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From Finite Covariance Windows to Modeling Filters: A Convex Optimization Approach*

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Abstract. The trigonometric moment problem is a classical moment problem with numerous applications in mathematics, physics, and engineering. The rational covariance extension problem is a constrained version of this problem, with the constraints arising from the physical realizability of the corresponding solutions. Although the maximum entropy method gives one well-known solution, in several applications a wider class of solutions is desired. In a seminal paper, Georgiou derived an existence result for a broad class of models. In this paper, we review the history of this problem, going back to Carathéodory, as well as applications to stochastic systems and signal processing. In particular, we present a convex optimization problem for solving the rational covariance extension problem with degree constraint. Given a partial covariance sequence and the desired zeros of the shaping filter, the poles are uniquely determined from the unique minimum of the corresponding optimization problem. In this way we obtain an algorithm for solving the covariance extension problem, as well as a constructive proof of Georgiou's existence result and his conjecture, a generalized version of which we have recently resolved using geometric methods. We also survey recent related results on constrained Nevanlinna–Pick interpolation in the context of a variational formulation of the general moment problem.

Key words. rational covariance extension, interpolation, partial stochastic realization, trigonometric moment problem, spectral estimation, speech processing, stochastic modeling, general moment problem

AMS subject classifications. 30E05, 42A15, 49N15, 60G35, 62M15, 65K10, 93A30, 93E12

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1. Introduction.

In [16] a solution to the problem of parameterizing all rational extensions of a given window of covariance data was given. This problem has a long history, with antecedents going back to potential theory in the work of Carathéodory

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[19, 20], Toeplitz [72], and Schur [69] and continuing in the work of Kalman [43], Georgiou [28], Kimura [46], and others. It has been of more recent interest due to its significant interface with problems in signal processing and speech processing [24, 18, 56, 44] and in stochastic realization theory and system identification [4, 73, 53]. Indeed, the recent geometric solution to this problem extends a result by Georgiou and confirms one of his conjectures [27, 28] in a stronger form, viz. that the solution exists, is unique, and depends smoothly on the problem data. In other words, the main result of [16] is that the problem of determining a shaping filter from its zeros and a covariance window is well-posed, in the sense of Hadamard. This result has also shed some light on the stochastic (partial) realization problem through the development of an associated Riccati-type equation, whose unique positive semidefinite solution has as its rank the minimum dimension of a stochastic linear realization of the given rational covariance extension [13]. In both its form as a complete parameterization of rational extensions to a given covariance sequence and as an indefinite Riccati-type equation, one of the principal problems which remained open was that of developing effective computational methods for the approximate solution of this problem. In [10], motivated by the effectiveness of interior point methods for solving nonlinear convex optimization problems, we recast the fundamental problem as such an optimization problem. That paper forms the basis of the current paper, which we have expanded to survey recent results and to be more accessible to a wider audience.

In section 2 we describe the history and the principal results for the rational covariance extension problem. We also set the notation we shall need throughout. The only solution to this problem for which there have been simple computational procedures is the so-called *maximum entropy* solution, which is the particular solution that maximizes the entropy gain. In section 3, motivated by ideas from signal processing, we give a recent generalization [6] of the maximal entropy gain to a form that will generate all solutions, and we demonstrate that the infinite-dimensional optimization problem for determining such a solution has a simple finite-dimensional dual. This motivates the introduction in section 4 of a nonlinear, strictly convex functional defined on a closed convex set and naturally related to the covariance extension problem. We first show that any solution of the rational covariance extension problem lies in the interior of this convex set and that, conversely, an interior minimum of this convex functional will correspond to the unique solution of the covariance extension problem.

Concerning the existence of a minimum, we show that this functional is proper and bounded from below, i.e., that the sublevel sets are compact. From this, it follows that there exists a minimum. Since uniqueness of the minimum follows from strict convexity of the functional, the central issue which needs to be addressed in order to solve the rational covariance extension problem is whether, in fact, this minimum is an interior point. Indeed, our formulation of the convex functional, which contains a barrier-like term, was inspired by interior point methods. However, in contrast to interior point methods, the barrier function we have introduced does not become infinite on the boundary of our closed convex set. Nonetheless, we are able to show that the gradient, rather than the value, of the convex functional becomes infinite on the boundary. The existence of an interior point which minimizes the functional then follows from this observation.

Our interest in this convex optimization problem is, in fact, twofold: as a starting point for the computation of an explicit solution and as a means of providing an alternative proof of the rational covariance extension theorem. In section 5, we include a new construction [15] of a closed 1-form from the geometric formulation of the rational covariance extension problem and show that it must be exact. We then use the resulting path integral to give an alternative derivation of the dual of the primal problem given by the generalization of the maximal entropy gain in section 3. We thereby prove that the convex minimization problem is well-posed in the sense of Hadamard and then apply these convex minimization techniques to obtain well-posedness of the rational covariance extension problem.

In section 6, we report some computational results and present some simulations. In section 2, we note that the rational covariance extension problem is a classical moment problem, with some complexity constraints. In section 7, motivated by [10] and the recent convex optimization results [8] for Nevanlinna–Pick interpolation with degree constraints, we show how to obtain a convex functional for the generalized moment problem. Following [15], the functional is constructed as a path integral of an exact 1-form derived from the general moment problem, specializing to the 1-form in section 5 in the case of the trigonometric moment problem. We conclude with a discussion of the results and applications of [8, 9] on the constrained Nevanlinna–Pick interpolation problem.

2. The Rational Covariance Extension Problem. It is well known that the spectral density Φ of a purely nondeterministic, second-order, stationary random process $\{y(t)\}$ with zero mean is given by the Fourier expansion

$$\Phi(e^{i\theta}) = \sum_{-\infty}^{\infty} c_k e^{ik\theta} \tag{2.1}$$

on the unit circle, where the covariance lags

$$c_k = \mathrm{E}\{y_{t+k} y_t\}, \quad k = 0, 1, 2, \ldots, \tag{2.2}$$

are the Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta. \tag{2.3}$$

In spectral estimation [18], identification [4, 53, 73, 22, 57], speech processing [24, 56, 58, 66], and several other applications in signal processing and systems and control, we are faced with the inverse problem of finding a spectral density which is *coercive*, i.e., positive on the unit circle, given only

$$c = (c_0, c_1, \ldots, c_n), \tag{2.4}$$

which is a *partial covariance sequence* positive in the sense that the Toeplitz matrix

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} \tag{2.5}$$

is positive definite.

In fact, the covariance lags (2.2) are usually estimated from an approximation

$$\frac{1}{N-k+1} \sum_{t=0}^{N-k} y_{t+k} y_t$$

of the ergodic limit

$$c_k = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} y_{t+k} y_t,$$

since only a finite string

$$y_0, y_1, y_2, y_3, \dots, y_N \tag{2.6}$$

of observations of the process $\{y(t)\}$ is available, and therefore we can only estimate a finite partial covariance (2.4), where $n << N$.

The corresponding inverse problem is a version of the *trigonometric moment problem* [1, 34]: Given a sequence (2.4) of real numbers satisfying the positivity condition $T_n > 0$, find a coercive spectral density Φ such that (2.3) is satisfied for $k = 0, 1, 2, \dots, n$. Of course there are infinitely many such solutions, and we shall shortly specify some additional properties which we would like the solution to have.

The trigonometric moment problem, as stated above, is equivalent to the *Carathéodory extension problem* to determine an extension

$$c_{n+1}, c_{n+2}, c_{n+3}, \cdots \tag{2.7}$$

with the property that the function

$$f(z) = \frac{1}{2}c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots \tag{2.8}$$

is *strictly positive real*, i.e., is analytic in the complement of the open unit disc (so that the Laurent expansion (2.8) holds for all $|z| \geq 1$) and satisfies

$$f(z) + f(z^{-1}) > 0 \quad \text{on the unit circle.} \tag{2.9}$$

In fact, given such an f,

$$\Phi(z) = f(z) + f(z^{-1}) \tag{2.10}$$

is a solution to the trigonometric moment problem. Conversely, any coercive spectral density Φ uniquely defines a strictly positive real function f via (2.10). For later reference, we note that a rational strictly positive real function is always *stable* and *minimum-phase*, i.e., both its poles and its zeros are located in the open unit disc. In fact, if f is strictly positive real, then so is $1/f$.

These problems are classical and go back to Carathéodory [19, 20], Toeplitz [72], and Schur [69] at the beginning of the 20th century. In fact, Schur parameterized all solutions in terms of what are now known as the *Schur parameters* or, more commonly in the circuits and systems literature, as the *reflection coefficients*, and which are easily determined from the covariance lags via the Levinson algorithm [65]. More precisely, modulo the choice of c_0, there is a one-to-one correspondence between infinite covariance sequences $c_0, c_1, c_2, \dots$ and Schur parameters $\gamma_0, \gamma_1, \dots$ such that

$$|\gamma_t| < 1 \quad \text{for } t = 0, 1, 2, \dots, \tag{2.11}$$

under which partial sequences (2.4) correspond to partial sequences $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ of Schur parameters. Therefore, covariance extension (2.7) amounts precisely to finding a continuation

$$\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \cdots \tag{2.12}$$

of Schur parameters satisfying (2.11). Each such solution is only guaranteed to yield an f which is meromorphic.

In the first half of the 20th century, positive real functions were also intensively studied in circuit theory as the driving point impedance of RLC circuits [36] where, however, these functions are rational with degree equal to the number of active components, i.e., inductors and capacitors. Generalizing these observations, rationality of a transfer function became a key concept in the development of finite-dimensional models (or realizations) in systems theory in the 1950s and 1960s. By 1968, Kalman [40] formulated the (deterministic) partial realization problem, which was to describe all rational functions of a bounded (or of minimal) degree which match a given window of Laurent coefficients.

In contrast, the rational covariance extension problem asks for a rational solution, of bounded degree, which is also positive real. Indeed, suppressing the rationality leads to the classically solved Carathéodory extension problem, while suppressing positivity leads to the deterministic partial realization problem, a now classical solved problem in mathematical systems theory [40, 41, 42, 67, 33]. The simultaneous imposition of rationality and positivity leads to a nontrivial and highly nonlinear problem, since generally there is no method to see which choices of free Schur parameters will yield rational solutions of at most degree n. Indeed, this amounts to explicitly describing a $2n$-dimensional submanifold of the Hilbert cube!

In a seminal paper in 1980, Kalman [43] formulated the problem of parameterizing all such filters in terms of finding solutions of the partial stochastic realization problem. Indeed, a choice of a rational positive real function f of at most degree n is equivalent to a choice of a rational spectral density Φ of at most degree $2n$. If f matches the covariance window, then the unique rational, stable, minimum-phase function w having the same degree as f and satisfying

$$w(z)w(z^{-1}) = \Phi(z) \tag{2.13}$$

in some annulus containing the unit circle is the transfer function of a *shaping filter*, which shapes white noise into a random process with the first $n + 1$ covariance lags given by (2.4); see, e.g., [16, 13] for more details. In this way, the problem of parameterizing all such shaping filters yields all solutions of the partial stochastic realization problem. In 1982, it was recognized [24] that existing methods of designing such shaping filters correspond to certain choices in the classical moment problem of Carathéodory.

One such choice leads to a particular solution which is ubiquitous because of its simplicity and ease of computation. Setting all free Schur parameters (2.12) equal to zero, which clearly satisfies the positivity condition (2.11), yields a rational solution

$$\Phi(z) = \frac{1}{a(z)a(z^{-1})}, \tag{2.14}$$

where $a(z)$ is a polynomial given by

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0 > 0), \tag{2.15}$$

which is easily computed via the Levinson algorithm [65]. This so-called *maximum entropy solution* is an all-pole or AR solution, and the corresponding shaping filter

$$w(z) = \frac{z^n}{a(z)} \tag{2.16}$$

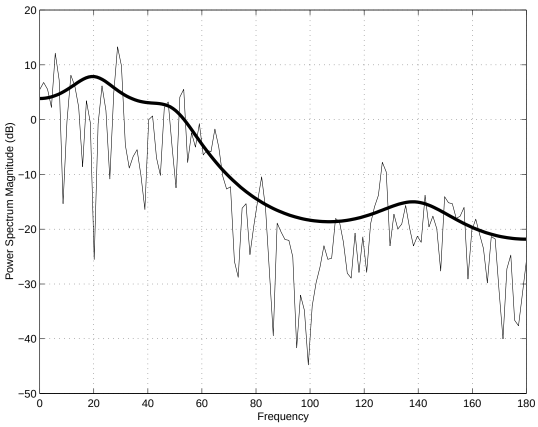

Fig. 2.1 *Spectral envelope of a maximum entropy solution corresponding to* $n = 6$.

has all its zeros at the origin. The term "maximum entropy" will be explained in section 3.

However, in many applications a wider variety in the choice of zeros in the spectral density Φ is required. To illustrate this point, consider in Figure 2.1 a spectral density in the form of a periodogram determined from a speech signal sampled over 30 milliseconds (in which time interval it represents a stationary process) together with a maximum entropy solution corresponding to $n = 6$. As can be seen, the latter yields a rather flat spectrum which is unable to approximate the valleys, or the "notches," in the speech spectrum. For this reason, in speech synthesis the maximum entropy solution is known to result in "machine" speech which can sound flat. This is a manifestation of the fact that all the zeros of the maximum entropy filter (2.16) are located at the origin and thus do not give rise to a frequency where the power spectrum is small. For this reason, by the 1970s it was widely appreciated in the signal and speech processing community that very high quality regeneration of human speech required the design of filters having nontrivial zeros [5, p. 1726], [58, pp. 271–272], [66, pp. 76–78]. Indeed, while all-pole filters can reproduce many human speech sounds, the acoustic theory teaches that nasals and fricatives require both zeros and poles [58, pp. 271–272], [66, p. 105]. In fact, the 30 ms speech segment used in producing Figure 2.1 is acquired during the formation of the voiced nasal [ng]. For such a signal, even a maximum entropy solution of degree as high as 20 will fail to model the deep valley in the spectrum, as seen in Figure 2.2.

Therefore, one is interested in shaping filters

$$w(z) = \frac{\sigma(z)}{a(z)}, \tag{2.17}$$

for which (2.15) and

$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n \tag{2.18}$$

are *Schur polynomials*, i.e., polynomials with all roots in the open unit disc. While the maximum entropy solution corresponds to the default choice $\sigma(z) = z^n$, we are

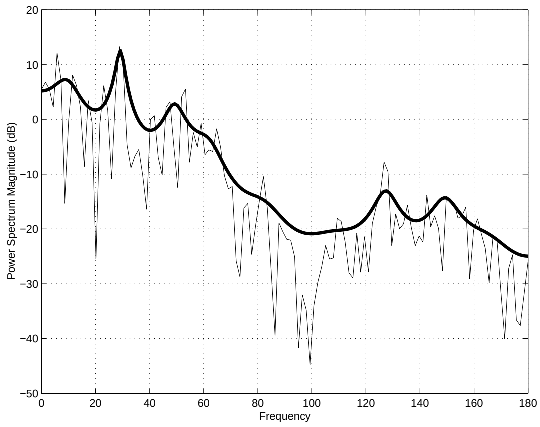

Fig. 2.2 *Spectral envelope of a maximum entropy solution corresponding to* $n = 20$.

particularly interested in how much flexibility there is in the choice of zeros and poles while matching the covariance window.

Not surprisingly, the first breakthrough, after the maximum entropy method, used nonlinear methods. In 1983, using degree theory [27] (see also [28]), Georgiou proved that for any prescribed zero polynomial $\sigma(z)$ there exists a shaping filter w and conjectured that this correspondence would yield a complete parameterization of all rational solutions of at most degree n, i.e., that the correspondence between f and a choice of positive sequence (2.4) and a choice of Schur polynomial (2.18) would be a bijection.

A decade later, this long-standing conjecture was generalized and resolved in [16]. The conjecture was generalized by insisting, following Hadamard, that the problem be *well-posed*; i.e., that for each σ a shaping filter w exists, is unique, and depends smoothly on the coefficients of $\sigma(z)$ and on the covariance window. This conjecture was then verified by proving the following theorem as a corollary of a more general theorem asserting that fixing σ and fixing the covariance window, respectively, define the leaves of two foliations on the space of all rational positive real functions of degree at most n, and that these foliations are complementary. This enables one to refine the degree theoretic calculations to see that the correspondence between σ and w is an analytic diffeomorphism.

THEOREM 2.1 (see [16]). *Given any partial covariance sequence* (2.4) *and Schur polynomial* (2.18), *there exists a unique Schur polynomial* (2.15) *such that* (2.17) *is a minimum-phase spectral factor of a spectral density* Φ *satisfying*

$$\Phi(z) = f_0 + \sum_{k=1}^{\infty} f_k(z^k + z^{-k})$$

in some annulus containing the unit circle, where

$$f_k = c_k \quad \textit{for} \quad i = 0, 1, \ldots, n.$$

In particular, the solutions of the rational positive extension problem are in one-to-one correspondence with self-conjugate sets of n points (counted with multiplicity) lying in *the open unit disc, i.e., with all possible zero structures of shaping filters. Moreover, this correspondence is bianalytic.*

Since this "zero-assignability" theorem appeared, there have been several enhancements and simplifications. In [13] we connected this solution to a certain Riccati-type matrix equation that sheds further light on the structure of the partial stochastic realization problem. An alternative proof of the theorem, based on Hadamard's global inverse function theorem, was given in [12]. A simplified proof that this correspondence is a homeomorphism was developed in [11].

All these proofs are nonconstructive. The first constructive algorithm which, given the partial covariance sequence (2.4) and the desired zero polynomial (2.18), computes the unique pole polynomial (2.15) was given in [10], on which this paper is based. Since then, several applications of convex optimization were developed for problems involving interpolation by classes of rational functions with a degree constraint [6, 8]. Extensions of both the geometric and the optimization approaches to the rational Nevanlinna–Pick interpolation problem will be discussed in section 7 in the context of the general moment problem.

In [10] the convex optimization problem was to minimize the value of the function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{\infty\}$, defined by

$$\begin{aligned}\varphi(q_0, q_1, \dots, q_n) = {} & c_0 q_0 + c_1 q_1 + \cdots + c_n q_n \\ & - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log Q(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta\end{aligned} \tag{2.19}$$

over all $q_0, q_1, \dots, q_n$ such that

$$Q(e^{i\theta}) = q_0 + q_1 \cos\theta + q_2 \cos 2\theta + \cdots + q_n \cos n\theta \geq 0 \quad \text{for all } \theta. \tag{2.20}$$

Using this convex optimization problem, a sixth-degree shaping filter with zeros at the appropriate frequencies can be constructed for the speech segment represented by the periodogram of Figure 2.1. In fact, Figure 2.3 illustrates the same periodogram together with the spectral density of such a filter. As can be seen, this filter yields a much better description of the notches than does the maximum entropy filter.

3. The Generalized Maximum Entropy Problem. Before turning to the main topic of this paper, the convex optimization problem for solving the rational covariance extension problem for arbitrarily assigned zeros, we shall provide a motivation for this approach in terms of entropy maximization.

This circle of ideas is closely connected to prediction theory. Given the stationary random process $\{y(t)\}$ defined above, let $\hat{y}(t)$ be the one-step predictor, i.e., the linear least-squares estimate of $y(t)$ given $y(0), y(1), \dots, y(t-1)$. It is well known and follows from the derivation of the Levinson algorithm that the square of the prediction error, i.e., $r_t = \mathrm{E}\{|y(t) - \hat{y}(t)|^2\}$, can be expressed in terms of the Schur parameters via the recursion

$$r_{k+1} = r_k(1 - \gamma_k^2), \quad r_0 = c_0.$$

Here we shall be interested in the steady-state prediction error

$$r_\infty = c_0 \prod_{k=0}^{\infty} (1 - \gamma_k^2). \tag{3.1}$$

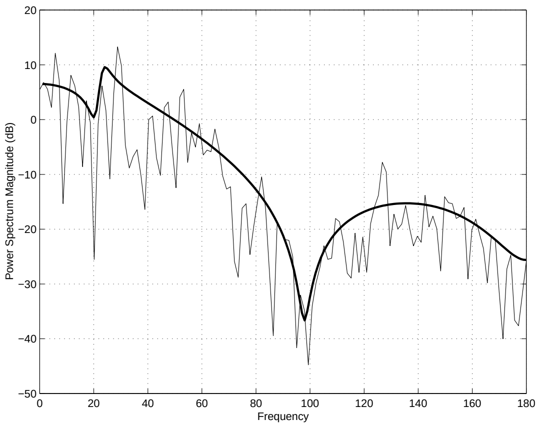

Fig. 2.3 *Spectral envelope of degree $n = 6$ obtained with appropriate choice of zeros.*

On the other hand, it is well known (see, e.g., [35, p. 74]) that this prediction error can also be expressed by the Szegö formula

$$r_\infty = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\Phi(e^{i\theta})d\theta\right\}. \tag{3.2}$$

In this context, the maximum entropy extension discussed in section 2 is the "most random" solution to the covariance extension problem in the sense that it maximizes the prediction error r_∞. In fact, from (3.1) we see that

$$\log r_\infty = \log c_0 + \log\prod_{k=0}^{n-1}(1-\gamma_k^2) + \log\prod_{k=n}^{\infty}(1-\gamma_k^2),$$

where the first two terms are fixed, and the last term will depend on the particular choice of extension (2.12). This variable term is always nonpositive and attains its maximal value zero when choosing the free Schur parameter equal to zero. Hence the maximum entropy solution is the extension which maximizes $\log r_\infty$ or, equivalently,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\Phi(e^{i\theta})d\theta, \tag{3.3}$$

as can be seen from (3.2).

More generally, by Theorem 2.1 there is exactly one solution for each choice of numerator polynomial $\sigma(z)$. We claim that this solution can be determined by maximizing the functional

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\Phi(e^{i\theta})\left|\sigma(e^{i\theta})\right|^2 d\theta. \tag{3.4}$$

As it turns out, there are compelling reasons for this choice from a signal processing perspective. Indeed, another way of representing the distribution of the stationary process is via the so-called *cepstrum*

$$\log\Phi(e^{i\theta}) = v_0 + \sum_{k=1}^{\infty} v_k\left(e^{ik\theta}+e^{-ik\theta}\right). \tag{3.5}$$

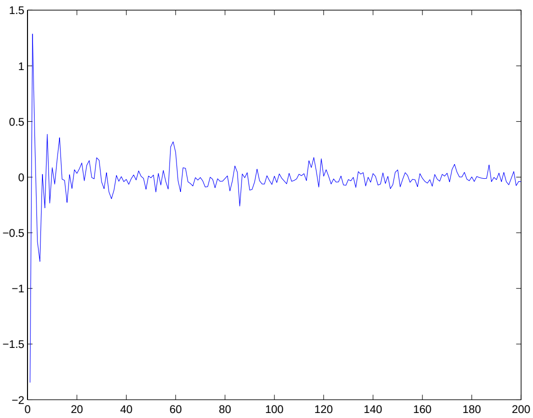

Fig. 3.1 *Cepstrum of voice speech signal.*

The Fourier coefficients

$$v_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log\Phi(e^{i\theta})d\theta \tag{3.6}$$

are known as the *cepstral coefficients* [64]. The basic observation which motivates this construct is the nature of the frequency response of a shaping filter driven by an excitation signal. As the Fourier transform of a convolution, the contributions of the shaping filter and the excitation signal to the spectral estimate are multiplicative. On the other hand, if we consider the cepstrum, the contribution of the excitation signal is additively superimposed on that of the shaping filter.

For example, Figure 3.1 shows the estimated cepstral coefficients of a frame of voiced speech. A contribution of the excitation signal is seen as spikes at multiples of the pitch period, corresponding to approximately $n_0 = 57$ in Figure 3.1. The spectral envelope can be estimated from a finite window

$$v_0, v_1, \dots, v_n \tag{3.7}$$

of cepstral coefficients, where $n < n_0$.

In this context, the entropy gain (3.3) is precisely the zeroth cepstral coefficient

$$v_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\Phi(e^{i\theta})d\theta.$$

However, in cepstral analysis, one is interested not only in v_0 but in a finite window (3.7) of cepstral coefficients. It is therefore natural to maximize instead a "positive" linear combination

$$p_0v_0 + p_1v_1 + \dots + p_nv_n \tag{3.8}$$

of the cepstral coefficients in the window (3.7). In view of (3.6), this may be written as a generalized entropy gain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\log\Phi(e^{i\theta})d\theta, \tag{3.9}$$

where P is the symmetric pseudopolynomial

$$P(z) = p_0 + \frac{1}{2}p_1(z + z^{-1}) + \cdots + \frac{1}{2}p_n(z^n + z^{-n}), \tag{3.10}$$

which we take to be positive definite on the unit circle, following [6, 7]. In particular, we see that if we take P to be

$$P(z) = \sigma(z)\sigma(z^{-1}), \tag{3.11}$$

we obtain (3.4). We note that for $\sigma(z) = z^n$, the numerator polynomial of the maximum entropy solution, the generalized entropy gain (3.4) reduces to the entropy gain (3.3).

To formulate a constrained optimization problem, we begin by setting up the appropriate spaces. To this end, first recall that

$$\Phi(e^{i\theta}) = \frac{1}{2}\left[f(e^{i\theta}) + f(e^{-i\theta})\right] = \mathrm{Re}\{f(e^{i\theta})\}, \tag{3.12}$$

where the (proper, rational) positive real function

$$f(z) = \frac{1}{2}f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots$$

is analytic and bounded in the complement of the open unit disc and hence belongs to the Hardy space H^∞ with respect to this region. We denote by $\mathcal{C}$ the Carathéodory class of real functions in H^∞ that take values with a nonnegative real part in the complement of the closed unit disc. Let $\mathcal{C}_+$ be the subclass of all $f \in \mathcal{C}$ that are coercive in the sense that, for some $\epsilon > 0$, $\mathrm{Re}\{f(e^{i\theta})\} \geq \epsilon$ for all $\theta \in [-\pi, \pi]$. Then $f \in \mathcal{C}_+$ if and only if $f^{-1} \in \mathcal{C}_+$.

Then introducing the functional $\psi : \mathcal{C}_+ \to \mathbb{R}$, defined by

$$\psi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\mathrm{Re}\, f(e^{i\theta})\right] \left|\sigma(e^{i\theta})\right|^2 d\theta, \tag{3.13}$$

we consider the *relaxed* constrained optimization problem

$$\max_{f \in \mathcal{C}_+} \psi(f) \tag{3.14}$$

subject to the constraints

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\mathrm{Re}\left\{f(e^{i\theta})\right\} d\theta = c_k \quad \text{for} \quad i = 0, 1, \ldots, n. \tag{3.15}$$

This optimization problem is relaxed in that the class $\mathcal{C}_+$ contains not only rational functions of degree at most n, but also rational functions of higher degree as well as nonrational functions. As it turns out, by analyzing the dual problem, one can see that the optimal solution is rational of degree at most n. In fact, we have the following theorem, which implies the result reported in [10] and is a variation of more recent applications of convex optimization to interpolation problems. (See [6, Theorem 3.3] and [8, Theorem 4.1].)

THEOREM 3.1. *The optimization problem to maximize* (3.13) *over* $\mathcal{C}_+$ *subject to the constraints* (3.15) *has a unique optimal solution* $\hat{f} \in \mathcal{C}_+$*, and it is rational of degree less than or equal to* n*. The corresponding shaping filter is given by*

$$w(z) = \frac{\sigma(z)}{a(z)},$$

where

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0 > 0) \tag{3.16}$$

is the unique stable polynomial (all roots in the open unit disc) satisfying

$$Re\,\hat{f}(e^{i\theta}) = \frac{\left|\sigma(e^{i\theta})\right|^2}{\left|a(e^{i\theta})\right|^2}. \tag{3.17}$$

This is an optimization problem in an infinite-dimensional space. The number of constraints, however, is finite, namely $n+1$, so the dual problem (in the sense of mathematical programming) should be a convex optimization problem with $n+1$ variables [59]. The dual problem will also be the key to proving Theorem 3.1. The following construction is a generalization of that in [10].

Duality theory amounts to forming the Lagrangian

$$L(f,q) = \psi(f) + \sum_{k=0}^{n} q_k \left[c_k - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \mathrm{Re}\left\{ f(e^{i\theta}) \right\} d\theta \right]$$

and determining the Lagrange multipliers $q \in \mathbb{R}^{n+1}$ by minimizing the dual functional

$$\rho(q) = \sup_{f \in \mathcal{C}_+} L(f,q).$$

Introducing the pseudopolynomial

$$Q(z) = q_0 + \frac{1}{2} q_1 (z + z^{-1}) + \cdots + \frac{1}{2} q_n (z^n + z^{-n}), \tag{3.18}$$

we can write the Lagrangian as

$$L(f,q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left[\mathrm{Re}\, f(e^{i\theta})\right] \left|\sigma(e^{i\theta})\right|^2 d\theta + c^{\mathsf{T}} q - \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(e^{i\theta}) \left[\mathrm{Re}\, f(e^{i\theta})\right] d\theta. \tag{3.19}$$

Clearly, $\rho(q) < \infty$ only if

$$Q(e^{i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi]. \tag{3.20}$$

We shall denote by $\mathcal{D}_n^+$ the class of all $q \in \mathbb{R}^{n+1}$ satisfying the positivity condition (3.20). If the function $f \mapsto L(f,q)$ has a maximum in the open subset $\mathcal{C}_+$ of H^∞, then

$$\frac{\partial L}{\partial f_k} = 0, \quad k = 0, 1, 2, \ldots,$$

in the maximizing point. This stationarity condition can be written

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \left[\frac{\left|\sigma(e^{i\theta})\right|^2}{\mathrm{Re}\left\{f(e^{i\theta})\right\}} - Q(e^{i\theta}) \right] d\theta = 0, \quad k = 0, 1, 2, \ldots,$$

which is satisfied if and only if

$$\mathrm{Re}\, f(e^{i\theta}) = \frac{\left|\sigma(e^{i\theta})\right|^2}{Q(e^{i\theta})} \tag{3.21}$$

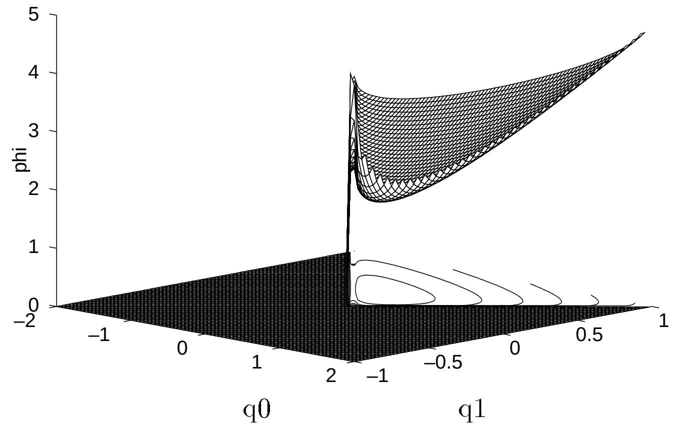

Fig. 3.2 *A typical cost function $\varphi(q)$ in the case $n = 1$.*

or, equivalently,

$$f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{\left|\sigma(e^{i\theta})\right|^2}{Q(e^{i\theta})}d\theta \tag{3.22}$$

holds. Inserting this into (3.19) yields the dual functional

$$\rho(q) = \varphi(q) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sigma(e^{i\theta})\right|^2\left[\log\left|\sigma(e^{i\theta})\right|^2 - 1\right]d\theta, \tag{3.23}$$

where

$$\varphi(q) = c_0q_0 + c_1q_1 + \cdots + c_nq_n - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sigma(e^{i\theta})\right|^2\log Q(e^{i\theta})d\theta. \tag{3.24}$$

Since the last term in (3.23) does not depend on q, we shall call the optimization problem to minimize $\varphi(q)$ over all q in the closure $\overline{\mathcal{D}_n^+}$ of $\mathcal{D}_n^+$, i.e.,

$$\min_{q\in\overline{\mathcal{D}_n^+}}\varphi(q), \tag{3.25}$$

the *dual problem.* The functional (3.24) is strictly convex, and therefore the minimum is unique, provided one exists. The dual problem is a finite-dimensional convex optimization problem, which is simpler than the original (primal) problem. Figure 3.2 depicts a typical cost function φ in the case $n = 1$. As can be seen, it attains its optimum in an interior point. In fact, the following theorem will be proven in section 4.

THEOREM 3.2. *The dual problem has a unique solution, and it belongs to $\mathcal{D}_n^+$.*

Given Theorem 3.2, we are now in a position to prove Theorem 3.1. To this end, let $\hat{q} \in \mathcal{D}_n^+$ be the unique solution to the dual problem, let $\hat{Q}$ be the corresponding pseudopolynomial (3.18), and let

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{\left|\sigma(e^{i\theta})\right|^2}{\hat{Q}(e^{i\theta})}d\theta.$$

Moreover, let $a(z)$ be the unique stable polynomial (3.16) satisfying

$$a(z)a(z^{-1}) = \hat{Q}(z). \tag{3.26}$$

Then

$$\operatorname{Re} \hat{f}(e^{i\theta}) = \hat{f}_0 + \sum_{k=1}^{\infty} \hat{f}_k \left(e^{ik\theta} + e^{-ik\theta}\right) = \frac{\left|\sigma(e^{i\theta})\right|^2}{\left|a(e^{i\theta})\right|^2},$$

which is precisely (3.17). Clearly, $\hat{f} \in \mathcal{C}_+$. Since $\hat{q}$ is an interior point,

$$\frac{\partial \varphi}{\partial q_k} = c_k - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \frac{\left|\sigma(e^{i\theta})\right|^2}{Q(e^{i\theta})} d\theta \tag{3.27}$$

equals zero at $Q = \hat{Q}$ for $k = 0, 1, \dots, n$. Consequently, the covariance matching condition (3.15) is fulfilled for $f = \hat{f}$, and therefore $\psi(\hat{f}) = L(\hat{f}, \hat{q})$. However, by the construction above,

$$L(\hat{f}, \hat{q}) = \sup_{f \in \mathcal{C}_+} L(f, \hat{q}) \geq L(f, \hat{q})$$

for all $f \in \mathcal{C}_+$. Then, for any $f \in \mathcal{C}_+$ which satisfies the covariance matching condition (3.15),

$$\psi(f) = L(f, \hat{q}) \leq \psi(\hat{f}),$$

which establishes the optimality of $\hat{f}$.

4. Interior Critical Points and Solutions of the Rational Covariance Extension Problem. Consider the dual functional $\varphi : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{\infty\}$ defined by

$$\varphi(q) = c^{\mathsf{T}} q - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log Q(e^{i\theta}) |\sigma(e^{i\theta})|^2 d\theta. \tag{4.1}$$

An interesting aspect of the functional comes in part from a barrier-like integral, which is analogous to the barrier terms arising in interior point methods. As it turns out, by a theorem of Szegö the logarithmic integrand is in fact integrable for nonzero Q having zeros on the boundary of the unit circle, so that $\varphi(q)$ does not become infinite on the boundary of the convex set $\mathcal{D}_n^+$. However, its gradient is infinite on the boundary. As we shall see, from this property it follows that minimizing (4.1) yields precisely, via

$$Q(z) = a(z)a(z^{-1}), \tag{4.2}$$

the unique $a(z)$ which corresponds to $\sigma(z)$. We begin with the existence and uniqueness of a minimum.

PROPOSITION 4.1. *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, the functional φ has a unique minimum on $\overline{\mathcal{D}_n^+}$.*

Proof. We shall show that φ has compact sublevel sets in $\overline{\mathcal{D}_n^+}$, so that φ achieves a minimum. Since φ is strictly convex and $\overline{\mathcal{D}_n^+}$ is convex, it follows that such a minimum is unique.

It is clear that if $q \in \mathcal{D}_n^+$, then $\varphi(q)$ is finite. Moreover, $\varphi(q)$ is also finite when Q has finitely many zeros on the unit circle, as can be seen from the following lemma.

LEMMA 4.2. *The functional φ is finite and continuous at any $q \in \overline{\mathcal{D}_n^+}$ except at zero. The functional is infinite, but continuous, at $q = 0$. Moreover, φ is a C^∞ function on $\mathcal{D}_n^+$.*

Proof. We want to prove that $\varphi(q)$ is finite when $q \neq 0$. Then the rest follows by inspection. Clearly, $\varphi(q)$ cannot take the value $-\infty$; hence, it remains to prove that $\varphi(q) < \infty$. Since $q \neq 0$,

$$\mu := \max_\theta Q(e^{i\theta}) > 0.$$

Then setting $R(z) := \mu^{-1}Q(z)$,

$$\log R(e^{i\theta}) \leq 0 \tag{4.3}$$

and

$$\varphi(q) = c^{\mathsf{T}}q - \frac{1}{2\pi}\log\mu\int_{-\pi}^{\pi}|\sigma(e^{i\theta})|^2 d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log R(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta,$$

and hence the question of whether $\varphi(q) < \infty$ is reduced to determining whether

$$-\int_{-\pi}^{\pi}\log R(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta < \infty.$$

However, since $|\sigma(e^{i\theta})|^2 \leq M$ for some bound M, this follows from

$$\int_{-\pi}^{\pi}\log R(e^{i\theta})d\theta > -\infty, \tag{4.4}$$

which is the well-known Szegö condition: (4.4) is a necessary and sufficient condition for R to have a stable spectral factor [35]. However, since P is a symmetric pseudopolynomial which is nonnegative on the unit circle, there is a polynomial $\pi(z)$ such that $\pi(z)\pi(z^{-1}) = R(z)$. But then $w(z) = \frac{\pi(z)}{z^n}$ is a stable spectral factor, and hence (4.4) holds. ☐

LEMMA 4.3. *The functional φ is strictly convex and defined on a closed, convex domain.*

Proof. We first note that $q = 0$ is an extreme point, but it can never be a minimum of φ since $\varphi(0)$ is infinite. In particular, in order to check the strict inequality

$$\varphi(\lambda q^{(1)} + (1-\lambda)q^{(2)}) < \lambda\varphi(q^{(1)}) + (1-\lambda)\varphi(q^{(2)}), \tag{4.5}$$

where one of the arguments is zero, we need only consider the case that either $q^{(1)}$ or $q^{(2)}$ is zero, in which case the strict inequality holds. We can now assume that none of the arguments is zero, in which case the strict inequality in (4.5) follows from the strict concavity of the logarithm. Finally, it is clear that $\overline{\mathcal{D}_n^+}$ is a closed convex subset. ☐

LEMMA 4.4. *Let $q \in \overline{\mathcal{D}_n^+}$, and suppose $q \neq 0$. Then $c^{\mathsf{T}}q > 0$.*

Proof. Consider an arbitrary covariance extension of c such as, for example, the maximum entropy extension, and let Φ be the corresponding spectral density (2.10). Then c is given by (2.3), which may also be written

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}(e^{ik\theta} + e^{-ik\theta})\Phi(e^{i\theta})d\theta, \quad k = 0, 1, \ldots, n.$$

Therefore, in view of (3.18),

$$c^{\mathsf{T}} q = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(e^{ik\theta}) \Phi(e^{i\theta}) d\theta, \tag{4.6}$$

which is positive whenever $Q(z) \geq 0$ on the unit circle and $q \neq 0$. □

REMARK 4.5. *The condition* $c^{\mathsf{T}} q > 0$ *is a direct consequence of the fact that the Toeplitz matrix* T_n, *defined by* (2.5), *is positive definite. In fact,*

$$c^{\mathsf{T}} q = a^{\mathsf{T}} T_n a,$$

where $a := (a_0, a_1, \ldots, a_n)^{\mathsf{T}}$ *is the vector of coefficients of the stable polynomial factor* (3.16) *in* (4.2). *To see this, merely replace* $Q(z)$ *by* $a(z)a(z^{-1})$ *in* (4.6).

PROPOSITION 4.6. *For all* $r \in \mathbb{R}$, $\varphi^{-1}(-\infty, r]$ *is compact. Thus* φ *is proper (i.e.,* $\varphi^{-1}(K)$ *is compact whenever* K *is compact) and bounded from below.*

Proof. Suppose $q^{(k)}$ is a sequence in $M_r := \varphi^{-1}(-\infty, r]$. It suffices to show that $q^{(k)}$ has a convergent subsequence. Each $Q^{(k)}$ may be factored as

$$Q^{(k)}(z) = \lambda_k \bar{a}^{(k)}(z) \bar{a}^{(k)}(z^{-1}) = \lambda_k \bar{Q}^{(k)}(z),$$

where λ_k is positive and $\bar{a}^{(k)}(z)$ is a monic polynomial, all of whose roots lie in the closed unit disc. The corresponding sequence of the (unordered) set of n roots of each $\bar{a}^{(k)}(z)$ has a convergent subsequence, since all (unordered) sets of roots lie in the closed unit disc. Denote by $\bar{a}(z)$ the monic polynomial of degree n which vanishes at this limit set of roots. By reordering the sequence if necessary, we may assume the sequence $a^{(k)}(z)$ tends to $\bar{a}(z)$. Therefore, the sequence $q^{(k)}$ has a convergent subsequence if and only if the sequence λ_k does, which will be the case provided the sequence λ_k is bounded from above and from below away from zero. Before proving this, we note that the sequences $c^{\mathsf{T}} \bar{q}^{(k)}$, where $\bar{q}^{(k)}$ is the vector corresponding to the pseudopolynomial $\bar{Q}^{(k)}$, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \bar{Q}^{(k)}(e^{i\theta}) |\sigma(e^{i\theta})|^2 d\theta \tag{4.7}$$

are both bounded from above and from below, respectively, away from zero and $-\infty$. The upper bounds come from the fact that $\{\bar{a}^{(k)}(z)\}$ are Schur polynomials and hence have their coefficients in the bounded Schur region. As for the lower bound of $c^{\mathsf{T}} \bar{q}^{(k)}$, note that $c^{\mathsf{T}} \bar{q}^{(k)} > 0$ for all k (Lemma 4.4) and $c^{\mathsf{T}} \bar{q}^{(k)} \to \alpha > 0$. In fact, $\bar{Q}^{(k)}(e^{i\theta}) \to |\bar{a}(e^{i\theta})|^2$, where $\bar{a}(z)$ has all its zeros in the closed unit disc, and hence it follows from (4.6) that $\alpha > 0$. Then, since $\varphi(q) < \infty$ for all $q \in \overline{\mathcal{D}_n^+}$ except $q = 0$ (Lemma 4.2), (4.7) is bounded away from $-\infty$. Next, observe that

$$\varphi(q^{(k)}) = \lambda_k c^{\mathsf{T}} \bar{q}^{(k)} - \frac{1}{2\pi} \log \lambda_k \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \bar{Q}^{(k)}(e^{i\theta}) |\sigma(e^{i\theta})|^2 d\theta.$$

From this we can see that if a subsequence of λ_k were to tend to zero, then $\varphi(q^{(k)})$ would exceed r. Likewise, if a subsequence of λ_k were to tend to infinity, φ would exceed r, since linear growth dominates logarithmic growth. □

This concludes the proof of existence and uniqueness. □

We now turn to the existence of interior minimizers. The next result describes an interesting systems-theoretic consequence of the existence of such interior minima.

THEOREM 4.7. *Fix a partial covariance sequence c and a Schur polynomial $\sigma(z)$. If $\hat{q} \in \mathcal{D}_n^+$ is a minimum for φ, then*

$$\hat{Q}(z) = a(z)a(z^{-1}), \tag{4.8}$$

where $a(z)$ is the solution of the rational covariance extension problem.

Proof. Suppose that $\hat{q} \in \mathcal{D}_n^+$ is a minimum for φ. Then

$$\frac{\partial \varphi}{\partial q_k}(\hat{q}) = 0 \quad \text{for } k = 0, 1, 2, \dots, n, \tag{4.9}$$

which, in view of (3.27), is equivalent to

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \frac{|\sigma(e^{i\theta})|^2}{\hat{Q}(e^{i\theta})} d\theta \quad \text{for } k = 0, 1, \dots, n, \tag{4.10}$$

where $\hat{Q}$ is the pseudopolynomial (3.18) corresponding to $\hat{q}$. But, in view of (2.13) and (2.17), this is precisely the interpolation condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta = c_k \quad \text{for} \quad i = 0, 1, \dots, n,$$

provided (4.8) holds. □

We now state the converse result, underscoring our interest in this particular convex optimization problem.

THEOREM 4.8. *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, suppose that $a(z)$ gives a solution to the rational covariance extension problem. If*

$$\hat{Q}(z) = a(z)a(z^{-1}), \tag{4.11}$$

then the corresponding $(n+1)$-vector $\hat{q}$ lies in $\mathcal{D}_n^+$ and is a unique minimum for φ.

Proof. Let $a(z)$ be the solution of the rational covariance extension problem corresponding to c and $\sigma(z)$, and let $\hat{Q}$ be given by (4.11). Then c satisfies the interpolation condition (4.10), which is equivalent to (4.9), as seen from the proof of Theorem 4.7. However, since $a(z)$ is a Schur polynomial, $\hat{Q}(z) > 0$ on the unit circle, and thus $\hat{q} \in \mathcal{D}_n^+$. Since φ is strictly convex on $\mathcal{D}_n^+$, (4.10) implies that $\hat{q}$ is a unique minimum for φ. □

Since the existence of a solution to the rational covariance extension problem was established in [28] (see also [16]), we do in fact know the existence of interior minima for this convex optimization problem. On the other hand, we know from Proposition 4.1 that φ has a minimum for some $\hat{q} \in \overline{\mathcal{D}_n^+}$, so to show that φ has a minimum in the interior $\mathcal{D}_n^+$ it remains to prove the following lemma.

LEMMA 4.9. *The functional φ never attains a minimum on the boundary $\partial\mathcal{D}_n^+$.*

Proof. Denoting by $D_r\varphi(q)$ the directional derivative of φ at q in the direction r, it is easy to see that

$$D_r\varphi(q) := \lim_{\epsilon \to 0} \frac{\varphi(q + \epsilon r) - \varphi(q)}{\epsilon} \tag{4.12}$$

$$= c^{\mathsf{T}} p - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R(e^{i\theta})}{Q(e^{i\theta})} |\sigma(e^{i\theta})|^2 d\theta, \tag{4.13}$$

where R is the pseudopolynomial

$$R(z) = r_0 + \frac{1}{2}r_1(z + z^{-1}) + \frac{1}{2}r_2(z^2 + z^{-2}) + \cdots + \frac{1}{2}r_n(z^n + z^{-n})$$

corresponding to the vector $r \in \mathbb{R}^{n+1}$. In fact,

$$\frac{\log(Q + \epsilon R) - \log Q}{\epsilon} = \frac{R}{Q}\log\left[\left(1 + \epsilon\frac{R}{Q}\right)^{\frac{1}{\epsilon}\frac{Q}{R}}\right] \to \frac{R}{Q}$$

as $\epsilon \to +0$, and hence (4.12) follows by dominated convergence.

Now, let $q \in \mathcal{D}_n^+$ and $\bar{q} \in \partial\mathcal{D}_n^+$ be arbitrary. Then the corresponding pseudopolynomials Q and $\bar{Q}$ have the properties

$$Q(e^{i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi]$$

and

$$\bar{Q}(e^{i\theta}) \geq 0 \quad \text{for all } \theta \text{ and } \bar{Q}(e^{i\theta_0}) = 0 \text{ for some } \theta_0.$$

Since $q_\lambda := \bar{q} + \lambda(q - \bar{q}) \in \mathcal{D}_n^+$ for $\lambda \in (0, 1]$, we also have for $\lambda \in (0, 1]$ that

$$Q_\lambda(e^{i\theta}) := \bar{Q}(e^{i\theta}) + \lambda[Q(e^{i\theta}) - \bar{Q}(e^{i\theta})] > 0 \quad \text{for all } \theta \in [-\pi, \pi],$$

and we may form the directional derivative

$$D_{\bar{q}-q}\varphi(q_\lambda) = c^{\mathsf{T}}(\bar{q} - q) + \frac{1}{2\pi}\int_{-\pi}^{\pi} h_\lambda(\theta)d\theta, \tag{4.14}$$

where

$$h_\lambda(\theta) = \frac{Q(e^{i\theta}) - \bar{Q}(e^{i\theta})}{Q_\lambda(e^{i\theta})}|\sigma(e^{i\theta})|^2.$$

Now,

$$\frac{d}{d\lambda}h_\lambda(\theta) = \frac{[Q(e^{i\theta}) - \bar{Q}(e^{i\theta})]^2}{Q_\lambda(e^{i\theta})^2}|\sigma(e^{i\theta})|^2 \geq 0,$$

and hence $h_\lambda(\theta)$ is a monotonically nondecreasing function of λ for all $\theta \in [-\pi, \pi]$. Consequently, h_λ tends pointwise to h_0 as $\lambda \to 0$. Therefore,

$$\int_{-\pi}^{\pi} h_\lambda(\theta)d\theta \to +\infty \quad \text{as } \lambda \to 0. \tag{4.15}$$

In fact, if

$$\int_{-\pi}^{\pi} h_\lambda(\theta)d\theta \to \alpha < \infty \quad \text{as } \lambda \to 0, \tag{4.16}$$

then $\{h_\lambda\}$ is a Cauchy sequence in $L^1(-\pi, \pi)$ and hence has a limit in $L^1(-\pi, \pi)$ which must equal h_0 almost everywhere. However, h_0, having poles in $[-\pi, \pi]$, is not summable and hence, as claimed, (4.16) cannot hold.

Consequently, by virtue of (4.14),

$$D_{q-\bar{q}}\varphi(q_\lambda) \to +\infty \quad \text{as } \lambda \to 0$$

for all $q \in \mathcal{D}_n^+$ and $\bar{q} \in \partial\mathcal{D}_n^+$, and hence, in view of Lemma 26.2 in [68], φ is essentially smooth. Then it follows from Theorem 26.3 in [68] that the subdifferential of φ is empty on the boundary of $\mathcal{D}_n^+$, and therefore φ cannot have a minimum there. □

Thus we have proven the following result.

THEOREM 4.10. *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, there exists an $(n+1)$-vector $\hat{q}$ in $\mathcal{D}_n^+$ which is a minimizing point for φ.*

Consequently, by virtue of Theorem 4.7, there does exist a solution to the rational covariance extension problem for each partial covariance sequence and zero polynomial $\sigma(z)$, and, in view of Theorem 4.8, this solution is unique. These theorems have the following corollary.

COROLLARY 4.11 (Georgiou's conjecture). *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, there is a unique Schur polynomial $a(z)$ such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\left|\frac{\sigma(e^{i\theta})}{a(e^{i\theta})}\right|^2 d\theta = c_k \quad \textit{for } k = 0, 1, \ldots, n.$$

5. Well-Posedness of the Optimization Problem. We wish to show that, for $\sigma(z)$ fixed, the convex minimization problem (3.25) is well-posed with respect to $c = (c_0, c_1, \ldots, c_n)$, in the sense of Hadamard. That is, a minimum exists for each c, is unique, and varies with c in a smooth (or continuous) way. Geometrically, well-posedness for a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ is characterized by Hadamard's global inverse function theorem [37, 38].

THEOREM 5.1 (Hadamard). *Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^k for $k \geq 1$. Then F is a C^k-diffeomorphism if and only if F is proper and $\det\{(\mathrm{Jac}F)(x)\}$ is nonzero for each $x \in \mathbb{R}^n$.*

For the convex optimization problem presented here, we have already established the existence and uniqueness of a minimizer in the previous section. Following [15], to see that the third property holds it is useful to ask the question: Why should there be a variational formulation of the rational covariance extension problem?

For $\sigma(z)$ fixed, we can express the rational covariance extension problem in terms of a map $F : \mathcal{D}_n^+ \to \mathcal{T}_n^+$, where $\mathcal{T}_n^+$ is the space of sequences $c = (c_0, c_1, \ldots, c_n)$ for which the Toeplitz matrix (2.5) is positive definite and where F is defined component wise via

$$F_k(q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})}d\theta, \quad k = 0, 1, \ldots, n.$$

Both $\mathcal{D}_n^+$ and $\mathcal{T}_n^+$ are convex, and hence connected. Since both are open in $\mathbb{R}^{n+1}$, each is diffeomorphic to $\mathbb{R}^{n+1}$.

We would like to solve the equation

$$F(q) = c, \quad \text{where } c = (c_0, c_1, \ldots, c_n) \in \mathcal{T}_n^+, \tag{5.1}$$

using a variational approach in which the functional is defined by the path integral

$$\varphi(\bar{q}) = \int_{e_0}^{\bar{q}} \omega, \quad \text{where } \omega = \sum_{k=0}^{n}[c_k - F_k(q)]dq_k,$$

and where e_0 is the unit vector $(1, 0, \dots, 0)^{\mathsf{T}}$ in $\mathbb{R}^{n+1}$.

For this expression to be well defined we need to know that the 1-form ω is closed. To say that $d\omega = 0$ is to say that

$$\frac{\partial F_k}{\partial q_j} = \frac{\partial F_j}{\partial q_k},$$

which, in turn, follows from

$$\frac{\partial F_k}{\partial q_j} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{ik\theta} + e^{-ik\theta})(e^{ij\theta} + e^{-ij\theta})\frac{|\sigma|^2}{Q^2}d\theta.$$

Therefore, by the Poincaré lemma [50, p. 137], the path integral of ω depends only on the end points.

We now compute the path integral:

$$\begin{aligned}
&\int_{e_0}^{\bar q}\sum_{k=0}^{n}\left[c_k - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{\left|\sigma(e^{i\theta})\right|^2}{Q(e^{i\theta})}d\theta\right] dq_k \\
&= \sum_{k=0}^{n}\left[c_k(\bar q_k - \delta_{k0}) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{e_0}^{\bar q} e^{ik\theta}\frac{1}{Q(e^{i\theta})}dq_k \left|\sigma(e^{i\theta})\right|^2 d\theta\right] \\
&= c^{\mathsf{T}}\bar q - c_0 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\int_1^{\bar Q} d(\log Q)\right]\left|\sigma(e^{i\theta})\right|^2 d\theta \\
&= c^{\mathsf{T}}\bar q - c_0 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log Q\Big|_1^{\bar Q}\left|\sigma(e^{i\theta})\right|^2 d\theta
\end{aligned}$$

and consequently, modulo a constant of integration,

$$\varphi(q) = c^{\mathsf{T}}q - \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma|^2\log Q\, d\theta.$$

Since the exterior differential is invariant under a change of coordinates, this gives an intrinsic explanation for why a variational formulation of the problem exists. Moreover, since the functional φ is strictly convex, the Jacobian matrix of F, which is symmetric, is positive definite and therefore everywhere invertible. In section 4, we established that the gradient of φ, viz. F, tends to infinity on the boundary of $\mathcal{D}_n^+$ and therefore F is proper. By Hadamard's theorem, F is a diffeomorphism: Since an analytic diffeomorphism has an analytic inverse, by the inverse function theorem, the change of coordinates from (σ, a) to (σ, c) is bianalytic on the space of shaping filters, thereby proving Theorem 2.1.

6. Computational Methods. One of the neat consequences of the well-posedness of the rational covariance extension problem, as developed in the previous section, is that, fixing $\sigma(z)$, the function $F : \mathcal{D}_n^+ \to \mathcal{T}_n^+$ is a proper map with no branch points and therefore the method of homotopy continuation yields a computational method for continuing the solution to $F(q^0) = c^0$ to the solution of $F(q) = c$ for any $c \in \mathcal{T}_n^+$. This was observed by Enqvist and is fully developed in [25] for the case in which $\sigma(z)$ varies. In this section, we shall describe an algorithm based on Newton's method.

In either case, one first needs to find an estimate for $\sigma(z)$ from the data. As it turns out, the cepstrum and the cepstral coefficients, introduced in section 3, provide an efficient method—known as cepstral windowing—for the estimation of the zeros of the shaping filter.

More precisely, in [6, 7] it is shown that there is a one-to-one correspondence between the $2n+1$ coefficients $c_1, c_2, \dots, c_n, v_0, v_1, \dots, v_n$ and the $2n+1$ coefficients $a_0, a_1, \dots, a_n, \sigma_1, \sigma_2, \dots, \sigma_n$ of the denominator polynomial $a(z)$ and numerator polynomial $\sigma(z)$ of the corresponding modeling filter (2.17), provided w has *exactly* degree n.

THEOREM 6.1 (see [6]). *Each modeling filter* (2.17) *of degree n determines and is uniquely determined by its window $c_0, c_1, \dots, c_n$ of covariance lags and its window $v_1, v_2, \dots, v_n$ of cepstral coefficients.*

It is, of course, clear that $c_0, c_1, c_2, \dots, c_n, v_1, \dots, v_n$ is determined by a modeling filter (2.17) of degree n. Conversely, given such a covariance-cepstral window, in [6] it was proposed to minimize the convex functional

$$\begin{aligned} J(s,q) &= c_0q_0 + c_1q_1 + \cdots + c_nq_n - v_1p_1 - v_2p_2 - \cdots - v_np_n \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta}) \log \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\theta \end{aligned} \tag{6.1}$$

with the pseudopolynomials (3.10) and (3.18), given by (3.11) and (4.2), respectively, ranging over the closed convex region in $\mathbb{R}^{2n+1}$ of variables $p_1, p_2, \dots, p_n, q_0, q_1, \dots, q_n$ such that the pseudopolynomials P and Q are nonnegative on the unit circle.

As before, J always has a minimum since it is a convex function defined on a closed convex set and any interior minimum of J must define a modeling filter that matches the covariance and the cepstral window. Moreover, when $a(z)$ and $\sigma(z)$ are coprime polynomials, i.e., when the filter is of degree precisely n, the modeling filter is uniquely determined either by the spectral density

$$\Phi(z) = \frac{P(z)}{Q(z)}$$

or by the covariance and the cepstral windows. It follows, by similar arguments to those in the previous section, that the covariance and the cepstral windows form a globally defined set of functions which form a local coordinate system about each shaping filter having degree precisely n.

REMARK 6.2. *One might also consider fixing a covariance window and a window of Markov parameters of w, as is done in* [54, 48, 3] *for w a rational filter with stable poles but arbitrary zeros. As it turns out, for shaping filters, there is a close relation between the cepstral coefficients and the Markov parameters of the corresponding shaping filter w. To establish these relations, using both the stability and minimum phase properties of the shaping filter we form a subset Ω of the complex plane, which is the intersection between an annulus containing the unit circle but none of the zeros of $w(z)$ or $w(z^{-1})$ and a sector containing the positive real axis. The corresponding Laurent expansions for*

$$\log \Phi(z) = \log w(z) + \log w(z^{-1})$$

on Ω yield a bianalytic change of coordinates between the cepstral coefficients and the Markov parameters. In particular [7], *for the shaping filters considered here the leaves of the foliation defined by the Markov parameters coincide with the leaves of the foliation defined by the cepstral windows. In* [7] *it was also shown that both the covariance windows and the Markov windows define foliations of the space of stable* *rational filters w of degree less than or equal to n. From the geometry of these foliations it is shown that the covariance and the Markov windows form a globally defined set of functions which form a local coordinate system about each shaping filter having degree precisely n. In particular, there is at most one stable filter having given covariance and Markov windows.*

Since the cepstral window and the covariance window characterize the shaping filter, one can expect to estimate the zeros of the shaping filter in terms of the cepstral window. We now proceed to describe the method for zero estimation proposed in [6]. Recall (see Figure 3.1) that the cepstrum is the additive superposition of the (lower frequency) effect of the shaping filter and the (higher frequency) effect of the driving signal. Windowing the cepstrum so as to neglect higher frequencies and smooth lower frequency contributions, i.e., cepstral smoothing, provides an estimate of the cepstrum at given frequencies [64, pp. 494–495]:

$$\hat{\Phi}(e^{i\theta_k}), \quad k = 1, \dots, N.$$

Given these estimates, find pseudopolynomials P and Q such that

$$\max_k |Q(e^{i\theta_k})\hat{\Phi}(e^{i\theta_k}) - P(e^{i\theta_k})|$$

is minimized. This leads to a standard linear programming problem in the $2n+2$ variables $\delta, p_1, \dots, p_n, q_0, q_1, \dots, q_n$, namely, to find δ, P, Q that minimize δ subject to the $4N$ constraints that

$$\begin{aligned} Q(e^{i\theta_k})\hat{\Phi}(e^{i\theta_k}) - P(e^{i\theta_k}) - \delta \le 0, \\ -Q(e^{i\theta_k})\hat{\Phi}(e^{i\theta_k}) + P(e^{i\theta_k}) - \delta \le 0, \\ P(e^{i\theta_k}) \ge \varepsilon, \\ Q(e^{i\theta_k}) \ge \varepsilon \end{aligned}$$

hold for $k = 1, 2, \dots, N$. Here the design parameter $\varepsilon > 0$ must be chosen large enough to ensure that P and Q are positive on the unit circle. The corresponding numerator polynomial, $\sigma(z)$, and the denominator polynomial, $a(z)$, can be obtained from P and Q by spectral factorization. However, the resulting shaping filter does not necessarily match the desired covariance window. For this reason, we use the method for finding a $\sigma(z)$ and initializing Newton's method with $a(z)$.

Given an arbitrary partial covariance sequence $c_0, c_1, \dots, c_n$ and an arbitrary zero polynomial $\sigma(z)$, we compute the gradient of the cost functional φ, which, as we saw in section 4, is given by

$$\frac{\partial \varphi}{\partial q_k}(q_0, q_1, \dots, q_n) = c_k - \bar{c}_k, \tag{6.2}$$

where

$$\bar{c}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})}d\theta \quad \text{for } k = 0, 1, 2, \dots, n \tag{6.3}$$

are the covariances corresponding to a process with spectral density

$$\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})} = \bar{c}_0 + 2\sum_{k=1}^{\infty} \bar{c}_k \cos(k\theta). \tag{6.4}$$

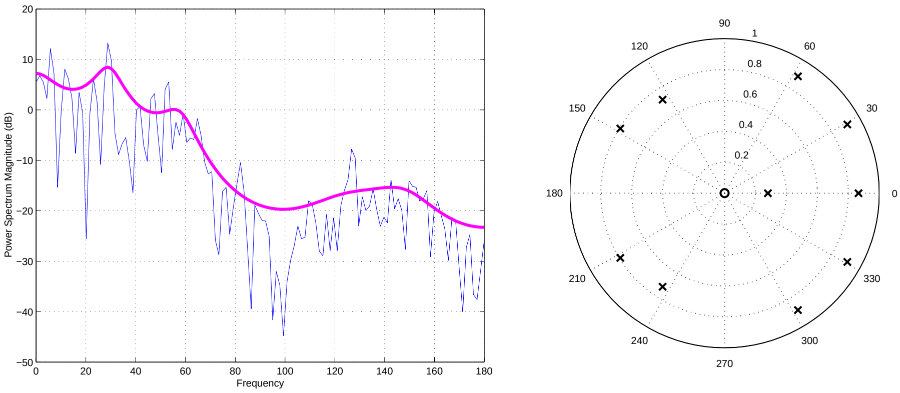

Fig. 6.1 *Spectral envelope and pole/zero locations for a maximum entropy solution* ($n = 10$).

The gradient is thus the difference between the given partial covariance sequence $c_0, c_1, \ldots, c_n$ and the partial covariance sequence corresponding to the choice of variables $q_0, q_1, \ldots, q_n$ at which the gradient is calculated. The minimum is attained when this difference is zero.

The following simulations were done by Enqvist using Newton's method (see, e.g., [55, 59]), which of course also requires computing the Hessian (second-derivative matrix) in each iteration. A straightforward calculation shows that the Hessian is the sum of a Toeplitz and a Hankel matrix. More precisely,

$$H_{ij}(q_0, q_1, \ldots, q_n) = \frac{1}{2}(d_{i+j} + d_{i-j}), \quad i, j = 0, 1, 2, \ldots, n, \tag{6.5}$$

where

$$d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})^2}d\theta \quad \text{for } k = 0, 1, 2, \ldots, 2n \tag{6.6}$$

and $d_{-k} = d_k$. Moreover, $d_0, d_1, d_2, \ldots, d_{2n}$ are the $2n+1$ first Fourier coefficients of the spectral representation

$$\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})^2} = d_0 + 2\sum_{k=1}^{\infty} d_k \cos(k\theta). \tag{6.7}$$

The gradient and the Hessian can be determined from (6.2) and (6.5), respectively, by applying the inverse Levinson algorithm (see, e.g., [65]) to the appropriate polynomial spectral factors of $Q(z)$ and $Q(z)^2$, respectively, and then solving the resulting linear equations for $\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_n$ and $d_0, d_1, d_2, \ldots, d_{2n}$.

To illustrate the procedure, let us consider two tenth-order spectral envelopes for the same signal as in Figures 2.1 and 2.3 together with the corresponding zeros and poles. Hence, Figure 6.1 illustrates the periodogram for a section of speech data together with the corresponding tenth-order maximum entropy spectrum, which, since it lacks finite zeros, becomes rather "flat." The locations of the corresponding poles (marked by $\times$) in the unit circle are shown next to it. The zeros (marked by $\circ$) of course all lie at the origin.

Now, using cepstral smoothing we obtain the zeros indicated to the right in Figure 6.2, and using Newton's method we obtain the poles as marked, and the corresponding

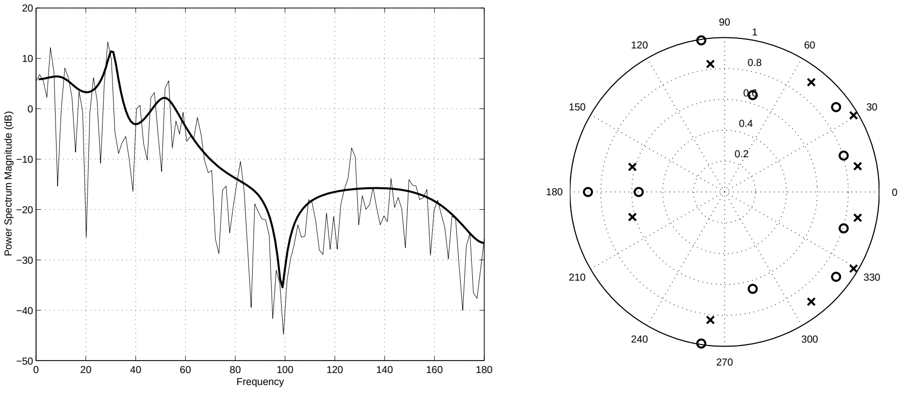

Fig. 6.2 *Spectral envelope and pole/zero locations for a solution of degree* $n = 10$.

tenth-order shaping filter produces the spectral envelope to the left in Figure 6.2. We see that the second solution has a spectral density that is less flat and would seem to provide a better approximation, reflecting the fact that the filter was designed so that its transmission zeros are influenced by the cepstral window and indeed are located near the minima of the periodogram. In general, in [6, 7] this method was compared with some existing system identification algorithms using three different lengths of Monte Carlo runs to generate data, and a quantitative improvement in the design of a shaping filter matching the desired covariances and cepstral windows has been observed.

7. Conclusions and Further Directions. In [27, 28] Georgiou proved that to each choice of partial covariance sequence and numerator polynomial of the shaping filter there exists a rational covariance extension yielding a pole polynomial for the shaping filter. He also conjectured that this extension is unique so that it provides a complete parameterization of all rational covariance extensions. In [16] we proved this long-standing conjecture in the more general context of a duality between filtering and interpolation.

In [10] we presented a constructive proof of Georgiou's conjecture, which for the first time provided an algorithm for solving the problem of determining the unique pole polynomial corresponding to the given partial covariance sequence and the desired zeros. In the present paper, following [15], this minimization problem is shown to be well-posed in the sense of Hadamard. Indeed, combining the strictly convex minimization problem, the existence of interior points, and Hadamard's global inverse function theorem yields an alternative geometric approach to the proof of Georgiou's conjecture given in [16].

The interior point argument involves a constrained convex optimization problem, which can be solved without explicitly computing the values of the cost function and which has the interesting property that the cost function is finite on the boundary but the gradient is not. In this context, Georgiou's conjecture is equivalent to establishing that there is a unique minimum in the *interior* of the feasible region. This optimization problem is shown to be a dual in the sense of mathematical programming to a primal problem motivated by cepstral analysis of speech. This primal problem amounts to maximizing a generalized entropy gain subject to covariance matching constraints.

As pointed out in section 2 (also see [24]), the rational covariance extension problem is a classical moment problem with a complexity constraint (a bound on the degree of the interpolant). As has recently been observed in [15], the methods developed in this paper and its predecessors can be applied to the general moment problem with a complexity constraint.

Briefly, in the general moment problem, one is given a sequence of complex numbers $c_0, c_1, \dots, c_n$ and a sequence of continuous, linearly independent complex-valued functions $\alpha_0, \alpha_1, \dots, \alpha_n$ defined on the real interval $[a, b]$. The moment problem is then to find all monotone, nondecreasing functions μ of bounded variation such that

$$\int_a^b \alpha_k(t) d\mu(t) = c_k, \quad k = 0, 1, \dots, n, \tag{7.1}$$

where the sequence $c_0, c_1, \dots, c_n$ is positive in the following sense. Assuming that α_0 is real-valued, let $\mathcal{P}$ be the subspace of $C[a, b]$ spanned by the functions

$$\alpha_{-n}, \dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_n,$$

where $\alpha_{-k} = \overline{\alpha_k}$, and let $\mathcal{P}_+$ be the subset of $p \in \mathcal{P}$ that are positive on $[a, b]$. It is typically assumed that $\mathcal{P}_+$ is nonempty. Then the sequence $c_0, c_1, \dots, c_n$ is *positive* if and only if

$$\langle c, q \rangle := \operatorname{Re} \sum_{k=0}^{n} q_k c_k > 0 \tag{7.2}$$

for all $q := (q_0, q_1, \dots, q_n) \in \mathbb{C}^{n+1}$ such that

$$\operatorname{Re} \sum_{k=0}^{n} q_k \alpha_k \in \mathcal{P}_+. \tag{7.3}$$

There is a vast literature on this subject (see, e.g., [1, 2, 49, 34]), in part because so many problems and theorems in pure and applied mathematics, physics, and engineering can be formulated as moment problems. For example, if $[a, b] = [-\pi, \pi]$ and $\alpha_k(\theta) = e^{ik\theta}$, we obtain the trigonometric moment problem. It is important to note that in this case (7.2) is precisely the linear term in the objective function (3.24) and, as noted in Remark 4.5, we have

$$\langle c, q \rangle = a^{\mathsf{T}} T_n a,$$

where T_n is the Toeplitz matrix (2.5) and $a := (a_0, a_1, \dots, a_n)^{\mathsf{T}}$ is the vector of coefficients of the stable polynomial factor (3.16) in (4.2). This term will play a fundamental role in a variational formulation of the general moment problem.

In [15] it is assumed that $\mathcal{P}_+$ is also open. Moreover, we introduce the complexity constraint

$$\frac{d\mu}{dt} = \Phi(t) = \frac{P(t)}{Q(t)}, \quad P, Q \in \mathcal{P}_+. \tag{7.4}$$

Generalizing the rational covariance extension problem, in our paper [15] all solutions of constrained complexity can be parameterized by the choice of $P \in \mathcal{P}_+$. Indeed, setting

$$F_k(\mu) = \int_a^b \alpha_k(t) d\mu(t)$$

and constructing the 1-form

$$\omega_c = \sum_{k=0}^{n} [c_k - F_k(\mu)] dq_k,$$

we observed that ω_c is closed. Therefore, by the Poincaré lemma, there exists a smooth function J such that

$$J = \int \omega_c,$$

with the integral being independent of the path between two end points. Computing the path integral along the lines in section 5, one finds [15] that

$$J(Q) = \langle c, q \rangle - \int_a^b P \log Q \, dt, \tag{7.5}$$

which is strictly convex and bounded from below for positive sequences $c_0, c_1, \ldots, c_n$.

This approach to the general moment problem was inspired by the application of convex analysis to both the rational covariance extension problem [10] and the Nevanlinna–Pick interpolation problem with degree constraint [8]. The Nevanlinna–Pick interpolation problem is a moment problem for the choice $\alpha_k = C_k$, where $C_0, C_1, \ldots, C_n$ are Cauchy kernels. In this case, a sequence $c_0, c_1, \ldots, c_n$ is positive if and only if the Pick matrix is positive definite. Imposing the complexity constraint amounts to imposing a degree constraint on a rational interpolant. Given the synthesis of a positive real rational function as the impedance of a circuit with finitely many active components, it is not surprising that important problems involving interpolation by positive real functions at points in the finite complex plane emerged in circuit theory [74, 23, 39]. They also abound in robust stabilization and control, as we will discuss below. Georgiou [29] published the first result toward developing a parameterization of all solutions of the Nevanlinna–Pick interpolation problem with a degree constraint, again showing that there exists at least one positive real interpolant for which the shaping filter would have a prespecified numerator. In [30], Georgiou adopted the method of proof in [11] to prove that this parameterization is also injective. In [14], geometric methods are used to verify that this problem is well-posed. A constructive proof was given in [8] in terms of minimizing a convex objective function, to which (6.1) specializes in the case of Nevanlinna–Pick interpolation.

One of our interests in Nevanlinna–Pick interpolation is a new approach to spectral estimation, introduced in [8] and fully developed in [9], based on passing an observed signal through a bank of one-dimensional, stable filters with transfer functions

$$G_k(z) = \frac{1}{z - p_k}, \quad k = 0, 1, \ldots, n,$$

as depicted in Figure 7.1. The basic idea is to estimate the spectral density Φ of the input process y, or, equivalently, the positive real function f satisfying

$$\operatorname{Re} f(e^{i\theta}) = \Phi(e^{i\theta}),$$

from output data. In fact, denoting by $u_0, u_1, \ldots, u_n$ the corresponding output processes, it was shown in [9] that

$$f(p_k^{-1}) = (1 - p_k^2) \mathrm{E}\{u_k^2\}, \quad k = 0, 1, \ldots, n.$$

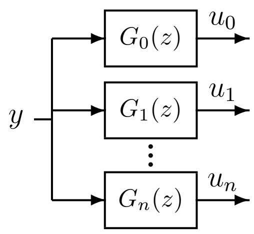

Fig. 7.1 *Filter bank.*

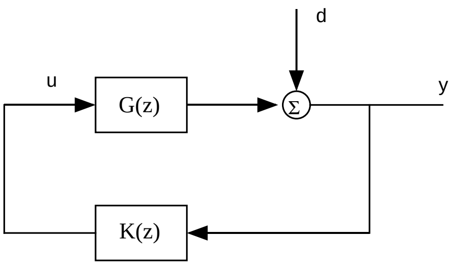

Fig. 7.2 *A feedback system.*

Since zeroth-order covariances $c_0(u_k) := \mathrm{E}\{u_k^2\}, k = 0, 1, \ldots, n$, can be determined via ergodic estimates, the zeroth-order covariance data for the outputs of the filter bank supply the interpolation constraints for f. An advantage of this approach is that interpolation of the spectrum can be chosen closer to the unit circle in precisely the frequency band where higher resolution is desired (see [9] for more details and examples). The choice of numerator P in (7.4) provides an additional set of tuning parameters.

Another of our interests is in robust control. Indeed, during the last two decades it has been discovered that analytic interpolation theory is closely related to several robust control problems, for example, the gain-margin maximization problem [70, 71, 45], the robust stabilization problem [47], sensitivity shaping in feedback control, simultaneous stabilization [32], the robust regulation problem [21], and the general H_∞ control problem [26].

To illustrate this point, let us consider the following example taken from [8]. Figure 7.2 depicts a feedback system with u denoting the control input to the plant $G(z)$ to be controlled, d representing a disturbance, and y being the resulting output, which is fed back through a compensator $K(z)$ to be designed.

Internal stability and the robustness of the output with respect to input disturbances can be characterized in terms of the *sensitivity function*

$$S(z) = [1 - G(z)K(z)]^{-1}.$$

In fact, the feedback system is internally stable if and only if $S(z)$ has all its poles inside the unit disc and satisfies the interpolation conditions

$$S(z_j) = 1, \quad j = 1, 2, \ldots, r, \qquad S(p_k) = 0, \quad k = 1, 2, \ldots, \ell,$$

where $z_1, z_2, \ldots, z_r$ and $p_1, p_2, \ldots, p_\ell$ are the zeros and poles, respectively, of the

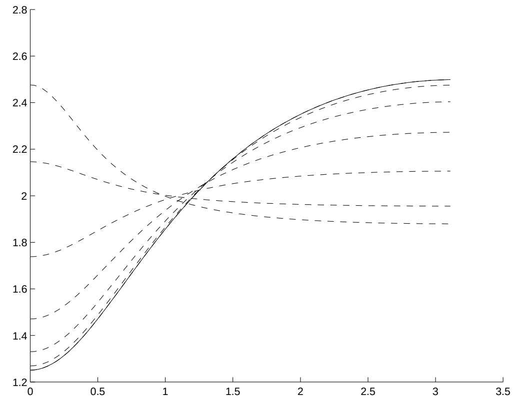

Fig. 7.3 *The graphs of $|S|$ for different choices of P.*

plant $G(z)$ outside the unit disc. Moreover, for disturbance attenuation, S needs to be bounded. The lowest bound,

$$\alpha_{\text{opt}} = \inf_{S(z_i)=1, S(p_j)=0} \|S\|_\infty, \tag{7.6}$$

is attained for an S such that $|S(e^{i\theta})| = \alpha_{\text{opt}}$ for all $\theta \in [-\pi, \pi]$.

However, in practice one wants to achieve lower sensitivity $|S(e^{i\theta})|$ in selected frequency bands. To satisfy such design specifications, a standard approach has been to solve a weighted optimization problem, a procedure that typically increases the dimension of the sensitivity function and hence of the compensator. The optimization approach of [8] provides a new procedure of satisfying the design specifications while bounding the degree of the sensitivity function by one less than the number of interpolation conditions. Indeed, allowing a higher upper bound $\alpha > \alpha_{\text{opt}}$, we obtain a complete parameterization of all such S in terms of the numerator function P in (7.4). In fact, the admissible sensitivity functions S are such that $\frac{1}{\alpha}S$ maps the exterior of the disc into the unit disc. Therefore, $f = (\alpha + S)(\alpha - S)^{-1}$ is a positive real function, of the same degree as S, satisfying the interpolation conditions

$$f(z_j) = \frac{\alpha+1}{\alpha-1}, \quad j = 1, 2, \ldots, r, \qquad f(p_k) = 1, \quad k = 1, 2, \ldots, \ell.$$

Again following [8], we consider the simple example for which $G(z) = \frac{1}{z-2}$. This system has one pole and one zero outside the unit disc, and hence the bound on the degree of S is one. Figure 7.3 depicts the corresponding one-parameter family of sensitivity functions. By choosing P appropriately, we can determine the most suitable compensator. Selection rules for making this choice have been developed by Nagamune, who is also applying the techniques of [8] to many of the robust control problems mentioned above [60]; also see [61, 62, 63].

Finally, one might ask why should the choice of spectral zeros play such a central and crucial role? For example, why not parameterize solutions to these interpolation problems by the choice of spectral poles? The short answer to the second question is that counterexamples to that approach exist in great profusion [12, p. 125], but this does not explain the intrinsic importance of the spectral zeros as parameters.

In fact, the foliation of the space of positive real functions of degree less than or equal to n by the leaves of those systems with fixed spectral zeros actually has a longer history and an intrinsic meaning beyond the scope of this problem. As it turns out, it is possible to reformulate the Kalman filtering equations as a "fast filtering algorithm" [51, 52] that evolves as a nonlinear dynamical system on this space of positive real functions [17] in such a way that the stable manifolds of the (manifold of) equilibria foliate this space and coincide with the leaves obtained by fixing the spectral zeros. Indeed, the geometric verification of well-posedness given in [14] uses this dynamical systems interpretation in a crucial way.

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REFERENCES

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner, New York, 1965.
[2] N. I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments*, AMS, Providence, RI, 1962.
[3] B. D. O. Anderson and R. E. Skelton, *q-Markov covariance equivalent realizations*, Internat. J. Control, 44 (1986), pp. 1477–1490.
[4] M. Aoki, *State Space Modeling of Time Series*, Springer-Verlag, Berlin, 1987.
[5] C. G. Bell, H. Fujisaki, J. M. Heinz, K. N. Stevens, and A. S. House, *Reduction of speech spectra by analysis-by-synthesis techniques*, J. Acoust. Soc. Amer., 33 (1961), pp. 1725–1736.
[6] C. I. Byrnes, P. Enqvist, and A. Lindquist, *Cepstral coefficients, covariance lags and pole-zero models for finite data strings*, IEEE Trans. Signal Process., SP-50 (2001), pp. 677–693.
[7] C. I. Byrnes, P. Enqvist, and A. Lindquist, *Identifiability and well-posedness of shaping-filter parameterizations: A global analysis approach*, SIAM J. Control Optim., submitted.
[8] C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A generalized entropy criterion for Nevanlinna-Pick interpolation with degree constraint*, IEEE Trans. Automat. Control, AC-46 (2001), pp. 822–839
[9] C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to spectral estimation: A tunable high-resolution spectral estimator*, IEEE Trans. Signal Process., SP-49 (2000), pp. 3189–3205.
[10] C. I. Byrnes, S. V. Gusev, and A. Lindquist, *A convex optimization approach to the rational covariance extension problem,* SIAM J. Control Optim. 37 (1998), pp. 211–229.
[11] C. I. Byrnes, H. J. Landau, and A. Lindquist, *On the well-posedness of the rational covariance extension problem*, in Current and Future Directions in Applied Mathematics, M. Alber, B. Hu, and J. Rosenthal, eds., Birkhäuser, Basel, 1997, pp. 83–106.
[12] C. I. Byrnes and A. Lindquist, *On duality between filtering and control*, in Systems and Control in the Twenty-First Century, C. I. Byrnes, B. N. Datta, D. S. Gilliam, and C. F. Martin, eds., Birhäuser, Basel, 1997, pp. 101–136.
[13] C. I. Byrnes and A. Lindquist, *On the partial stochastic realization problem*, IEEE Trans. Automat. Control, AC-42 (1997), pp. 1049–1070.
[14] C. I. Byrnes and A. Lindquist, *On the duality between filtering and Nevanlinna–Pick interpolation*, SIAM J. Control Optim., 39 (2000), pp. 757–775.
[15] C. I. Byrnes and A. Lindquist, *Interior Point Solutions of Variational Problems and Global Inverse Function Theorems*, preprint.
[16] C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Mateev, *A complete parametrization of all positive rational extensions of a covariance sequence*, IEEE Trans. Automat. Control, AC-40 (1995), pp. 1841–1857.
[17] C. I. Byrnes, A. Lindquist, and Y. Zhou, *On the nonlinear dynamics of fast filtering algorithms*, SIAM J. Control Optim., 32 (1994), pp. 744–789.
[18] J. A. Cadzow, *Spectral estimation: An overdetermined rational model equation approach*, Proc. IEEE, 70 (1982), pp. 907–939.
[19] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*, Math. Ann., 64 (1907), pp. 95–115.

[20] C. Carathéodory, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Functionen*, Rend. Circ. Mat. Palermo (2), 32 (1911), pp. 193–217.
[21] M. K. K. Cevik and J. M. Schumacher, *The Robust Regulation Problem with Robust Stability*, Technical Report MAS-R9827, CWI, Amsterdam, 1999.
[22] A. Dahlén, A. Lindquist, and J. Mari, *Experimental evidence showing that stochastic subspace identification methods may fail*, Systems Control Lett., 34 (1998), pp. 303–312.
[23] Ph. Delsarte, Y. Genin, and Y. Kamp, *On the role of the Nevanlinna-Pick problem in circuits and system theory*, Circuit Theory Appl., 9 (1981), pp. 177–187.
[24] P. Delsarte, Y. Genin, Y. Kamp, and P. van Dooren, *Speech modelling and the trigonometric moment problem*, Philips J. Res., 37 (1982), pp. 277–292.
[25] P. Enqvist, *Spectral Estimation by Geometric, Topological and Optimization Methods*, Ph.D. dissertation, Optimization and Systems Theory, Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden, 2001.
[26] B. A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, New York, 1987.
[27] T. T. Georgiou, *Partial Realization of Covariance Sequences*, Ph.D. thesis, Center for Mathematical Systems Theory, University of Florida, Gainesville, FL, 1983.
[28] T. T. Georgiou, *Realization of power spectra from partial covariance sequences*, IEEE Trans. Acoust. Speech Signal Process., ASSP-35 (1987), pp. 438–449.
[29] T. T. Georgiou, *A topological approach to Nevanlinna–Pick interpolation*, SIAM J. Math. Anal., 18 (1987), pp. 1248–1260.
[30] T. T. Georgiou, *The interpolation problem with a degree constraint*, IEEE Trans. Automat. Control, 44 (1999), pp. 631–635.
[31] Y. L. Geronimus, *Orthogonal Polynomials*, Consultants Bureau, New York, 1961.
[32] B. K. Ghosh, *Transcendental and interpolation methods in simultaneous stabilization and simultaneous partial pole placement problems*, SIAM J. Control Optim., 24 (1986), pp. 1091–1109.
[33] W. B. Gragg and A. Lindquist, *On the partial realization problem*, Linear Algebra Appl., 50 (1983), pp. 277–319.
[34] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, University of California, Berkeley, CA, 1958.
[35] U. Grenander and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Almqvist & Wiksell, Stockholm, Sweden, 1956.
[36] E. A. Guillemin, *Synthesis of Passive Networks*, Wiley, New York, 1957.
[37] J. Hadamard, *Sur les transformations ponctuelles*, Bull. Soc. Math. France, 34 (1906), pp. 71–84; Oeuvres, pp. 349–363.
[38] J. Hadamard, *Sur les correspondances ponctuelles*, Oeuvres, Editions du Centre Nationale de la Researche Scientifique, Paris, 1968, pp. 383–384.
[39] J. W. Helton, *Non-Euclidean analysis and electronics*, Bull. Amer. Math. Soc., 7 (1982), pp. 1–64.
[40] R. E. Kalman, *On partial realizations of a linear input/output map*, Guillemin Anniversary Volume, Holt, Rinehart and Winston, 1968.
[41] R. E. Kalman, *On minimal partial realizations of a linear input/output map*, in Aspects of Network and System Theory, R. E. Kalman and N. de Claris, eds., Holt, Rinehart and Winston, 1971, pp. 385–408.
[42] R. E. Kalman, *On partial realizations, transfer functions and canonical forms*, Acta Polytech. Scand., MA31 (1979), pp. 9–39.
[43] R. E. Kalman, *Realization of covariance sequences*, in Proceedings of the Toeplitz Centennial, Tel Aviv, Israel, 1981, Oper. Theory Adv. Appl. 4, Birkhäuser, Basel, 1982, pp. 331–342.
[44] S. M. Kay and S. L. Marple, Jr., *Spectrum analysis—A modern perspective*, Proc. IEEE, 69 (1981), pp. 1380–1419.
[45] P. P. Khargonekar and A. Tannenbaum, *Non-Euclidean metrics and robust stabilization of systems with parameter uncertainty*, IEEE Trans. Automat. Control, AC–30 (1985), pp. 1005–1013.
[46] H. Kimura, *Positive partial realization of covariance sequences*, in Modelling, Identification and Robust Control, C. I. Byrnes and A. Lindquist, eds., North–Holland, Amsterdam, 1987, pp. 499–513.
[47] H. Kimura, *Robust stabilizability for a class of transfer functions*, IEEE Trans. Automat. Control, AC–29 (1984), pp. 788–793.
[48] A. M. King, U. B. Desai, and R. E. Skelton, *A generalized approach to Q-Markov covariance equivalent realizations for discrete systems*, Automatica, 53 (1988), pp. 507–515.
[49] M. G. Krein and A. A. Nudelman, *The Markov Moment Problem and Extremal Problems*, AMS, Providence, RI, 1977.

[50] S. Lang, *Fundamentals of Differential Geometry*, Springer-Verlag, New York, 1999.
[51] A. Lindquist, *A new algorithm for optimal filtering of discrete-time stationary processes*, SIAM J. Control, 12 (1974), pp. 736–746.
[52] A. Lindquist, *Some reduced-order non-Riccati equations for linear least-squares estimation: the stationary, single-output case*, Internat. J. Control, 24 (1976), pp. 821–842.
[53] A. Lindquist and G. Picci, *Canonical correlation analysis, approximate covariance extension, and identification of stationary time series*, Automatica J. IFAC, 32 (1996), pp. 709–733.
[54] K. Liu and R. E. Skelton, *A new formulation of Q-Markov covariance equivalent realization*, Appl. Math. Comput., 53 (1993), pp. 83–95.
[55] D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd ed., Addison–Wesley, Reading, MA, 1984.
[56] J. Makhoul, *Linear prediction: A tutorial review*, Proc. IEEE, 63 (1975), pp. 561–580.
[57] J. Mari, A. Dahlén, and A. Lindquist, *A covariance extension approach to identification of time series*, Automatica, 36 (2000), pp. 379–398.
[58] J. D. Markel and A. H. Gray, *Linear Prediction of Speech*, Springer-Verlag, Berlin, 1976.
[59] M. Minoux, Jr., *Mathematical Programming: Theory and Algorithms*, Wiley, New York, 1986.
[60] R. Nagamune, Forthcoming Ph.D. dissertation, Optimization and Systems Theory, Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden.
[61] R. Nagamune, *Sensitivity reduction for SISO systems using the Nevanlinna-Pick interpolation with degree constraint*, in Proceedings of the 14th International Symposium of the Mathematical Theory of Networks and Systems, Perpignan, France, 2000, available only on CD-ROM.
[62] R. Nagamune, *A robust solver using a continuation method for Nevanlinna-Pick interpolation with degree constraint*, in Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, 2001, to appear.
[63] R. Nagmune and A. Lindquist, *Sensitivity shaping in feedback control and analytic interpolation theory*, in Optimal Control and Partial Differential Equations, J. L. Medaldi et al., eds., IOS Press, Amsterdam, 2001, pp. 404–413.
[64] A. V. Oppenheim and R. W. Shafer, *Digital Signal Processing*, Prentice–Hall, London, 1975.
[65] B. Porat, *Digital Processing of Random Signals*, Prentice–Hall, Englewood Cliffs, NJ, 1994.
[66] L. R. Rabiner and R. W. Schafer, *Digital Processing of Speech Signals*, Prentice–Hall, Englewood Cliffs, NJ, 1978.
[67] J. Rissanen, *Recursive identification of linear systems*, SIAM J. Control, 9 (1971), pp. 420–430.
[68] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
[69] I. Schur, *On power series which are bounded in the interior of the unit circle* I *and* II, J. Reine Angew. Math., 148 (1918), pp. 122–145.
[70] A. Tannenbaum, *Feedback stabilization of linear dynamical plants with uncertainty in the gain factor*, Internat. J. Control, 32 (1980), pp. 1–16.
[71] A. Tannenbaum, *Modified Nevanlinna-Pick interpolation of linear plants with uncertainty in the gain factor*, Internat. J. Control, 36 (1982), pp. 331–336.
[72] O. Toeplitz, *Über die Fouriersche Entwicklung positiver Funktionen*, Rend. Circ. Mat. Palermo (2), 32 (1911), pp. 191–192.
[73] P. van Overschee and B. De Moor, *Subspace algorithms for stochastic identification problem*, Automatica, 29 (1993), pp. 649–660.
[74] D. C. Youla and M. Saito, *Interpolation with positive-real functions*, J. Franklin Institute, 284 (1967), pp. 77–108.