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ON FEEDBACK CONTROL OF LINEAR STOCHASTIC SYSTEMS*

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Abstract. Feedback control of linear continuous-time stochastic systems of general type is discussed. Various types of (classical) information patterns with both complete and partial observations (white and colored measurement noise) are considered. The cost functional is quadratic. A class of admissible control laws is defined which includes all linear and nonlinear control policies for which our problem makes sense, i.e., existence, uniqueness etc. are secured. Then, we determine the optimal control law by an imbedding procedure which amounts to solving a problem without a feedback loop. We investigate under what conditions the optimal control law is linear in the data.

1. Introduction. In recent years there has been a considerable interest in feedback control of linear continuous-time stochastic systems. However, as pointed out by Witsenhausen [20], the difficulties created by the feedback loop have frequently been overlooked, and therefore many results have appeared which as yet have not been rigorously justified. On the other hand, as one might expect, many rigorous proofs suffer from undesired technical restrictions.

The most well-known problem of this type is the stochastic linear-quadratic regulator problem with noisy measurements, for which various versions of the "separation theorem" hold. These versions usually differ in the way in which the set of admissible control laws is defined. By confining ourselves to control laws which are linear in the data, we can easily avoid the difficulties mentioned above. However, we usually want to compare them with nonlinear control laws even when such a comparison rules in favor of a linear one. To the author's knowledge the first fully rigorous proof along these lines appeared in the book [16] by Kushner, where only control laws satisfying a uniform Lipschitz condition in a certain state estimate are admitted. The state estimate is assumed to be generated by a linear Kalman filter to which the nonlinear feedback loop is added, but it is shown that this estimate is indeed the expected value of the current state given past observations, as long as we confine ourselves to admissible control laws. In the well-known paper [21] by Wonham the class of admissible control laws is defined in a more straightforward way, first excluding the possibility that the "information" carried by the observation process is control-dependent by requiring the control to be Lipschitz in this process. Then, the admissible control laws are defined to be Lipschitz continuous functions of the conditional expectation of the state given past observations. Moreover, the separation theorem is generalized to hold for nonquadratic cost functionals. (Also see [22].)

Of course the Lipschitz conditions are imposed to insure that there exist *unique* solutions of the feedback equations. Otherwise the problem would not

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make sense. Unfortunately, these conditions exclude many control laws which are not sufficiently smooth but for other reasons are natural to admit. (See § 3.) In order to get rid of these technical restrictions, Davis and Varaiya [5] defined a new concept of solution using a theorem of Girsanov [11] to eliminate the control dependence. For other contributions in this spirit see Beneš [2] and Davis and Varaiya [6]. In a recent paper by Lindquist [17] on optimal control of linear stochastic systems (primarily devoted to stochastic functional differential equations) most technical restrictions mentioned above are dispensed with without renouncing the usual concept of solution. Also in a paper by Bensoussan [3] on the separation principle for distributed parameter systems the set of control laws is defined so as to avoid undesirable restrictions. However, contrary to [5] and [21], in both these papers the cost functional is quadratic.

In this paper we consider feedback control of linear stochastic systems of general type. Various types of information patterns with both complete and partial observations are considered. The cost functional is quadratic, but it is of a more general type than usually encountered in the literature. The approach is the same as that of [17], but the objective of this paper is somewhat different. In [17] our prime purpose was to determine explicit feedback solutions for linear stochastic time-lag systems. But, since for such systems the conditional expectation of the current state given past observations is no longer a sufficient statistic, we could not adhere to the approach of [21]. Thus we had to define our set of admissible control laws with a minimum of technical restrictions. However, rather than to discuss the problems of feedback, our main effort was to demonstrate that stochastic time-lag problems of the most general type can be handled in a rigorous way. Therefore, in this paper we shall present a more detailed discussion of our feedback approach, and at the same time we shall be able to present some extensions. In order to avoid obscuring our exposition, we have used technically less complicated examples than in [17] to illustrate our basic ideas. Nevertheless, in certain aspects they will be more general.

In § 3 we discuss the problems of feedback in a general context. We define the concepts of *stochastic open loop* (SOL) problem and *feedback* (FB) problem. A SOL problem is usually easy to solve but what we want is a solution of a FB problem. Therefore, our basic method is to imbed our FB problem in a suitable SOL problem, and to this end, in § 4, we derive an identity for the cost functional. In § 5, we investigate what conditions we have to impose *on the system* in order that the *optimal* control law be linear. This is to simplify the imbedding procedure and also to enable the practical implementation of the optimal control law. Thus we define our system so that among all nonlinear control laws which make sense (conditions of existence, uniqueness, etc. are fulfilled) the optimal one is linear. For stochastic systems of the type discussed above, this amounts to requiring the perturbing noise process to be a martingale in the case of complete observations and a Wiener process for partial observations. We may well be able to solve a SOL problem without these conditions, but the solution is usually of limited interest to us, since we do not know of any method to decide whether an arbitrary nonlinear control law is admissible for our FB problem. Finally, in § 6 we give some simple examples to illustrate our method. For instance, we prove the separation theorem for colored measurement noise and for time delay in the control.

For more explicit control and filtering solutions of systems with delay in the state process we refer the reader to [17] and [18].

2. Preliminaries. Let $x_0(t)$ be a (fixed) measurable[1] n-dimensional stochastic process with bounded second order moments, and let $K(t, s)$ be an $n \times m$ matrix function such that $\int_0^T |K(t, s)|^2\, ds$ is bounded. ($|\cdot|$ is the Euclidean norm.) We shall define three vector functions taking values in R^m, R^n and R^k, respectively, namely the *input* or *control* $u(t)$, the *state* $x(t)$, and the *output* or *observation* $z(t)$. These functions are related to each other in the following way:

$$x(t) = x_0(t) + \int_0^t K(t, s)u(s)\, ds, \tag{2.1}$$

$$z(t) = Hx(t), \tag{2.2}$$

where H is a constant $k \times n$ matrix. In the sequel, we shall often use the following shorthand notation:

$$x = x_0 + Ku, \tag{2.1'}$$

$$z = Hx. \tag{2.2'}$$

Therefore whenever u is a measurable stochastic process such that $E|u(t)|^2 < \infty$ is integrable, x and z are also measurable processes, and they have bounded second order moments.

Our object, however, will be to construct a feedback system. At each time t, $u(t)$ should be formed as a functional of observations received so far: $\{z(s); 0 \leqq s \leqq t\}$ in such a way as to minimize

$$EV_0(x, u), \tag{2.3}$$

where

$$V_s(x, u) = \int_s^T x'(t)Q_1(t)x(t)\, d\alpha(t) + \int_s^T u'(t)Q_2(t)u(t)\, dt. \tag{2.4}$$

Here Q_1 and Q_2 are bounded matrix functions which are nonnegative definite and positive definite respectively, ′ denotes transpose and E expectation, and α is a monotone nondecreasing bounded function which is continuous on the right and thus defines a finite Borel measure μ_α. Moreover, Q_2 has a bounded inverse Q_2^{-1}.

In order to facilitate the formulation of this problem in more precise mathematical terms, we shall define a few concepts: Let P^k be the set of all measurable k-dimensional stochastic processes, and S^m the set of m-dimensional stochastic variables. Then the function

$$\pi : [0, T] \times P^k \to S^m$$

is a *nonanticipative* function of z if $\pi(t, z)$ is a function of $\{z(s); 0 \leqq s \leqq t\}$ only for

[1] In this paper a *measurable* n-dimensional stochastic process will be a $\mathscr{B} \times \mathfrak{S}$-measurable function $[0, T] \times \Omega \to R^n$, where $\mathscr{B}$ and $\mathfrak{S}$ are the sigma fields of Borel sets and events respectively. Then we have assumed an underlying complete probability space $(\Omega, \mathfrak{S}, P)$, where as usual Ω is the sample space with elements ω and P is the probability measure. As usual we shall write $x_0(t)$ instead of $x_0(t, \omega)$. All deterministic functions defined in this paper are Borel measurable.

each t and defines an element in P^m. The measurable process x is a *stochastic B-solution* of the equation

$$x(t) = x_0(t) + \int_0^t K(t, s)\pi(s, Hx)\,ds \tag{2.5}$$

if for each $t \in [0, T]$ it satisfies (2.5) with probability 1 and $E|x(t)|^2$ is *bounded*. In this paper we shall make no distinction between equivalent processes, i.e., processes which for each t are equal with probability 1.

Our model (2.1) of the controlled system is sufficiently general to include linear dynamic systems such as stochastic differential equations and stochastic functional differential equations. Since our prime interest is in differential systems of this type, the technical assumptions of boundedness imposed above are natural and convenient, but it should be pointed out that they are in no way crucial.

3. Feedback in linear stochastic systems. Let $\{\mathfrak{S}_t \subset \mathfrak{S}; 0 \leqq t \leqq T\}$ be a family of sigma fields and let $\mathscr{U}$ be the set of all m-dimensional stochastic processes such that:

(i) $u(t, \omega)$ is measurable (t, ω);
(ii) $\int_0^T E|u|^2\,dt < \infty$;
(iii) $u(t)$ is $\mathfrak{S}_t$-measurable for almost all t.

Consider the problem of finding a $u^* \in \mathscr{U}$ so as to minimize

$$EV_0(x_0 + Ku, u).$$

It will be shown in the Appendix that there indeed exists a unique $u^* \in \mathscr{U}$ for which the minimum is attained. Following [17], such a problem will be called a *stochastic open loop* (SOL) problem and $\mathscr{U}$ a SOL class. If all $\mathfrak{S}_t \equiv \mathfrak{S}_0$, we have an *open loop* (OL) problem, which is essentially an ordinary variational problem, but in general the SOL problem corresponds to the situation where the available amount of "information" (given by $\mathfrak{S}_t$) varies (usually increases) with time but is unaffected by the choice of u.

However, we are primarily interested in problems where information about the state process is provided by the observation process

$$z = Hx. \tag{3.1}$$

The problem is to determine a control law, that is, to design a "black box" in which the observations received so far are filtered and fed back into the system as a control signal (Fig. 1). Then we have a *feedback* (FB) problem. The "black box" will be described mathematically by a nonanticipative function $\pi:(t, z) \to u(t) = \pi(t, z)$, and we shall use the shorthand notation:

$$u = \pi z. \tag{3.2}$$

Of course, we have to define the set of admissible π in such a way that there exists a unique solution of the stochastic functional equation created by the feedback loop. (To this end, Wonham [21] only admitted π for which a certain Lipschitz condition is fulfilled. However, for technical reasons which will be revealed below we do not choose to formulate our problem in this way.) To avoid these rather intricate problems of existence, we could instead of our FB problem solve the

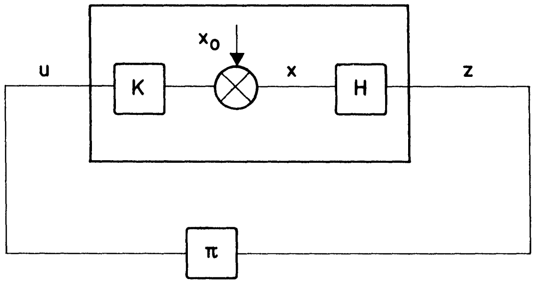

FIG. 1. *The FB problem*

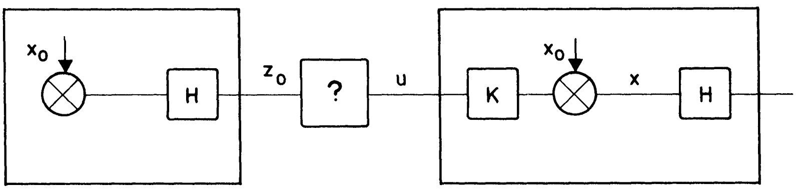

FIG. 2. *A SOL problem*

SOL problem (Fig. 2):

(3.3a)
$$\min_{u \in \mathscr{U}_0} EV_0(x_0 + Ku, u),$$

where $\mathscr{U}_0$ is the SOL class defined by

(3.3b)
$$\mathfrak{S}_t = \sigma\{z_0(s); 0 \leqq s \leqq t\};$$

that is, the family of sigma fields generated by

(3.4)
$$z_0 = Hx_0.$$

Methods along these lines have been proposed [23], [24]. In fact, z_0 can be determined by subtracting HKu from z. However, a control system designed in this way will not be a proper closed loop system, and it will obviously lack some desirable properties associated with the concept of feedback. Also, the reader is warned against exchanging z_0 for z in (3.3), for then we cannot a priori assume that $\{\mathfrak{S}_t\}$ is constant with respect to variations of the control, and moreover questions of existence have to be settled.

Now, we define our class of admissible control laws in the following way: Π is the class of all nonanticipative functions

$$\pi:[0, T] \times P^k \to S^m$$

which are *measurable* in the sense that $\pi(t, y)$ is $\sigma\{y(s); 0 \leqq s \leqq t\}$-measurable for all (t, y) for which π is defined, and which fulfill the following conditions:

(i) there exists a unique stochastic B-solution x_π of

(3.5)
$$x = x_0 + K\pi Hx;$$

(ii) $u = \pi Hx_\pi \in \mathscr{U}_0$.

Then our problem is to determine a $\pi^* \in \Pi$ which minimizes

$$EV_0(x_\pi, \pi H x_\pi),$$

but in general we do not know whether there really exists an optimal π. However, if we define $\mathscr{U}_\Pi$ to be the set of *stochastic processes*

$$\mathscr{U}_\Pi = \{\pi H x_\pi : \pi \in \Pi\}, \tag{3.6}$$

it is clear that $\mathscr{U}_\Pi \subset \mathscr{U}_0$ and that

$$\begin{aligned} \inf_{\pi\in\Pi} EV_0(x_\pi, \pi H x_\pi) &= \inf_{u\in\mathscr{U}_\Pi} EV_0(x_0 + Ku, u) \\ &\geqq \min_{u\in\mathscr{U}_0} EV_0(x_0 + Ku, u). \end{aligned} \tag{3.7}$$

So if we can find an optimal u^* for the problem (3.3) so that $u^* \in \mathscr{U}_\Pi$, then we have found a solution of our FB problem provided that we can also determine a $\pi^* \in \Pi$ such that $u^* = \pi^* H x_{\pi^*}$.

At first sight it seems quite reasonable to assume that the class Π of admissible control laws includes all π for which our problem makes sense. The only point on which this claim could be questioned is the condition that $u(t) = \pi(t, Hx_\pi)$ be $\sigma\{z_0(s); 0 \leqq s \leqq t\}$-measurable for almost all t. However, it should be noted that this condition is true whenever the solution x_π of (3.5) is such that $z_\pi = Hx_\pi$ can be constructed as the limit in probability of a sequence of (measurable) nonanticipative functions of z_0. Therefore, for all practical purposes we can safely ignore all π which do not belong to Π.

As an *example* let us consider the following stochastic functional differential equation:

$$\begin{aligned} dx = [A_1(t)x(t) + A_2(t)x(t-h) &+ \int_{t-h}^{t} A_0(t,s)x(s)\,ds \\ &+ B_1(t)u(t) + B_2(t)u(t-h)]\,dt + C(t)\,dv \quad \text{for } t \geqq 0; \\ x(t) = \xi(t) \quad &\text{for } t \leqq 0, \end{aligned} \tag{3.8}$$

where A_0, A_1, A_2, B_1, B_2 and C are bounded matrix functions, $B_2 \equiv 0$ for $t < h$, the delay $h > 0$, v is a stochastic vector process with orthogonal stationary increment such that

$$Ev(t) = 0; \qquad E\{v(s)v'(t)\} = I \min(s,t) \tag{3.9}$$

and ξ is a process with bounded second order moments. The processes ξ and v are independent.

Problems of this type have been studied under more general conditions in [17], where it was shown that (3.8) can be written in the following equivalent form:

$$x(t) = x_0(t) + \int_0^t K(t,s)u(s)\,ds, \tag{3.10}$$

where

$$\begin{aligned} x_0(t) = \Phi(t,0)\xi(0) &+ \int_{-h}^{0}\left[\Phi(t,s+h)A_2(s+h) + \int_0^h \Phi(t,\tau)A_0(\tau,s)\,d\tau\right]\xi(s)\,ds \\ &+ \int_0^t \Phi(t,s)C(s)\,dv(s), \end{aligned} \tag{3.11}$$

(3.12) $$K(t,s) = \Phi(t,s)B_1(s) + \Phi(t,s+h)B_2(s+h)$$

and Φ is the transition matrix function:

(3.13) $$\frac{\partial \Phi}{\partial t}(t,s) = A_1(t)\Phi(t,s) + A_2(t)\Phi(t-h,s) + \int_{t-h}^{t} A_0(t,\tau)\Phi(\tau,s)\,d\tau \quad \text{for } t \geqq 0;$$
$$\Phi(s,s) = I; \quad \Phi(t,s) = 0 \quad \text{for } t < 0.$$

Now we have transformed our problem into the type discussed above, and it is not hard to see that the unique solution x_π of $x = x_0 + K\pi Hx$ ($\pi \in \Pi$) is also the unique solution of the feedback equation obtained when $u(t) = \pi(t, Hx)$ is inserted in (3.8). In fact, it is demonstrated in [17, Theorem 5.3] that the two equations can be transformed into each other, so that any (stochastic B-) solution of one is also a solution of the other.

Now, let Y^k be the space of all k-dimensional stochastic processes y which can be represented in the following form:

(3.14) $$y(t) = \int_0^t q(s)\,ds + \int_0^t D(s)\,dw(s),$$

where q is a measurable stochastic process such that $E|q(t)|^2$ is integrable, D is a matrix function with square-integrable elements, and w is a stochastic vector process of type (3.9) with orthogonal increments. Then, putting $u = 0$ in (3.8), it is clear that $x_0 \in Y^n$ and $z_0 \in Y^k$. Define $\mathscr{L}$ to be the class of all functions

$$\varphi : [0, T] \times Y^k \to S^m$$

such that

($\mathscr{L}$) $$\varphi(t,y) = f(t) + \int_0^t F(t,s)\,dy(s),$$

where f is an L_2 vector function and F is an L_2 matrix kernel ($\iint |F|^2\,ds\,dt < \infty$). If there is a stochastic B-solution x_φ of (3.8) with $u = \varphi z$, we must clearly have $x_\varphi \in Y^n$ and consequently $z_\varphi \in Y^k$.

LEMMA 3.1. *For the dynamic system* (3.8) *we have*: $\mathscr{L} \subset \Pi$.

Proof (cf. [17]). First observe that if $\psi(t)$ is an L_2 matrix function,

(3.15) $$\int_0^t \psi(s)\varphi(s,y)\,ds = \int_0^t \psi(s)f(s)\,ds + \int_0^t \int_\tau^t \psi(s)F(s,\tau)\,ds\,dy(\tau)$$

a.s. for all y for which φ is defined, that is, we can change the order of integration. In fact, considering (3.14) we can divide the last term of (3.15) into two and apply the usual Fubini theorem to the first term (for $\int |q|^2\,ds < \infty$ a.s.) and the stochastic Fubini theorem ([7, p. 431], [10, p. 197]) to the second. Now, inserting

$$K(t,s) = B_1(s) + B_2(s+h)\theta(t-s-h) + \int_s^t \Gamma(\tau,s)\,d\tau$$

(where Γ is an L_2 matrix kernel such that $\Gamma(t,s) = (\partial K/\partial t)(t,s)$ for $t \neq s + h$ and θ is the unit step function) into (3.10) and changing the order of integration, we obtain

(3.16) $$x(t) = x_0(t) + \int_0^t \left[B_1(s)u(s) + B_2(s)u(s-h) + \int_0^s \Gamma(s,\tau)u(\tau)\,d\tau \right] ds.$$

Then after inserting $u(t) = \varphi(t, z)$ into (3.16), applying (3.15) and multiplying by H we have an expression of the following type:

$$dz = dz_0 + \left[\int_0^t G(t,s)\,dz(s) + g(t)\right] dt, \tag{3.17}$$

where G is an L_2 matrix kernel and g is an L_2 vector function.

Now, let the L_2 matrix kernel R be defined by the Volterra resolvent equation

$$G(t,s) = R(t,s) - \int_s^t R(t,\tau)G(\tau,s)\,d\tau, \tag{3.18}$$

exchange G in (3.17) for the right member of (3.18), and change the order of integration. Then we obtain

$$dz = dz_0 + \left[\int_0^t R(t,s)\,dz_0 + \int_0^t R(t,s)g(s)\,ds + g(t)\right] dt$$

which inserted into (3.16) with $u = \varphi z$ yields the unique solution x_φ. Evidently $\varphi H x_\varphi \in \mathscr{U}_0$. (We consider a measurable version of $\varphi(t, z)$. See [7, p. 430] or [10, p. 196].) This concludes the proof.

By prescribing some conditions of regularity on the sample functions of z such as continuity or boundedness, we could define π as a function of individual sample functions of z rather than the whole stochastic process. For example, if z has continuous sample functions, following Wonham [21] we could define the class Ψ of all functions

$$\psi : [0, T] \times C \to R^m$$

such that $\psi(t, \zeta)$ is a function of $\{\zeta(s); 0 \leqq s \leqq t\}$ and satisfies a Lipschitz condition

$$|\psi(t,\zeta_1) - \psi(t,\zeta_2)| \leqq \gamma\|\zeta_1 - \zeta_2\|,$$

where $\|\cdot\|$ denotes the sup norm in the space C of continuous functions on $[0, T]$ with values in R^k. Let C^k be the space of all k-dimensional stochastic processes with continuous sample functions, and define Π_{LIP} to be the class of all functions

$$(t, z) \in [0, T] \times C^k \to \psi(t, z) \in S^m,$$

where $\psi \in \Psi$. Then it can be shown that $\Pi_{\text{LIP}} \subset \Pi$. (See [21] and [22].) However, for the control of systems of type (3.8), $\mathscr{L}$ is often a very natural class of control laws. Indeed, below we shall introduce some further conditions so that the optimal control law $\pi^* \in \mathscr{L}$, but in general we have to impose still further conditions in order that $\pi^* \in \Pi_{\text{LIP}}$. In fact, $\mathscr{L} \not\subset \Pi_{\text{LIP}}$, for usually a stochastic integral cannot be defined samplewise. This is only possible if the functions $s \to F(t, s)$ are of bounded variation (and z has continuous sample functions). Then we can integrate by parts to obtain

$$\varphi(t,z) = F(t,t)z(t) - F(t,0)z(0) - \int_0^t d_sF(t,s)z(s).$$

With a few additional conditions on F this control law will belong to Π_{LIP}.

4. An identity. Let $w(t)$ be a p-dimensional martingale[2] with finite second order moments, zero mean ($w(0) = 0$) and incremental covariances:

$$E\{dw_i(t)\,dw_j(t)\} = \begin{cases} d\beta_i(t), & j = i, \\ 0, & j \neq i, \end{cases} \tag{4.1}$$

where w_i ($i = 1, 2, \cdots, p$) are the components of w, and β_i are monotone nondecreasing bounded functions which are continuous from the right. In the Hilbert space H, with inner product $(\xi, \eta) = E\{\xi, \eta\}$, of all (real-valued) stochastic variables with finite second order moments, define H_t to be the subspace of stochastic variables which are measurable with respect to

$$\mathfrak{S}_t = \sigma\{w(s); 0 \leqq s \leqq t\} \tag{4.2}$$

and $\hat{H}_t$ to be the closed linear hull of $\{w_i(s); 0 \leqq s \leqq t, i = 1, 2, \cdots, p\}$ together with all constants, i.e., the set of all stochastic variables ξ which can be represented in the following way:

$$\xi = \bar{\xi} + \int_0^t f'(s)\,dw(s),$$

where $\bar{\xi}$ is a constant, f is an L_2 vector function, and integration is with respect to the stochastic measure

$$\mu((t_1, t_2]) = w(t_2) - w(t_1)$$

(cf. [10, p. 194]). Let $E_t\xi$ and $\hat{E}_t\xi$ denote the projections of $\xi \in H$ onto H_t and $\hat{H}_t$ respectively, i.e., the conditional and wide sense conditional expectations of ξ given $\{w(s); 0 \leqq s \leqq t\}$. Since $\hat{H}_t \subset H_t$, we can form the orthogonal complement $H_t \ominus \hat{H}_t$ of $\hat{H}_t$ in H_t. Finally, let $\mathcal{U}_w$ be the SOL class with $\{\mathfrak{S}_t\}$ given by (4.2).

LEMMA 4.1. *If $u \in \mathcal{U}_w$ is given by*

$$u(t) = \bar{u}(t) + \sum_{i=1}^{p} \int_0^t u_i(t, s)\,dw_i(s) + \tilde{u}(t) \tag{4.3}$$

where $\int |\bar{u}(t)|^2\,dt < \infty$, $\iint |u_i(t, s)|^2\,d\beta_i(s)\,dt < \infty$ ($i = 1, 2, \cdots, p$) and $\tilde{u} \in \mathcal{U}_w$ is a stochastic process such that $\tilde{u}_i(t) \in H_t \ominus \hat{H}_t$ for almost all t ($i = 1, 2, \cdots, m$), then

$$EV_0(x_0 + Ku, u) = V_0(\bar{x}, \bar{u}) + \sum_{i=1}^{k} \int_0^T V_s(x_i(\cdot, s), u_i(\cdot, s))\,d\beta_i(s) + EV_0(\tilde{x}, \tilde{u}),$$

$\bar{x}_0, x_i(\cdot, s)$ and $\tilde{x}$ being defined in the following way:

$$\bar{x}(t) = \bar{x}_0(t) + \int_0^t K(t, \tau)\bar{u}(\tau)\,d\tau,$$

$$x_i(t, s) = m_i(t, s) + \int_s^t K(t, \tau)u_i(\tau, s)\,d\tau,$$

$$\tilde{x}(t) = \tilde{x}_0(t) + \int_0^t K(t, \tau)\tilde{u}(\tau)\,d\tau,$$

[2] $E\{w(s)|w(\tau); 0 \leqq \tau \leqq t\} = w(t)$ for $t < s$. Since $E|w(t)|^2 < \infty$, $w(t)$ has *orthogonal increments.*

where[3] $\bar{x}_0(t) = Ex_0(t)$, $m_i(t,s) = (\partial/\partial\beta_i)E\{x_0(t)w_i(s)\}$, *and* $\tilde{x}_0(t) = x_0(t) - \hat{E}_t x_0(t)$.

Proof (cf. [17]). According to Lemma B.1 (see Appendix),

$$\hat{x}_0(t) = \bar{x}_0(t) + \sum_{i=1}^{k} \int_0^t m_i(t,s)\,dw_i(s)$$

is (a version of) the wide sense conditional mean $\hat{E}_t x_0(t)$. Then, inserting $x_0(t) = \hat{x}_0(t) + \tilde{x}_0(t)$ and (4.3) into $x_0 + Ku$, we have:

$$\begin{aligned} x(t) &= \bar{x}_0(t) + \int_0^t K(t,\tau)\bar{u}(\tau)\,d\tau \\ &\quad + \sum_i \left[\int_0^t m_i(t,s)\,dw_i(s) + \int_0^t K(t,\tau) \int_0^\tau u_i(\tau,s)\,dw_i(s)\,d\tau \right] \\ &\quad + \tilde{x}_0(t) + \int_0^t K(t,\tau)\tilde{u}(\tau)\,d\tau \\ &= \bar{x}(t) + \sum_i \int_0^t x_i(t,s)\,dw_i(s) + \tilde{x}(t), \end{aligned} \tag{4.4}$$

where we have used the stochastic Fubini theorem.

Now due to the martingale property,[4] $\tilde{u}(s) \perp \hat{H}_t$ for almost all $s \leqq t$, and therefore (since $\tilde{x}_0(t) \perp \hat{H}_t$), $\tilde{x}(t) \perp \hat{H}_t$. In fact, if

$$\xi_n = \int_s^t f_n'(s)\,dw(s), \tag{4.5}$$

where f_n is a vector step function, $E_s \xi_n = 0$, and therefore

$$E\{\tilde{u}(s)\xi_n\} = E\{\tilde{u}(s)E_s\xi_n\} = 0$$

for almost all $s \leqq t$, for $E = EE_s$ and $\tilde{u} \in \mathscr{U}_w$. But each $\xi \in \hat{H}_t \ominus \hat{H}_s$ can be represented as the limit in H of a fundamental $\{\xi_n\}$ of type (4.5), and therefore (for almost all $s \leqq t$) $E\{\tilde{u}(s)\xi\} = 0$ for all such ξ, and consequently $\tilde{u}(s) \perp \hat{H}_t \ominus \hat{H}_s$. But by definition, $\tilde{u}(s) \perp \hat{H}_s$ and hence $\tilde{u}(s) \perp \hat{H}_t$ (for almost all $s \leqq t$).

Since the terms of (4.4) are mutually orthogonal and the same is true for (4.3) for almost all t, we have

$$EV_0(x,u) = V_0(\bar{x},\bar{u}) + \sum_i EV_0(x_i w_i, u_i w_i) + EV_0(\tilde{x},\tilde{u}),$$

where $(x_i w_i)(t) = \int_0^t x_i(t,s)\,dw_i(s)$ and $u_i w_i$ are defined analogously. However,

$$\begin{aligned} EV_0(x_i w_i, u_i w_i) &= \int_0^T \int_0^t x_i'(t,s)Q_1(t)x_i(t,s)\,d\beta_i(s)\,d\alpha(t) \\ &\quad + \int_0^T \int_0^t u_i'(t,s)Q_2(t)u_i(t,s)\,d\beta_i(s)\,dt \\ &= \int_0^T V_s(x_i(\cdot,s),u_i(\cdot,s))\,d\beta_i(s), \end{aligned}$$

where we have used (4.1) and Fubini's theorem. This concludes the proof.

[3] $(\partial/\partial\beta_i)E\{x_0(t)w_i(s)\} = [(\partial/\partial\sigma)E\{x_0(t)w_i(\beta_i^{-1}(\sigma))\}]_{\sigma=\beta_i(s)}$ (see Appendix).

[4] $\tilde{u}(s) \perp \hat{H}_t$ means "all components of $\tilde{u}(s)$ are orthogonal to $\hat{H}_t$."

Remark. If w is a process of type (3.9), i.e., $\beta_i(t) = t$, we have

$$m_i(t, s) = \frac{\partial}{\partial s} E\{x_0(t) w_i(s)\}.$$

If $x_0(t) \in \hat{H}_t$, we have

$$x_0(t) = \bar{x}_0(t) + \sum_i \int_0^t m_i(t, s)\, dw_i(s).$$

5. The imbedding procedure. In order to exploit the identity derived in the previous section, we note that for many stochastic systems of interest it is possible to represent the uncontrolled observation process z_0 defined in § 3 in the following way:

$$z_0(t) = \bar{z}_0(t) + \int_0^t N(t, s)\, dw(s), \tag{5.1}$$

where w is an r-dimensional martingale defined as in § 4, $\bar{z}_0$ is a bounded deterministic function, and N is a $k \times r$ matrix function with columns N_i such that $\int |N_i(t, s)|^2\, d\beta_i(s)$ are bounded. With $\mathscr{U}_\Pi$, $\mathscr{U}_0$ and $\mathscr{U}_w$ defined as in §§ 3 and 4, we obtain

$$\mathscr{U}_\Pi \subset \mathscr{U}_0 \subset \mathscr{U}_w \tag{5.2}$$

and therefore we have imbedded the set $\mathscr{U}_\Pi$ of all control processes generated by admissible control laws in the SOL class defined by a martingale process. The following lemma will explain our basic method to construct optimal control laws.

LEMMA 5.1. *Let u^* be the optimal solution of the SOL problem*:

$$\min_{u \in \mathscr{U}_w} EV_0(x_0 + Ku, u). \tag{5.3}$$

If there is a $\pi^* \in \Pi$ such that

$$u^*(t) = \pi^*(t, z_0 + HKu^*), \tag{5.4}$$

then there exists an optimal solution of the FB problem

$$\min_{\pi \in \Pi} EV_0(x_\pi, \pi H x_\pi)$$

and it is provided by π^*.

Proof. Let $z^* = z_0 + HKu^*$. Then, according to (5.4), $u^* = \pi^* z^*$ and therefore $z^* = z_0 + HK\pi^* z^*$. But then z^* must be the unique solution z_{π^*} of $z = z_0 + HK\pi^* z$, and consequently $u^* = \pi^* z_{\pi^*}$. Now, due to (5.2),

$$\inf_{u \in \mathscr{U}_\Pi} EV_0(x_0 + Ku, u) \geqq EV_0(x_0 + Ku^*, u^*);$$

but since $u^* \in \mathscr{U}_\Pi$, equality holds. This concludes the proof of the lemma.

Of course this procedure can only be successful if w is so defined that $u^* \in \mathscr{U}_\Pi$. It is therefore desirable that $\mathscr{U}_w$ be as small as possible. If the transformation (5.1) is causally invertible, i.e., w can be represented as a measurable nonanticipative function of z_0, we have $\mathscr{U}_w = \mathscr{U}_0$. Then (5.1) is a *canonical representation* and w will be called an *innovation process.* (See, e.g., [4], [12], [13], [9], [14].) However, in general it is no trivial problem to decide whether u^* really belongs to $\mathscr{U}_\Pi$. Often a nonanticipative function expressing u^* in terms of $z^* = z_0 + HKu^*$ is quite easily determined (e.g., in the form of conditional expectations), but it still remains to show that this function belongs to Π, i.e., that *there exists a unique solution of the corresponding feedback system.* In order to reduce these difficulties and also to make full use of Lemma 4.1, we shall investigate under what conditions on the pair (x_0, w), the optimal SOL control u^* is *linear* in w.

THEOREM 5.2. *If the pair* (x_0, w) *fulfills the condition*

$$E_t x_0(t) = \hat{E}_t x_0(t) \quad \mu_\alpha\text{-a.e. on } [0, T], \tag{5.5}$$

i.e., the conditional and wide sense conditional expectations of $x_0(t)$ *given* $\{w(s); 0 \leqq s \leqq t\}$ *coincide for* μ_α*-almost all* t *on* $[0, T]$, *then the optimal solution* u^* *of the SOL problem* (5.3) *is given by*

$$u^*(t) = \bar{u}^*(t) + \sum_{i=1}^{r} \int_0^t u_i^*(t, s)\, dw_i(s), \tag{5.6}$$

where $\bar{u}^*$ *is the optimal* L_2 *solution of the problem*

$$\min_{\bar{u}} V_0(\bar{x}_0 + K\bar{u}, \bar{u}) \tag{5.7}$$

and $u_i^*(\cdot, s)$ $(i = 1, 2, \cdots, r; 0 \leqq s \leqq T)$ *are the optimal* L_2 *solutions of*

$$\min_{u_i(\cdot, s)} V_s(m_i(\cdot, s) + Ku_i(\cdot, s), u_i(\cdot, s)) \tag{5.8}$$

subject to the constraints $u_i(t, s) \equiv 0$ *for* $t < s$. *Here,*

$$\bar{x}_0(t) = Ex_0(t) \quad \text{and} \quad m_i(t, s) = \frac{\partial}{\partial \beta_i} E\{x_0(t) w_i(s)\}.$$

Moreover, if $x^* = x_0 + Ku^*$, *for all* $(\tau, t) \in [0, T] \times [0, T]$ *we have*

$$\hat{E}_\tau x^*(t) = \bar{x}^*(t) + \sum_{i=1}^{r} \int_0^\tau x_i^*(t, s)\, dw_i(s), \tag{5.9}$$

where $\bar{x}^* = \bar{x}_0 + K\bar{u}^*$ *and* $x_i^*(\cdot, s) = m_i(\cdot, s) + Ku_i^*(\cdot, s)$.

Proof. Let $\tilde{u}$ and $\tilde{x}_0$ be defined as in Lemma 4.1. Then, $\tilde{u}_i(s) \in H_s \subset H_t$ for almost all $s \leqq t$ and therefore $(K\tilde{u})_i(t) \in H_t$. Now, condition (5.5) implies that

$$E_t \tilde{x}_0(t) = E_t\{x_0(t) - \hat{E}_t x_0(t)\} = E_t x_0(t) - \hat{E}_t x_0(t) = 0 \quad \mu_\alpha\text{-a.e.},$$

and consequently $\tilde{x}_0(t) \perp H_t$ μ_α-a.e. Therefore $\tilde{x}_0(t)$ and $(K\tilde{u})(t)$ are orthogonal for μ_α-almost all t on $[0, T]$, and hence

$$EV_0(\tilde{x}, \tilde{u}) = EV_0(\tilde{x}_0, 0) + EV_0(K\tilde{u}, \tilde{u}) \geqq EV_0(\tilde{x}_0, 0),$$

where equality holds for $\tilde{u} = 0$. Therefore our assertion (5.6) follows from Lemma 4.1, for problems (5.7) and (5.8) indeed have unique solutions such that u^* defined by (5.6) is a measurable stochastic process (see [17]). To obtain (5.9), insert (5.6) into $x^* = x_0 + Ku^*$, change the order of integration (stochastic Fubini theorem) and apply Lemma B.1. This concludes the proof.

Note that we do not a priori assume that u^* is linear in w, but the linearity is a consequence of condition (5.5) and the martingale property. It should be clear from the proof of Lemma 4.1 that if we confine our set of admissible controls to those which are linear in w, we only need to assume that w have orthogonal increments to insure that (5.6) is optimal.

COROLLARY 5.2. *Either one of the following two conditions is sufficient for* (5.6) *to be the optimal solution of problem* (5.3):

(i) $x_0(t) \in \hat{H}_t$ *on* $[0, T]$;

(ii) x_0 *and* w *are jointly Gaussian.*

When condition (i) *holds,* $x^*(t)$ *can be obtained from* (5.9) *by putting* $\tau = t$, *and when condition* (ii) *is fulfilled,* $\hat{x}^*(t|\tau) = E_\tau x^*(t)$ *is given by* (5.9).

Proof. Both conditions are sufficient for (5.5) to hold. As for condition (ii), see, e.g., [10, pp. 228–229].

Remark. Conditions (i) and (ii) can be weakened so as to exploit the fact that (5.5) only needs to hold for μ_α-almost all t.

Now, if u^* is given by (5.6), we have

$$z^* = Nw + HKu^*, \tag{5.10}$$

$$u^* = U^*w, \tag{5.6'}$$

where N and U^* denote the affine transformations of (5.1) and (5.6) respectively. Then, if we can eliminate w from this system to obtain a nonanticipative function (5.4) expressing u^* in terms of z^*, there should be no problem in establishing whether this control law really belongs to Π, for it is linear in z^*. Once the optimal control law π^* has been determined, there remains the problem of implementing it. We shall refer to this problem as the *filtering problem*, since it often amounts to constructing a linear filter whose transfer property is described by π^*.

As an example, let us return to the dynamic system defined by (3.8). It is clear from (3.11) that if ξ is a known (deterministic) function and v is a martingale, we have a representation of type (5.1). Also, with $w = v$, condition (i) of Corollary 5.2 holds and consequently u^* is given by (5.6). Indeed, in § 6, we shall use this representation for the case with complete state information ($H = I$), but in general the SOL class $\mathscr{U}_v$ is too large. However, if v is a Wiener-process, ξ is Gaussian, $\{z(t); t \leqq 0\}$ is deterministic, and $HC(t)$ is a square matrix with a bounded inverse, we have an innovation process

$$\begin{aligned} dw(t) &= (HC(t))^{-1}[dz_0(t) - H\hat{q}(t)\,dt], \\ w(0) &= 0, \end{aligned} \tag{5.11}$$

where

$$\hat{q}(t) = E\{q(t)|z_0(s); 0 \leqq s \leqq t\}, \tag{5.12}$$

$$q(t) = A_1(t)x_0(t) + A_2(t)x_0(t-h) + \int_{t-h}^{t} A_0(t,s)x_0(s)\,ds. \tag{5.13}$$

The innovation process w is a Wiener-process, w and z_0 are related to each other by invertible and nonanticipative linear transformations, and the families of sigma fields generated by the two processes are identical. (See, e.g., [14], where other references are also given, or [17, Lemma 3.2].) Therefore,

$$\hat{q}(t) = \bar{q}(t) + \int_0^t Q(t,s)\,dw, \tag{5.14}$$

where the L_2 vector function $\bar{q}$ and L_2 matrix kernel Q are given by Lemma A.2, and hence z_0 can be represented in the form (5.1). Moreover, since x_0 and w are jointly Gaussian, condition (ii) of Corollary 5.2 holds, and u^* is given by (5.6). Now, from (3.16) we have

$$(5.15)\quad dz^* = dz_0 + \left[HB_1u^* + HB_2u^*(t-h) + \int_0^t H\Gamma(t,s)u^*(s)\,ds\right]dt.$$

Then, inserting z_0 given by (5.11) and (5.14) and u^* given by (5.6) into (5.15) and changing the order of integration, we obtain an expression of type

$$(5.16)\qquad (HC)^{-1}\,dz^* = dw + \int_0^t P(t,s)\,dw\,dt + p(t)$$

(where p and P are functions of the same type as $\bar{q}$ and Q). The resolvent technique previously used for equation (3.17) can now be applied to solve (5.16) for dw, which inserted into (5.6) yields

$$u^*(t) = \pi^*(t, z^*),$$

where $\pi^* \in \mathscr{L}$. Therefore, since $\mathscr{L}$ is contained in Π (Lemma 3.1), Lemma 5.1 implies that π^* is the optimal control law of our FB problem. We have not bothered to determine π^* explicitly but have only described how this can be done. The reason for this is that π^* is often more easily implemented by a linear filter in which w is formed as an intermediate process. Such a filter contains linear feedback loops, and we may ask whether the existence and uniqueness for the complete feedback system is preserved. However, this is the case, for mathematically these loops correspond to linear Volterra integral equations of either ordinary type or type (5.16), and therefore they can be resolved by reformulation using the resolvent equations. (Note that in this paper we allow no stochastic processes whose sample functions are not a.s. square integrable.)

6. Examples. In order to illustrate our basic technique, we shall apply the results of this paper to some simple and well-known problems, all of which will concern systems of type (3.8). However, we shall not consider delay in the state process since this would only introduce complications in notation without exposing any new ideas which cannot be found in [17].

Example 1. *Complete state information.* Consider the system:

$$(6.1)\qquad dx(t) = [A(t)x(t) + B_1(t)u(t) + B_2(t)u(t-h)]\,dt + C(t)\,dw,$$

where $x(0) = a$ is a deterministic vector, w is a martingale of type (4.1), and the matrix functions are defined as in § 3. The observation $z(t)$ is the state process $x(t)$ itself, and the problem is to determine a control law $\pi:(t,x) \to u(t) = \pi(t,x)$ in the class Π which minimizes

$$(6.2)\qquad E\left\{\int_0^T (x'Q_1x + u'Q_2u)\,dt + x'(T)Q_1(T)x(T)\right\}.$$

Here (6.2) is the cost functional (2.3) with $\alpha(t) = t$ for $t < T$ and $\alpha(t) = t + 1$ for $t \geqq T$.

Now, if Φ is defined as in (3.13) with $A_1 = A$ and $A_0 = A_2 = 0$, we have

$$x_0(t) = \Phi(t, 0)a + \int_0^t \Phi(t, s)C(s)\,dw(s) \tag{6.3}$$

and hence condition (i) of Corollary 5.2 holds. Therefore, (5.6) is the optimal solution of the SOL problem (5.3). The functions $\bar{x}_0$, m_i and K in Lemma 5.2 are given by

$$\bar{x}_0(t) = \Phi(t, 0)a, \tag{6.4}$$

$$m_i(t, s) = \Phi(t, s)c_i(s) \quad \text{for } t \geqq s, \tag{6.5}$$

$$K(t, s) = \Phi(t, s)B_1(s) + \Phi(t, s + h)B_2(s + h), \tag{3.12$'$}$$

where c_i is the ith column of C, and therefore problems (5.7) and (5.8) belong to the family of problems

$$\min \int_s^T (x'Q_1x + u'Q_2u)\,dt + x(T)Q_1(T)x(T) \tag{6.6}$$

when

$$\frac{dx}{dt} = Ax + B_1u(t) + B_2u(t - h) \quad \text{for } t > s, \qquad u(t) = 0 \quad \text{for } t < s,$$

where for (5.7), $s = 0$ and $x(0) = a$, and for (5.8), $x(s) = c_i(s)$. Now, according to Appendix C, we have the following feedback solutions:

$$\bar{u}^*(t) = P_0(t)\bar{x}^*(t) + \int_{t-h}^t P_1(t, \tau)\bar{u}^*(\tau)\,d\tau,$$

$$u_i^*(t, s) = P_0(t)x_i^*(t, s) + \int_{t-h}^t P_1(t, \tau)u_i^*(\tau, s)\,d\tau, \qquad t \geqq s,$$

which inserted into (5.6) yields, after applying the stochastic Fubini theorem,

$$u^*(t) = P_0(t)x^*(t) + \int_{t-h}^t P_1(t, \tau)u^*(\tau)\,d\tau, \tag{6.7}$$

for $x^*(t)$ is given by (5.9) with $\tau = t$ (Corollary 5.2), and $u_i^*(t, s) \equiv 0$ for $t < s$. Since P_1 is an L_2 matrix kernel and almost all sample functions of P_0x^* are square integrable, according to standard Volterra theory (see, e.g., [19]) there is an L_2 resolvent kernel P_2 such that

$$u^*(t) = P_0(t)x^*(t) + \int_0^t P_2(t, s)P_0(s)x^*(s)\,ds \tag{6.8}$$

which defines a nonanticipative function $\pi^* : (t, x^*) \to u^*$.

Now we can use a similar argument to show that $\pi^* \in \Pi$, and therefore, according to Lemma 5.1, (6.8) is an optimal FB solution. However, note that the filter described by (6.7) might prove to be a more suitable implementation of π^*. (See Fig. 6.1.) In fact, (6.7) only requires storing u^* on the interval $(t - h, t)$, while in (6.8) we need x^* on the whole interval $(0, t)$. It should be clear from the dis-

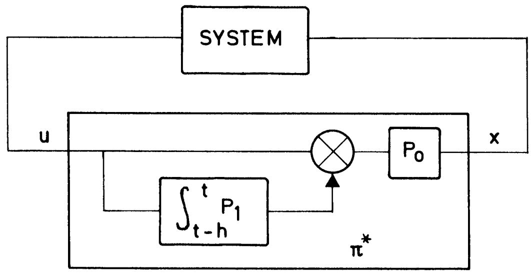

FIG. 3

cussion above that this implementation preserves the existence and uniqueness property of π^*.

Example 2. *Separation theorem*; *white measurement noise*. Consider the stochastic vector processes y and z defined by

$$dy(t) = [A(t)y(t) + B(t)u(t)]\,dt + C_1(t)\,dv_1(t), \tag{6.9}$$

$$dz(t) = D(t)y(t)\,dt + dv_2; \qquad z(0) = 0, \tag{6.10}$$

where $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector Wiener-process of type (3.9), $y(0)$ in Gaussian, $Ey(0) = a$, and v and $y(0)$ are independent. All matrix functions are bounded. The problem is to determine a nonanticipative function $\pi:(t, z) \to u(t) = \pi(t, z)$ so as to minimize

$$E\left\{\int_0^T (y'Q_3y + u'Q_2u)\,dt\right\}, \tag{6.11}$$

where Q_3 is nonnegative definite and bounded.

Now, this is clearly a problem of the type discussed in § 3. In fact, define x to be $\begin{pmatrix} y \\ z \end{pmatrix}$ and H to be $(0, I)$. Moreover, in $V_s(x, u)$, define Q_1 to be $\begin{pmatrix} Q_3 & 0 \\ 0 & 0 \end{pmatrix}$ and put $\alpha(t) = t$. Therefore, Π will be our class of admissible control laws from which a π minimizing (6.11) is to be selected.

For this problem we have an innovation process (5.11) given by

$$dw = dz_0 - D(t)\hat{y}_0(t)\,dt; \qquad w(0) = 0, \tag{6.12}$$

where $\hat{y}_0(t)$ is defined as in (5.12), and, according to Corollary 5.2 (condition (ii)), (5.6) is the optimal solution of our basic SOL problem (5.3) to determine a $u \in \mathscr{U}_w$ so as to minimize (6.11). Now, let $\bar{y}_0, \bar{y}^*, g_i$ and y_i^* be the subvectors in "y-position" of $\bar{x}_0, \bar{x}^*, m_i$ and x_i^* respectively, and let K_1 be the corresponding submatrix of K. Then, if Φ is defined as in Example 1, we have

$$\bar{y}_0(t) = \Phi(t, 0)a, \tag{6.13}$$

$$g_i(t, s) = \Phi(t, s)P(s)\,d_i(s) \quad \text{for } t \geqq s, \tag{6.14}$$

$$K_1(t, s) = \Phi(t, s)B(s), \tag{6.15}$$

where d_i is the ith column of D', and P is the conditional covariance

$$P(t) = E\{[y_0(t) - \hat{y}_0(t)][y_0(t) - \hat{y}_0(t)]'\}. \tag{6.16}$$

In fact, since y_0 and v_2 are independent, we have

$$\begin{aligned} E\{y_0(t)w'(s)\} &= \int_0^s E\{y_0(t)[y_0(\tau) - \hat{y}_0(\tau)]'\}D'(\tau)\,d\tau \\ &= \int_0^s \Phi(t,\tau)P(\tau)D'(\tau)\,d\tau, \end{aligned}$$

for the two last terms in

$$y_0(t) = \Phi(t,\tau)[y_0(\tau) - \hat{y}_0(\tau)] + \Phi(t,\tau)\hat{y}_0(\tau) + \int_\tau^t \Phi(t,\tau)C_1(\tau)\,dv_1$$

are orthogonal to $y_0(\tau) - \hat{y}_0(\tau)$, and therefore (6.14) follows from the definition of $m_i(t, s)$. Then, (5.7) and (5.8) belong to the family of problems:

$$\text{(6.17)} \quad \text{when} \quad \begin{aligned} &\min \int_s^T (y'Q_3y + u'Q_2u)\,dt \\ &\frac{dy}{dt} = Ay + Bu \quad \text{for } t \geqq s, \end{aligned}$$

where $s = 0$ and $y(0) = a$ for (5.7), and $y(s) = P(s)\,d_i(s)$ for (5.8).

Hence, we have the following feedback solutions:

$$\bar{u}^*(t) = L(t)\bar{y}^*(t), \tag{6.18}$$

$$u_i^*(t,s) = L(t)y_i^*(t,s), \tag{6.19}$$

where L can be found in any textbook on the linear-quadratic regulator problem. This inserted into (5.6) yields

$$u^*(t) = L(t)\hat{y}^*(t), \tag{6.20}$$

where $\hat{y}^*(t) = E_t y^*(t)$ is given by (5.9) (see Corollary 5.2), and therefore satisfies the stochastic differential equation

$$\text{(6.21)} \quad \begin{aligned} &d\hat{y}^*(t) = [A(t)\hat{y}^*(t) + B(t)u^*(t)]\,dt + P(t)D'(t)\,dw(t), \\ &\hat{y}^*(0) = a. \end{aligned}$$

Now, the innovation process (6.12) can also be written

$$dw = dz^* - D\hat{y}^*\,dt, \tag{6.22}$$

for the control-dependent terms of z^* and $\hat{y}^*$ cancel out. Then, if Ψ is the transition matrix

$$\text{(6.23)} \quad \begin{aligned} &\frac{\partial\Psi}{\partial t}(t,s) = [A(t) + B(t)L(t) - P(t)D'(t)D(t)]\Psi(t,s), \\ &\Psi(s,s) = I, \end{aligned}$$

we have a nonanticipative function $\pi^*:(t, z^*) \to u^*$ which is defined by

$$u^*(t) = L(t)\Psi(t, 0)a + \int_0^t L(t)\Psi(t, s)P(s)D'(s)\,dz^*(s) \tag{6.24}$$

and hence belongs to Π (Lemma 3.1). Therefore, according to Lemma 5.1, among all control laws in Π, the function π^* gives the smallest value to (6.11). However, it should be clear from the discussion at the end of § 5 that π^* can safely be implemented by the linear filter defined by (6.20), (6.21) and (6.22). Of course this is an advantage, for in this way there is no need to store old z^*.

Example 3. *Separation theorem*; *colored measurement noise*. We shall consider the preceding example (Example 2) modified in the following way: The observation process z is no longer defined by (6.10), but

$$z(t) = H_1 y(t) + n(t), \tag{6.25}$$

where H_1 is a constant matrix and n is a colored noise term generated by

$$dn(t) = D(t)n(t)\,dt + dv_2 \tag{6.26}$$

with $n(0) = 0$. Also $y(0) = a$ is assumed to be deterministic.

Again we have a problem of the type discussed in § 3, for define x to be $\begin{pmatrix} y \\ n \end{pmatrix}$, put $H = (H_1, I)$, and let Q_1 and α be defined as in the previous example. Therefore, the problem is to determine $\pi \in \Pi$ so as to minimize (6.11) when $u(t) = \pi(t, z)$.

Since $HC = I + H_1C_1$, we shall further assume that $(I + H_1C_1)^{-1}$ exists and is bounded. Then, we have an innovation process (5.11) given by

$$dw = (I + H_1C_1)^{-1}[dz_0 - H_1A\hat{y}_0\,dt - D\hat{n}\,dt], \tag{6.27}$$

where $w(0) = 0$ and $\hat{y}_0(t)$ and $\hat{n}(t)$ are defined as in (5.12).

Now, apart from the definition of the innovation process, we have the same problem as in the preceding example, and the optimal solution u^* of our basic SOL problem is given by (6.20) and (6.21). The innovation process can now be expressed in terms of z^* and $\hat{y}^*$:

$$dw = (I + H_1C_1)^{-1}[dz^* - Dz^*\,dt - (H_1A + H_1BL - DH_1)\hat{y}^*\,dt], \tag{6.28}$$

for due to (6.20) and $\hat{n}(t) = z^*(t) - H_1\hat{y}^*(t)$ which is an immediate consequence of (6.25), we have only added terms which cancel out. Then, because of (6.20), (6.21) and (6.28), we have

$$\begin{aligned} u^*(t) = {} & L(t)\Psi(t, 0)a + \int_0^t \Gamma(t, s)D(s)\,ds H_1 a \\ & + \int_0^t \left[\Gamma(t, s) + \int_s^t \Gamma(t, \tau)D(\tau)\,d\tau\right] dz^*(s), \end{aligned} \tag{6.29}$$

where Γ and the transition matrix Ψ are defined by

$$\frac{\partial\Psi}{\partial t}(t, s) = [A + BL - PD'(I + H_1C_1)^{-1}(H_1A + H_1BL - DH_1)]\Psi(t, s),$$

$$\Psi(s, s) = I,$$

$$\Gamma(t, s) = L(t)\Psi(t, s)P(s)D'(s)(I + H_1C(s))^{-1}.$$

To obtain (6.29) we have used the fact that

$$z^*(t) = H_1 a + \int_0^t dz^*$$

and applied the stochastic Fubini theorem. Now, (6.29) clearly defines a function $\pi^*:(t, z^*) \to u^*$ which belongs to $\mathscr{L} \subset \Pi$. Therefore, according to Lemma 5.1, π^* is an optimal control law in the class Π. However, as usual we will find it more convenient to implement π^* by the linear filter defined by (6.20), (6.21) and (6.28).

Appendix A.

LEMMA A.1. *The SOL problem posed in the beginning of* § 3 *has an optimal solution which is unique up to a* (t, ω)*-equivalence.*

Proof. We introduce the following notation:

$$J(u) = EV_0(x_0 + Ku, u).$$

Now, there is a sequence $u_n \in \mathscr{U}$, $n = 1, 2, 3, \cdots$, such that

$$\lim_{n\to\infty} J(u_n) = \inf_{u \in \mathscr{U}} J(u) = \rho \geqq 0.$$

Then, for each $\varepsilon > 0$ the parallelogram identity yields

$$\begin{aligned} EV_0(K(u_m - u_n), u_m - u_n) &= 2J(u_m) + 2J(u_n) - 4J\left(\frac{u_n + u_m}{2}\right) \\ &< 2\left(\rho + \frac{\varepsilon}{4}\right) + 2\left(\rho + \frac{\varepsilon}{4}\right) - 4\rho = \varepsilon \end{aligned}$$

for sufficiently large m and n. Therefore $\{u_n\}$ is a Cauchy sequence in $L_2([0, T] \times \Omega, \mathscr{B} \times \mathfrak{S}, \lambda \times P)$ (with norm $\|\cdot\| = (\int E|\cdot|^2\,dt)^{1/2}$; λ is the Lebesgue measure) defining a limit point u^* which, due to completeness, clearly satisfies conditions (i) and (ii) in the definition of $\mathscr{U}$. Moreover, since

$$\lim_{n\to\infty} \|u_n - u^*\| = 0, \qquad \lim_{n\to\infty} E|u_n(t) - u^*(t)|^2 = 0$$

for almost all t, and hence u^* satisfies condition (iii), too. Therefore, $u^* \in \mathscr{U}$. It remains to show that u^* is optimal. However this is the case, for it is not hard to see that $|J(u_n) - J(u^*)| \leqq \gamma\|u_n - u^*\|$, where γ is a constant, and hence $J(u^*) = \rho$. Moreover, if $u^0 \in \mathscr{U}$ and $J(u^0) = \rho$, the parallelogram identity implies that $\|u^* - u^0\| = 0$, for

$$J\left(\frac{u^* + u^0}{2}\right) \geqq \rho.$$

Therefore, $u^0 = u^*$ $\lambda \times P$-a.e., and hence the asserted uniqueness property is true.

Appendix B. Let $y(t)$ be a stochastic vector process with finite second order moments and mean $E\{y(t)\} = \bar{y}(t)$, and let $w(t)$ be a vector process with zero mean and orthogonal increments described by (4.1). The inverse functions $t \to \beta_i^{-1}(t)$ are uniquely defined except for at most enumerably many t. Then define

$$n_i(t, s) = \left[\frac{\partial}{\partial\sigma} E\{y(t) w_i(\beta_i^{-1}(\sigma))\}\right]_{\sigma = \beta_i(s)} \tag{B.1}$$

for which we shall use the shorthand notation

$$n_i(t,s) = \frac{\partial}{\partial \beta_i} E\{y(t)w_i(s)\}. \tag{B.2}$$

Now, we can formulate a lemma which slightly generalizes results which may be found in [13], [9] and [17], for example.

LEMMA B.1. *The wide sense conditional expectation of* $y(t)$ *with respect to* $\{w(s); 0 \leqq s \leqq \tau\}$ *is given by*

$$\hat{E}_\tau y(t) = \bar{y}(t) + \sum_i \int_0^\tau n_i(t,s)\, dw_i(s). \tag{B.3}$$

Proof (cf. [17, Lemma 2.1]). Since, by definition, $\hat{E}_\tau y(t)$ must have a representation of type (B.3), it only remains to show that n_i is given by (B.1). For $s \leqq \tau$, $w_i(s)$ and $y(t) - \hat{E}_\tau y(t)$ are orthogonal, and therefore

$$E\{y(t)w_i(s)\} = \int_0^s n_i(t,\tau)\, d\beta_i(\tau),$$

or, with $\sigma = \beta_i(s)$,

$$E\{y(t)w(\beta_i^{-1}(\sigma))\} = \int_{\beta_i(0)}^{\sigma} n_i(t, \beta_i^{-1}(\theta))\, d\theta,$$

which yields (B.1).

Appendix C. Problem (6.6) has an optimal feedback solution

$$u^*(t) = P_0(t)x^*(t) + \int_{t-h}^{t} P_1(t,s)u^*(s)\, ds, \tag{C.1}$$

where

$$P_1(t,s) = \Lambda(t, s+h)B_2(s+h) - \int_t^T R(t,\sigma,t)\Lambda(\sigma, s+h)\, d\sigma B_2(s+h),$$

$$P_0(t) = \Lambda(t,t) - \int_t^T R(t,\tau,t)\Lambda(\tau,t)\, d\tau.$$

Here,

$$\Lambda(t,s) = -Q_2^{-1}(t)\left[\int_t^T K'(\tau,t)Q_1(\tau)\Phi(\tau,s)\, d\tau + K'(T,t)Q_1(T)\Phi(T,s)\right],$$

where K is defined by (3.12) and R is the resolvent kernel given by

$$\begin{aligned} R(t,\tau,s) - P(t,\tau) &= -\int_s^T P(t,\sigma)R(\sigma,\tau,s)\, d\sigma \\ &= -\int_s^T R(t,\sigma,s)P(\sigma,\tau)\, d\sigma \end{aligned}$$

with

$$P(t, s) = \Lambda(t, s)B_1(s) + \Lambda(t, s + h)B_2(s + h).$$

In fact, by the method used in [17, § 4], we have

$$u^*(t) + \int_s^T P(t, \tau)u^*(\tau)\, d\tau = \Lambda(t, s)x^*(s) + \int_{s-h}^s \Lambda(t, \tau + h)B_2(\tau + h)u^*(\tau)\, d\tau$$

from which (C.1) follows by the same argument as in [17]. (Also see [1] and [15] where versions of this problem are discussed in detail.)

REFERENCES

[1] Y. Alekal, P. Brunovský, D. H. Chyung and E. B. Lee, *The quadratic problem for systems with time delays*, IEEE Trans. Automatic Control, AC-16 (1971), pp. 473–687.

[2] V. E. Beneš, *Existence of optimal stochastic control laws*, this Journal, 9 (1971), pp. 446–472.

[3] A. Bensoussan, *On the separation principle for distributed parameter systems*, IFAC Symp. Control of Distributed Parameter Systems, Banff, Canada, 1971.

[4] H. Cramér, *On the structure of purely non-deterministic stochastic processes*, Ark. Mat., 4 (1961), pp. 249–266.

[5] M. H. A. Davis and P. P. Varaiya, *Information states in linear stochastic systems*, J. Math. Anal. Appl., 37 (1972), pp. 384–402.

[6] ———, *Dynamic programming conditions for partially observable stochastic systems*, Memo. ERL-M304, Electronics Research Laboratory, University of California, Berkeley, 1971.

[7] J. L. Doob, *Stochastic Processes*, John Wiley, New York, 1953.

[8] W. H. Fleming, *Optimal continuous-parameter stochastic control*, SIAM Rev., 11 (1969), pp. 470–509.

[9] P. Frost, *Nonlinear estimation in continuous time systems*, Rep. 6304-4, Center for Systems Research, Stanford University, Stanford, Calif., 1968.

[10] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, W. B. Saunders, Philadelphia, 1969.

[11] I. V. Girsanov, *On transforming a certain class of stochastic processes by absolutely continuous substitution of measures*, Theor. Probability Appl., 5 (1960), pp. 285–301.

[12] T. Hida, *Canonical representations of Gaussian processes and their applications*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math., 33 (1960/61), pp. 109–155.

[13] T. Kailath, *An innovations approach to least squares estimation, Part I: Linear filtering in additive white noise*, IEEE Trans. Automatic Control, AC-13 (1968), pp. 646-655.

[14] ———, *The innovations approach to detection and estimation theory*, Proc. IEEE, 58 (1970), pp. 680–695.

[15] H. N. Koivo and E. B. Lee, *Controller syntheses for linear systems with retarded state and control variables and quadratic cost*, Automatica, 8 (1972), pp. 203–208.

[16] H. J. Kushner, *Stochastic Stability and Control*, Academic Press, New York, 1967.

[17] A. Lindquist, *Optimal control of linear stochastic systems with applications to time lag systems*, Information Sci., 4 (1972).

[18] ———, *A theorem on duality between estimation and control for linear stochastic systems with time delay*, J. Math. Anal. Appl., 37 (1972), pp. 516–536.

[19] F. Smithies, *Integral Equations*, Cambridge University Press, New York, 1958.

[20] H. S. Witsenhausen, *Separation of estimation and control for discrete time systems*, Proc. IEEE, 59 (1971), pp. 1557–1566.

[21] W. M. Wonham, *On the separation theorem of stochastic control*, this Journal, 6 (1968), pp. 312–326.

[22] ———, *Random differential equations in control theory*, Probabilistic Methods in Applied Mathematics, vol. 2, A. T. Bharucha-Reid, ed., Academic Press, New York, 1970, pp. 131–212.

[23] L. E. Zachrisson, *A proof of the separation theorem in control theory*, IOS Rep. R 23, Royal Institute of Technology, Stockholm, Sweden, 1968.

[24] ———, *On a separation theorem for almost separated systems*, to appear.