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A NEW ALGORITHM FOR OPTIMAL FILTERING OF DISCRETE-TIME STATIONARY PROCESSES*

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Abstract. An algorithm (which does not involve the usual Riccati-type equation) for computing the gain matrices of the Kalman filter is presented. If the dimension k of the state space is much larger than that of the observation process, the number of nonlinear equations to be solved in each step is of order k rather than k^2 as by the usual procedure.

1. Introduction. Let $\{x_n\}$ be a k-dimensional and $\{y_n\}$ an m-dimensional (wide sense) stationary stochastic vector process generated by the well-known model:

$$x_{n+1} = Fx_n + v_n, \tag{1.1}$$

$$y_n = Hx_n + w_n, \tag{1.2}$$

where, for convenience, x_0, $\{v_n\}$ and $\{w_n\}$ have zero mean and are pairwise uncorrelated, and

$$E\{x_0x_0'\} = P_0, \tag{1.3}$$

$$E\{v_iv_j'\} = P_1\delta_{ij}, \tag{1.4}$$

$$E\{w_iw_j'\} = P_2\delta_{ij} \tag{1.5}$$

(δ_{ij} is the Kronecker delta and $'$ denotes transpose). The matrices F, H, P_0, P_1 and P_2 are constant and have the appropriate dimensions. To simplify matters we assume that P_2 is positive definite (of course all P_i are nonnegative definite).

Now, it is well known that the linear least squares estimate $\hat{x}_n$ of x_n given $\{y_0, y_1, \cdots, y_{n-1}\}$ can be determined by the Kalman filter [5]:

$$\hat{x}_{n+1} = F\hat{x}_n + K_n(y_n - H\hat{x}_n) \tag{1.6}$$

with initial condition $\hat{x}_0 = 0$ and the gain matrix K_n given by

$$K_n = F\Sigma_nH'(H\Sigma_nH' + P_2)^{-1}. \tag{1.7}$$

Here the error covariance matrix

$$\Sigma_n = E\{(x_n - \hat{x}_n)(x_n - \hat{x}_n)'\} \tag{1.8}$$

can be recursively computed from the equation

$$\Sigma_{n+1} = F[\Sigma_n - \Sigma_nH'(H\Sigma_nH' + P_2)^{-1}H\Sigma_n]F' + P_1 \tag{1.9}$$

with initial condition $\Sigma_0 = P_0$. Therefore, this procedure requires computation of the (symmetric) $k \times k$ matrix Σ_n in each step in order to obtain the gain K_n,

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while actually the $k \times m$ matrix

$$Q_n = \Sigma_n H' \tag{1.10}$$

is needed. When, as is often the case, $m \ll k$, this amounts to computing plenty of unnecessary information.

In this paper we present an algorithm by which Q_n can be computed directly without using the Riccati-type equation (1.9). Instead of the $k(k + 1)/2$ equations of (1.9), we need only solve $2km + m(m + 1)/2$ equations, which is a major reduction when $m \ll k$. In the scalar output case ($m = 1$) we actually only need $2k$ equations.

[Before turning to the derivation of our algorithm we shall present some facts about the classical problem of determining the linear least squares estimate of y_n given $\{y_0, y_1, \cdots, y_{n-1}\}$ when $\{y_n\}$ is an *arbitrary* m-dimensional (wide sense) stationary stochastic sequence. Due to the stationarity it is possible to solve the normal equations recursively for the ($m \times m$ matrix) filter coefficients as n increases. For the scalar case ($m = 1$) such recursions can be found in [8], [9], [13] and also in the theory of orthogonal polynomials [1], [2].[1] The latter is of course no coincidence since the connection between prediction and orthogonal polynomials is well established [3]. When $m > 1$, the situation is somewhat more complicated (which accounts for the fact that the number of equations in our algorithm increases "discontinuously" as m becomes greater than 1). Recursive equations for this case can be found in [7], [12], [14]. However, a relation which is important for our purposes is missing in [7] and although this relation is mentioned in [12], [14], there is no proof for it. Therefore, in presenting a set of such equations we shall supply the reader with a short but complete proof. At the same time we shall be able to relate these equations to certain forward and backward prediction problems.]

In § 2 we shall introduce some notations and recall certain facts from estimation theory, in § 3 the abovementioned recursions for the filter coefficients will be developed, and in §§ 4 and 5 we shall return to what is the basic contribution of this paper, namely, the derivation of an algorithm for Q_n without making use of (1.9).

Independently, Kailath [4] has recently shown that (under certain conditions which are fulfilled in the stationary case) the Riccati equation for the continuous-time Kalman–Bucy filter can be factorized to yield equations similar to ours. Indeed, our method modified to the continuous-time case gives exactly the corresponding equations of Kailath, as we shall demonstrate in [10]. Similar results have also recently been announced by Rissanen [6], who, however, does not consider the model (1.1)–(1.2).

We have presented our algorithm for Q_n in connection with the one-step prediction problem (which is the standard problem in the literature). However, the algorithm can also be used for the pure filtering problem as pointed out in § 4.

[1] The equations in [1], [2] were first made known to us by R. E. Kalman, who suggested the theory of orthogonal polynomials studied in an algebraic context as a possible vehicle in obtaining a more effective algorithm. Our approach, however, is quite elementary in the sense that only facts of linear algebra are used. References [9], [12], [13], [14] were brought to our attention by a referee.

2. The forward and backward prediction problem. Let $y_0, y_1, y_2, y_3, \cdots$ be a wide sense stationary sequence of m-dimensional stochastic vectors with zero mean and covariances

$$C_i = E\{y_{n+i}y_n'\},$$

which, of course, are $m \times m$ matrices such that $C_{-i} = C_i'$. To simplify matters, we assume that this vector process has full rank in the sense that the generalized Toeplitz matrices

$$T_n = \begin{pmatrix} C_0 & C_1' & C_2' & \cdots & C_n' \\ C_1 & C_0 & C_1' & \cdots & C_{n-1}' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & C_0 \end{pmatrix},$$

$n = 0, 1, 2, \cdots$, are positive definite.

Now, for each $n \in \{1, 2, 3, \cdots\}$ we can define two problems of estimation, namely, the *forward* (P_n) and *backward* (P_n^*) one-step prediction problem.

Problem P_n. Find the linear least squares estimate $\hat{y}_n$ of y_n given $\{y_0, y_1, \cdots, y_{n-1}\}$.

Problem P_n^*. Find the linear least squares estimate $\hat{y}_n^*$ of y_0 given $\{y_1, y_2, \cdots, y_n\}$.

Clearly these estimates have the following form:

$$\hat{y}_n = -\sum_{i=0}^{n-1} \Phi_{ni}' y_i,$$

$$\hat{y}_n^* = -\sum_{i=1}^{n} \Phi_{ni}^{*\prime} y_i,$$

where Φ_{ni} and Φ_{ni}^* are $m \times m$ matrices. Then, by defining Φ_{nn} and Φ_{n0}^* to be unit matrices, the estimation errors can be written

$$\tilde{y}_n = y_n - \hat{y}_n = \sum_{i=0}^{n} \Phi_{ni}' y_i, \tag{2.1}$$

$$\tilde{y}_n^* = y_0 - \hat{y}_n^* = \sum_{i=0}^{n} \Phi_{ni}^{*\prime} y_i. \tag{2.2}$$

We introduce the following notations for the error covariances:

$$R_n = E\{\tilde{y}_n \tilde{y}_n'\}, \tag{2.3}$$

$$R_n^* = E\{\tilde{y}_n^* \tilde{y}_n^{*\prime}\}. \tag{2.4}$$

Now let E_n be the matrix formed by an infinite number of $m \times m$ matrices arranged in a vertical array with zero matrices in positions $0, 1, \cdots, n-1, n+1, n+2, \cdots$ and a unit matrix in position n:

$$E_n' = (0, \cdots, 0, I, 0, 0, \cdots).$$

Furthermore, given $X = \sum_{k=0}^{p} E_k X_k$ and $Y = \sum_{k=0}^{p} E_k Y_k$, where X_k, $k = 0, 1, \cdots, p < \infty$, and Y_k, $k = 0, 1, \cdots, p$, are $m \times m$ matrices (some of which may be zero), define the $m \times m$ matrix $[X, Y]$:

$$[X, Y] = X'TY,$$

where T is the infinite matrix T_∞. (Of course we could define $[X, Y]$ in terms of T_p, but we prefer an expression which is independent of p.) Then we have

$$[Y, X] = [X, Y]' \quad \text{and} \quad [E_i, E_j] = C_{i-j}. \tag{2.5}$$

Finally, introduce the shift operator σ:

$$\sigma^i X = \sum_{k=0}^{p} E_{k+i} X_k \qquad (i \geqq 0). \tag{2.6}$$

It is then easily seen that

$$[E_k, \sigma^i X] = [E_{k-i}, X] \qquad (k \geqq i) \tag{2.7}$$

and that

$$[\sigma X, \sigma Y] = [X, Y]. \tag{2.8}$$

We are now in a position to express some well-known orthogonality properties in terms of

$$\Phi_n = \sum_{k=0}^{n} E_k \Phi_{nk} \tag{2.9}$$

and

$$\Phi_n^* = \sum_{k=0}^{n} E_k \Phi_{nk}^*. \tag{2.10}$$

LEMMA 2.1.

$$[E_k, \Phi_n] = 0 \quad \text{for } 0 \leqq k < n, \tag{2.11}$$

$$[E_n, \Phi_n] = R_n, \tag{2.12}$$

$$[\Phi_k, \Phi_l] = R_k \delta_{kl}. \tag{2.13}$$

Proof. Equations (2.11) and (2.12) follow from

$$[E_k, \Phi_n] = \sum_{i=0}^{n} C_{k-i} \Phi_{ni} = \sum_{i=0}^{n} E\{y_k y_i'\} \Phi_{ni} = E\{y_k \tilde{y}_n'\},$$

which by orthogonality is 0 for $k < n$ and R_n for $k = n$. We obtain (2.13) by observing that

$$[\Phi_k, \Phi_l] = \sum_{i=0}^{k} \Phi_{ki}' [E_i, \Phi_l],$$

which, by (2.11) and (2.12), is 0 for $k < l$ and R_k for $k = l$. Then it follows from (2.5) that (2.13) holds for $k > l$ also.

LEMMA 2.2.

$$[E_k, \Phi_n^*] = 0 \quad \text{for } 0 < k \leqq n, \tag{2.14}$$

(2.15) $$[E_0, \Phi_n^*] = R_n^*,$$

(2.16) $$[\sigma^{n-k}\Phi_k^*, \sigma^{n-l}\Phi_l^*] = R_k^*\delta_{kl} \qquad (0 \leqq k, l \leqq n).$$

Proof. Equations (2.14) and (2.15) are obtained in the same way as (2.11) and (2.12), only replacing $\tilde{y}_n$ by $\tilde{y}_n^*$. To obtain (2.16), also observe (2.6) and (2.7) to see that (for $k \leqq l$)

$$[\sigma^{n-k}\Phi_k^*, \sigma^{n-l}\Phi_l^*] = \sum_{i=0}^{k} \Phi_{ki}^{*\prime}[E_{i+n-k}, \sigma^{n-l}\Phi_l^*] = \sum_{i=0}^{k} \Phi_{ki}^{*\prime}[E_{l-k+i}, \Phi_l^*].$$

Remark 2.1. Observe that Φ_n is *uniquely* determined by the system (2.11) of *normal equations*:

(2.17) $$\sum_{i=0}^{n-1} C_{k-i}\Phi_{ni} = -C_{k-n}, \qquad k = 0, 1, \cdots, n-1,$$

for the coefficient matrix T_{n-1} is nonsingular. Likewise, Φ_n^* is *uniquely* determined by (2.14):

(2.18) $$\sum_{i=1}^{n} C_{k-i}\Phi_{ni}^* = -C_k, \qquad k = 1, 2, \cdots, n,$$

which can also be written

(2.19) $$\sum_{i=0}^{n-1} C'_{k-i}\Phi_{n,n-i}^* = -C'_{k-n}, \qquad k = 0, 1, \cdots, n-1.$$

Remark 2.2. In the scalar case ($m = 1$) we have a particularly simple relationship between P_n and P_n^*, namely $\Phi_{ni}^* = \Phi_{n,n-i}$ and $R_n^* = R_n$. In fact, the first relation follows from (2.17) and (2.19) (for $C_i' = C_i$). Then the second relation is obtained by comparing (2.12) and (2.15).

Remark 2.3. Note that R_n and R_n^* are positive definite. In fact, observing that T_n is positive definite, this follows from $R_n = [\Phi_n, \Phi_n]$ and $R_n^* = [\Phi_n^*, \Phi_n^*]$. Clearly, R_n and R_n^* are also symmetric.

3. Difference equations for Φ_n and Φ_n^*.

LEMMA 3.1. *The following equations hold with the initial condition given by* $\Phi_0 = \Phi_0^* = E_0$:

(3.1) $$\Phi_{n+1} = \sigma\Phi_n - \Phi_n^*\Gamma_n^*,$$

(3.2) $$\Phi_{n+1}^* = \Phi_n^* - \sigma\Phi_n\Gamma_n,$$

where Γ_n *and* Γ_n^* *are* $m \times m$ *matrices defined by the following equations*:

(3.3) $$R_n\Gamma_n = (R_n^*\Gamma_n^*)' = S_n,$$

where

(3.4) $$S_n = [\sigma\Phi_n, \Phi_n^*]$$

(3.5) $$= [\sigma\Phi_n, E_0]$$

(3.6) $$= [E_{n+1}, \Phi_n^*].$$

Proof. Let S_n be defined by (3.4). Then

$$S_n = \sum_{k=0}^{n} \Phi'_{nk}[E_{k+1}, \Phi_n^*],$$

which is equal to (3.6) by Lemma 2.2. Also, due to (2.7),

$$S_n = [\sigma\Phi_n, E_0] + \sum_{k=1}^{n} [\Phi_n, E_{k-1}]\Phi_{nk}^*,$$

which, by Lemma 2.1, is equal to (3.5).

To prove (3.1), first observe that Φ_{n+1} can be represented in the following form:

$$\Phi_{n+1} = E_{n+1} + \sum_{k=0}^{n} \sigma^{n-k}\Phi_k^* B_{nk},$$

where B_{nk} are $m \times m$ matrices. (In fact, this equation is equivalent to the following system:

$$\sum_{i=k}^{n} \Phi_{i,i-k}^* B_{ni} = \Phi_{n+1,n-k}, \qquad k = 0, 1, \cdots, n,$$

which can be solved for the B_{ni} matrices, for $\Phi_{i0}^* = I$.) By Lemma 2.2, we have

$$[\sigma^{n-i}\Phi_i^*, \Phi_{n+1}] = [\sigma^{n-i}\Phi_i^*, E_{n+1}] + R_i^* B_{ni} \qquad (0 \leqq i \leqq n).$$

The left member, being equal to $\sum_{k=0}^{i} \Phi_{ik}^{*\prime}[E_{k+n-i}, \Phi_{n+1}]$, is zero by Lemma 2.1, and due to (2.7) the first term on the right side equals S_i' as given by (3.6). Hence, since R_i^* is nonsingular, by (3.3), $B_{ni} = -\Gamma_i^*$. Then form $\Phi_{n+1} - \sigma\Phi_n$ to see that (3.1) holds.

Similarly (3.2) can be proved by considering the representation

$$\Phi_{n+1}^* = E_0 + \sum_{k=0}^{n} \sigma\Phi_k B_{nk}.$$

Then Lemma 2.1 (together with (2.8)) and (3.5) imply

$$[\sigma\Phi_i, \Phi_{n+1}^*] = S_i + R_i B_{ni},$$

the left member of which is zero (Lemma 2.2), and therefore $B_{ni} = -\Gamma_i$. Thus (3.2) holds. This concludes the proof of the lemma.

In the case $m = 1$ (i.e., the process $\{y_n\}$ is scalar) we have $\Phi_{nk}^* = \Phi_{n,n-k}$, $R_n^* = R_n$ (see Remark 2.2) and, consequently, $\Gamma_n^* = \Gamma_n$. The corresponding versions of (3.1) and (3.2) can be found in the theory of orthogonal polynomials (see [1, p. 183] or [2, p. 155]). In the general case ($m > 1$) similar equations can be found in [7], [12], [13]. However, [7] does not contain relation (3.3), and although this relation is mentioned in [12], [14], there is no proof for it.

LEMMA 3.2. *The error covariances R_n and R_n^* satisfy the following difference equations with $R_0 = R_0^* = C_0$:*

$$R_{n+1} = R_n - \Gamma_n^{*\prime} R_n^* \Gamma_n^*, \tag{3.7}$$

$$R_{n+1}^* = R_n^* - \Gamma_n' R_n \Gamma_n \tag{3.8}$$

[*or the uncoupled equations*

$$R_{n+1} = R_n(I - \Gamma_n \Gamma_n^*), \tag{3.9}$$

$$R_{n+1}^* = R_n^*(I - \Gamma_n^* \Gamma_n). \tag{3.10}$$

Also the following relations hold:

$$R_{n+1}\Gamma_n = (R_{n+1}^* \Gamma_n^*)' \tag{3.11}$$

and

$$(R_{n+1})^{-1} = (R_n)^{-1} + \Gamma_n (R_{n+1}^*)^{-1} \Gamma_n', \tag{3.12}$$

$$(R_{n+1}^*)^{-1} = (R_n^*)^{-1} + \Gamma_n^* (R_{n+1})^{-1} \Gamma_n^{*\prime}.]^2 \tag{3.13}$$

Proof. From (3.1) and (3.6) we have

$$[E_{n+1}, \Phi_{n+1}] = [E_{n+1}, \sigma\Phi_n] - S_n \Gamma_n^*$$

which, by (2.7) and Lemma 2.1, is the same as (3.7) or (3.9), depending on which of the two expressions (3.3) for S_n is used. Likewise (3.2) and (3.5) yield

$$[E_0, \Phi_{n+1}^*] = [E_0, \Phi_n^*] - S_n' \Gamma_n,$$

which is the same as (3.8) or (3.10) (Lemma 2.2). To obtain (3.11), postmultiply (3.7) and (3.8) by Γ_n and Γ_n^* respectively, and use (3.3). Finally, from (3.9) we have

$$(R_n)^{-1} = (I - \Gamma_n \Gamma_n^*)(R_{n+1})^{-1}$$

which, by (3.11), is equal to (3.12). (To see this, transpose (3.11), premultiply by $(R_{n+1}^*)^{-1}$ and postmultiply by $(R_{n+1})^{-1}$.) Equation (3.13) is derived in the same way.

4. An algorithm for the gain matrix. We now return to the problem described in § 1. Thus the innovation process (2.1) will be

$$\tilde{y}_n = H(x_n - \hat{x}_n) + w_n. \tag{4.1}$$

Our object is to determine the gain matrix (1.7):

$$K_n = FQ_n(HQ_n + P_2)^{-1}, \tag{4.2}$$

where Q_n, defined by (1.10) and (1.8), can be written

$$Q_n = E\{x_n \tilde{y}_n'\}. \tag{4.3}$$

Here we have first used the orthogonality between $\hat{x}_n$ and $(x_n - \hat{x}_n)$ to obtain $Q_n = E\{x_n(x_n - \hat{x}_n)'\}H'$ and then (4.1) and the fact that x and w are uncorrelated. By a similar argument, we can express the error covariance $R_n = E\{\tilde{y}_n \tilde{y}_n'\}$, defined in § 2, in terms of Q_n:

$$R_n = HQ_n + P_2. \tag{4.4}$$

Of course, (4.2), (4.3) and (4.4) can easily and in a well-known fashion be derived directly, and our reference to the equations in § 1 is merely for the purpose of comparison. Note in particular that we make no use of the Riccati equation (1.9).

[2] The corresponding part of the proof should also be bracketed.

Now, since $E\{v_j x_i'\} = 0$ for $j \geqq i$ and $E\{x_i x_i'\} = P_0$ (stationarity), (1.1) yields

$$E\{x_n x_i'\} = F^{n-i}P_0 \qquad (n \geqq i), \tag{4.5}$$

and therefore, remembering that $y_i = Hx_i + w_i$, we have

$$E\{x_n y_i'\} = F^{n-i}P_0H' \qquad (n \geqq i) \tag{4.6}$$

(for x and w are uncorrelated) and

$$C_i = HF^iP_0H' + P_2\delta_{i0} \qquad (i \geqq 0). \tag{4.7}$$

Inserting (2.1) into (4.3) and observing (4.6), we obtain

$$Q_n = \sum_{i=0}^{n} F^{n-i}P_0H'\Phi_{ni}. \tag{4.8}$$

Then, if we define

$$Q_n^* = \sum_{i=0}^{n} F^{n+1-i}P_0H'\Phi_{ni}^*, \tag{4.9}$$

we can exploit Lemma 3.1 to obtain

$$Q_{n+1} = Q_n - Q_n^*\Gamma_n^*, \tag{4.10}$$

$$Q_{n+1}^* = FQ_n^* - FQ_n\Gamma_n, \tag{4.11}$$

where $Q_0 = P_0H'$ and $Q_0^* = FP_0H'$.

Furthermore, from (3.6) we have

$$S_n = \sum_{i=0}^{n} C_{n+1-i}\Phi_{ni}^*,$$

which by (4.7) and (4.9) equals

$$S_n = HQ_n^*. \tag{4.12}$$

This enables us to determine Γ_n and Γ_n^* from (3.3):

$$\Gamma_n = R_n^{-1}HQ_n^*, \tag{4.13}$$

$$\Gamma_n^* = (R_n^*)^{-1}Q_n^{*\prime}H'. \tag{4.14}$$

By (4.4), R_n can be expressed in terms of Q_n, while for R_n^* we must employ the recursion (3.8) of Lemma 3.2.

Hence we are now in a position to state our main result.

THEOREM 4.1. *The optimal gain matrix for the filter* (1.6) *can be determined in the following way*:

$$K_n = FQ_n(HQ_n + P_2)^{-1}, \tag{4.2}$$

where

$$Q_{n+1} = Q_n - Q_n^*(R_n^*)^{-1}Q_n^{*\prime}H', \tag{4.15}$$

$$Q_{n+1}^* = FQ_n^* - FQ_n(HQ_n + P_2)^{-1}HQ_n^*, \tag{4.16}$$

$$R_{n+1}^* = R_n^* - Q_n^{*\prime}H'(HQ_n + P_2)^{-1}HQ_n^*, \tag{4.17}$$

with initial conditions $Q_0 = P_0H'$, $Q_0^* = FP_0H'$ *and* $R_0^* = HP_0H' + P_2$.

Remark 4.1. Note that Q_n and Q_n^* are $k \times m$ matrices and R_n^* is a symmetric $m \times m$ matrix. Thus we have $2km + [m(m + 1)]/2$ equations to determine Q_n.

[*Remark* 4.2. Since only the inverse of R_n^* is needed, we may replace (4.17) by

$$(4.18)\quad (R_{n+1}^*)^{-1} = (R_n^*)^{-1} + (R_n^*)^{-1}Q_n^{*\prime}H'(HQ_{n+1} + P_2)^{-1}HQ_n^*(R_n^*)^{-1}.$$

To see this, just insert (4.14) and (4.4) into (3.13). Equation (4.18) can also be obtained directly from (4.17) by applying the matrix inversion lemma[3] and noticing that $R_n = HQ_n + P_2$ is given by

$$(4.19)\qquad R_{n+1} = R_n - HQ_n^*(R_n^*)^{-1}Q_n^{*\prime}H'$$

with initial condition $R_0 = HP_0H' + P_2$.]

Remark 4.3. Equations (4.15), (4.16) and (4.17) can also be used for the pure filtering problem to determine the linear least squares estimate of x_n given $\{y_0, y_1, \cdots, y_n\}$. In fact, it is well known that we now have the following filtering equation (which of course is derived without resort to the Riccati equation):

$$(4.20)\qquad \hat{x}_n = F\hat{x}_{n-1} + L_n(y_n - HF\hat{x}_{n-1})$$

with initial condition $\hat{x}_0 = 0$, where the gain L_n is given by

$$(4.21)\qquad L_n = Q_n(HQ_n + P_2)^{-1}.$$

[*Remark* 4.4. We have made an effort to present our algorithm for Q_n in a compact form using as few equations as possible. Equations (4.15), (4.16) and (4.17) contain all the information needed for determining the gain sequences K_n and L_n. However, as usual, a certain judgment has to be exercised in implementing our algorithm. Computational requirements call for minimizing the number of arithmetic operations (see, e.g., [11] for details), and different considerations have to be made for the one-step predictor and for the pure filter. The reader should convince himself that *in general* Table 1 describes the natural implementation of our algorithm (when $m \ll k$), although the number of equations has increased. For example, instead of computing the quantities $R_n = HQ_n + P_2$ from Q_n in each step, amending the projected equation (4.19) = (3.7) (which of course is contained in (4.15)) usually (but not always) reduces the number of arithmetic operations. Also, which is even more important, there should be a minimum of multiplications by the large matrix F. We have introduced some auxiliary variables in addition to the ones defined in the text, $U_n \equiv (R_n)^{-1}$, $U_n^* \equiv (R_n^*)^{-1}$, $\bar{Q}_n \equiv FQ_n$ and $\bar{Q}_n^* \equiv FQ_n^*$ (the last two used for the one-step predictor only). However, it should be noted that special properties of the system's matrices *may* call for some other implementation of the algorithm. For example, with a sparse F (e.g., a companion matrix) the multiplication by F becomes less critical.]

[3] The author would like to thank Prof. I. H. Rowe (among others) for suggesting this. (This remark was communicated to the editor on March 20, 1973.)

TABLE 1

Pure filtering		One-step prediction
$Q_0 = P_0H'$	$Q_0^* = FP_0H'$ $R_0 = HP_0H' + P_2$ $U_0 = U_0^* = R_0^{-1}$	$\bar{Q}_0 = FP_0H'$
$L_n = Q_nU_n$		$K_n = \bar{Q}_nU_n$
	$S_n = HQ_n^*$ $\Gamma_n^* = U_n^*S_n'$	
		$\bar{Q}_n^* = FQ_n^*$
$Q_{n+1} = Q_n - Q_n^*\Gamma_n^*$ $Q_{n+1}^* = F(Q_n^* - L_nS_n)$		$\bar{Q}_{n+1} = \bar{Q}_n - \bar{Q}_n^*\Gamma_n^*$ $Q_{n+1}^* = \bar{Q}_n^* - K_nS_n$
	$R_{n+1} = R_n - S_n\Gamma_n^*$ $U_{n+1} = (R_{n+1})^{-1}$ $U_{n+1}^* = U_n^* + \Gamma_n^*U_{n+1}\Gamma_n^{*\prime}$	

5. The scalar output case. In the case $m = 1$ we have a somewhat simpler situation. Since $R_n^* = R_n$, which is given by (4.4), equation (4.17) now becomes superfluous, and therefore we end up with $2k$ equations. (Also note that S_n is now a scalar.)

We can also write our equations directly in terms of the gain vector k_n without increasing the number of equations[4]

$$k_{n+1} = [1 - (h'k_n^*)^2]^{-1}[k_n - (h'k_n^*)Fk_n^*], \tag{5.1}$$

$$k_{n+1}^* = [1 - (h'k_n^*)^2]^{-1}[Fk_n^* - (h'k_n^*)k_n], \tag{5.2}$$

with initial conditions $k_0 = k_0^* = (h'P_0h + P_2)^{-1}FP_0h$, where we write H as h' to emphasize that it is a vector.

In fact, observe that $k_n = FQ_nR_n^{-1}$. Then define $k_n^* = Q_n^*R_n^{-1}$, from which we have $\Gamma_n = h'k_n^*$. Therefore, (3.8) gives

$$R_{n+1} = [1 - (h'k_n^*)^2]\, R_n,$$

and (4.10) and (4.11) yield the desired result. (Since R_{n+1} and R_n are both positive, so is $[1 - (h'k_n^*)^2]$, and therefore we can safely divide by this quantity.)

[The equations can be simplified at the expense of the "symmetry" by adding $(h'k_n^*)$ times (5.2) to (5.1):

$$k_{n+1} = k_n - (h'k_n^*)k_{n+1}^*, \tag{5.3}$$

$$k_{n+1}^* = [1 - (h'k_n^*)^2]^{-1}[Fk_n^* - (h'k_n^*)k_n], \tag{5.4}$$

but we should remember that computational requirements may call for retaining the original algorithm of § 4.]

Similar equations can be obtained for the pure filtering problem.

[4] [We can obtain similar equations for $m > 1$ if we amend the equations for both R_n and $R_n{}^*$.]

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