On-line trajectory planning using adaptive temporal discretization

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***Abstract*— Direct methods for trajectory optimization are traditionally based on *a priori* temporal discretization and collocation methods. In this work, the problem of node distribution is formulated as an optimization problem, which is to be included in the underlying NLP. The benefits of utilizing such adaptive temporal discretization method for trajectory optimization, are illustrated by a missile guidance example.**

I. Introduction

Consider the following optimal control problem (OCP):

$$\begin{array}{lrcl} \min & J & = & \int_0^T \mathcal{L}(x,u)\mathrm{d}t + \Psi(x(T)) \\ \text{s.t.} & \dot{x} & = & f(x,u) \\ & g(x,u) & \leq & 0 \\ & \Psi_i(x(0)) & \in & S_i \subseteq \mathbb{R}^n \\ & \Psi_f(x(T)) & \in & S_f \subseteq \mathbb{R}^n, \end{array}$$

where the state $x \in \mathbb{R}^n$, the control $u \in \mathbb{R}^m$, and the constraints $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Assuming the problem originates from a complex, real-world application, the existence of analytical solutions is disregarded, thus seeking fast computational algorithms for solving the OCP. To this end, the infinite-dimensional problem of choosing a control function, u, in a given space, have to be *transcribed* into a finite dimensional non-linear mathematical programming problem (NLP). This work focuses on *direct* transcription methods based on *temporal discretization*. In most direct methods (see *e.g.* [1]), the transcription is achieved by *a priori* partition of the time interval into a prescribed number of subintervals whose endpoints are called *nodes*. Generally, trajectory optimization run-times are critically depending on the number of variables in the NLP. These in turn, are proportional to the number of nodes or collocation points[1], N, in the temporal discretization. Therefore, it is extremely important to keep N as low as possible when aiming at constructing computationally efficient methods for trajectory optimization. In most cases however, there exist a trade-off between accuracy and efficiency in terms of computational effort. Here-within, the focus is on improving accuracy for a given efficiency requirement. More precisely, once the number of collocation points has been decided, the question of optimal node distribution is raised.

Although adaptive grid methods have been an active field for the last couple of decades, to the best of our knowledge, utilizing them for trajectory optimization has not been considered elsewhere.

In what follows, we advocate that in any computational efficient method, node distribution should be a part of the optimization process and show that the receding horizon (RH) approach is merely an outcome of such a paradigm. In Section III, the benefits of utilizing the suggested method are confirmed by a missile guidance example.

II. Adaptive temporal discretization

It is a well-established fact in numerical analysis, that a proper distribution of grid points is crucial for the accuracy of the approximating solution. By concentrating the nodes and hence computational effort in those parts of the grid that require most attention (*e.g.* areas with sharp non-linearities and large solution variations), it becomes possible to gain accuracy whilst retaining computational efficiency.

In general, there exist three types of grid adaption techniques [2]. However, as trajectory optimization run-times are critically depending on N, it is the idea of maintaining a fixed number of nodes, but relocating them strategically over the interval that suits the on-line trajectory optimization problem best (this is referred to as r-refinement). To this end, let $p = [t_1, \ldots, t_N] \in \mathbb{R}^N$ denote a partition of $[0, T]$,

$$0 = t_1 < t_2 < \cdots < t_{N-1} < t_N \leq T.$$

Adaptive grid methods are then based on either *equidistribution* of a *monitor function*, or *functional minimization* (FM) [2], [3]. The equidistribution principle (EP) requires a chosen positive definite monitor function (or weight), w, to be equidistributed over all subintervals. Mathematically, the EP can be expressed in various equivalent forms, *e.g.*:

$$m_i(p) = \int_{t_i}^{t_{i+1}} w\mathrm{d}t - \frac{\int_0^T w\mathrm{d}t}{N-1} = 0,\ i = 1, \ldots, N-1, \text{or}$$
$$m_i(p) = \int_{t_{i-1}}^{t_i} w\mathrm{d}t - \int_{t_i}^{t_{i+1}} w\mathrm{d}t = 0,\ i = 2, \ldots, N-1.$$

Commonly employed monitor functions include the "arclength monitor function", $w = \sqrt{\varepsilon + \dot{x}^2}$, and "curvature monitor function", $w = (\varepsilon + \ddot{x}^2)^{\frac{1}{4}}$. Regardless the choice of w, we remark that node allocation by the EP, can be determined by imposing a number of *grid constraints*, $m(p) \leq 0$.

The functional framework to grid generation (FM), is based on the principle of specifying a measure of the grid quality. Traditionally, principles as smoothness, orthogonality and clustering properties of the grid are included in the functional, $I(p)$ [3]. Minimizing $I(p)$, will produce an optimal partition with respect to the chosen grid quality measure.

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[1] Since the nodes and collocation points have the same cardinal number, they are here-whithin considered to be *conceptually* equivalent.

Based on the two existing frameworks for adaptive grid generation (EP and FM), we now outline a generalized approach. The idea is to formulate the problem of collocation point distribution as a constrained optimization problem:

$$\begin{array}{ll} \min & I(p) \\ \text{s.t.} & m(p) \le 0, \end{array} \tag{1}$$

which is to be augmented with the underlying NLP. From (1) it is plainly seen that EP and FM are merely special cases of the suggested approach. We conclude this section by giving examples of the usage of this approach.

Example 1: Setting $d_i = t_{i+1} - t_i, i = 1 \ldots, N-1$, the solution to the following optimization problem:

$$\begin{array}{llll} \min & I(d) & = & \sum_{i=1}^{N-1} d_i - \varepsilon \ln d_i \\ \text{s.t.} & m(d) & = & \sum_{i=1}^{N-1} d_i - T \le 0 \quad (d_i \ge 0), \end{array}$$

is the equidistant RH discretization scheme with ε deciding the step length (and hence planning horizon). This follows since if $(N-1)\varepsilon \le T$, then

$$\nabla_i I(d) = 1 - \frac{\varepsilon}{d_i} = 0 \Longrightarrow d_i = \varepsilon.$$

Example 2: The linear constraint

$$m(d) = \sum_{i=1}^{\varepsilon_1 (N-1)} d_i - \varepsilon_2 T \le 0,$$

reflects the objective of distributing ε_1 parts of the nodes in the first ε_2 parts of the time interval.

III. Design study: missile guidance

The 3DoF equations of motion in the pitch plane consider the rotation of a body-fixed coordinate frame, (X_b, Z_b) about an Earth-fixed inertial frame, (X_e, Z_e). The governing dynamic equations are

$$\begin{array}{rcl} \dot{u} & = & \dfrac{F_x}{m} - qw - g\sin\theta \\ \dot{w} & = & \dfrac{F_z}{m} + qu + g\cos\theta \\ \dot{q} & = & \dfrac{M}{I_y} \\ \dot{\theta} & = & q \\ \dot{x}_e & = & u\cos\theta + w\sin\theta \\ \dot{z}_e & = & -u\sin\theta + w\cos\theta, \end{array}$$

where u and w are the X_b and Z_b components of the velocity vector, x_e and z_e denote the position of the missile in the inertial frame (X_e, Z_e), q is the pitch angular rate, θ denotes the pitch angle, m is the missile mass, g is the gravitational force, while I_y denotes the pitching moment of inertia. The system inputs are the applied pitch moment, M, together with the aerodynamic forces, F_x, F_z, acting along the X_b and Z_b axis respectively. During the simulations, we set $m = 204.02\ kg$, $g = 9.8\ m/s^2$ and $I_y = 247.437\ kg\, m^2$.

Referring to Fig. 1 and 2, the first simulation shows the terminal guidance part of a missile trajectory optimization problem. The missile starts off horizontally from $(0, 10)$ aiming at a target in $(700, 0)$ with terminal aspect angle $-\frac{\pi}{2}$. Fig. 1 depicts the reference trajectories with the missile velocities (in the inertial frame) indicated by small line segments.

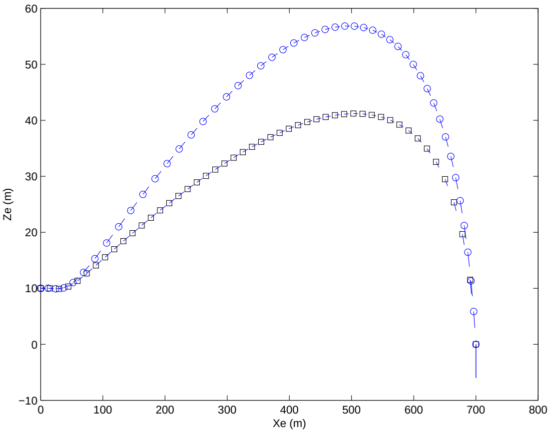

Fig. 1. Reference trajectories: static (○) and adaptive (□).

In the adaptive case, an EP based on the arclength monitor function together with a linear $I(p)$ is used. Seeing beyond the unequal axis scales, the nodes have been distributed more evenly over different path segments. In fact, there are 7 nodes/100 m path segment in the adaptive case, while the same figure varies between $5 - 13$ in the static case.

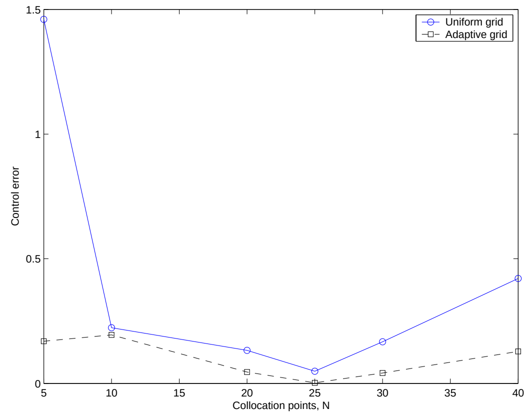

Fig. 2. The accuracy of the approximating control.

Fig. 2 shows the optimal control approximation error as a function of N. It can be noted that, for a given N, the extra degree of freedom provided by distributing the nodes is used constructively to improve accuracy. This illustrates the soundness of the proposed approach.

References

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