributed Gaussian vectors $q(t_i)$ and then the limit is Gaussian too (cf Ref. 2, p. 17), for a.s. and q.m. convergence imply weak convergence. By a similar argument it is seen that all finite dimensional distributions are Gaussian, and if the processes y and q have joint Gaussian distributions, the same is true for y and $\int q\, ds$. In Sec. 6, Stieltjes integrals of an a.s. sample continuous Gaussian process $x_0(t)$ are considered. These integrals can be regarded as Riemann-Stieltjes integrals which are a.s. limits of Riemann sums and, in virtue of the previous discussion, they are jointly Gaussian. Thus $y_0(t)$ and $q_0(t)$ defined in Sec. 6 are Gaussian, and the same is true for

$$\int_0^t q_0(s)\, ds = \int_0^t d_s\left\{\int_s^t H(\tau,s)\, d\tau\right\} x_0(s),$$

where we have used the unsymmetric Fubini theorem.

Finally, all stochastic integrals with Wiener measure are Gaussian, for by definition they are q.m. limits of Gaussian sequences.

5. Consider a process of type (3.11):

$$z(t) = \int_0^t q(s)\, ds + w(t).$$

The closed linear hull of $\{z_i(s); s \in [0,t], i = 1,2\ldots n_5; 1\}$ in the usual Hilbert space of second-order stochastic variables consists of integrals of type:

$$\bar{\xi} + \int_0^t f'(s)\, dz(s),$$

where $\bar{\xi}$ is a constant and f an L_2 vector function. In fact, if f_n is a step function with values c_i on $[t_{i-1}, t_i)$, then

$$\sum c_i[z(t_i) - z(t_{i-1})] = \sum c_i \int_{t_{i-1}}^{t_i} q(s)ds + \sum c_i \, [w(t_i) - w(t_{i-1})]$$

$$= \int_0^t f_n(s)\, q(s)\, ds + \int_0^t f_n(s)\, dw(s). \tag{A.2}$$

Then if we have a sequence of step functions such that: $\int |f_n - f_m|^2 ds \to 0$, defining a limit function f, the second term in Eq. (A.2) tends to $\int f dw$ in q.m. by definition and the first term tends to $\int fq\,ds$. In fact, by Schwarz's inequality:

$$E| \int_0^t [f_n(s) - f_m(s)]\, q(s)\, ds^2 \leqslant \int |f_n - f_m|^2 ds\, E \int |q|^2 ds,$$

which tends to zero when $\int E|q|^2 ds < \infty$.

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ADDENDUM TO "OPTIMAL CONTROL OF LINEAR STOCHASTIC SYSTEMS WITH APPLICATIONS TO TIME LAG SYSTEMS" BY ANDERS LINDQUIST.

In Sec. 2 and 5 of the above mentioned paper the stochastic process $v(t)$ was assumed to have independent stationary increments. However it is sufficient