Observability and Active Observers for Mobile Robotic Systems

David A. Anisi* and Xiaoming Hu

***Abstract*— An important class of non-uniformly observable systems come from applications in mobile robotics. In this paper, the problem of active observer design for such systems is considered. The set of feasible configurations and the set of output flow equivalent states is defined. It is shown that the inter-relation between these two sets serves as the basis for design of active observers. The proposed observer design methodology is illustrated by considering a unicycle robot model, equipped with a set of range-measuring sensors.**

I. Introduction

Since 1970's there has been an extensive study on the design of observers for nonlinear control systems, [1]–[7]. It is known that for such systems, observability does not only depend on the initial conditions, but also on the exciting control. Most current methods, such as observers with linearizable error dynamics [3] and high gain observers [6], [7], lead to the design of an exponential observer. As a necessary condition for the existence of a smooth exponential observer, the linearized pair must be detectable [5]. In fact, most of the existing non-linear observer design methods are only applicable to uniformly observable nonlinear systems. This is witnessed in [8], where it is pointed out that one of the key questions in nonlinear control is "how to design a nonlinear observer for nonlinear systems whose linearization is neither observable nor detectable".

An important class of non-uniformly observable systems come from applications in mobile robotics. For such systems, due to environmental restrictions and the way the sensors function, constraints have to be put on the control. This thus presents an interesting issue: how to design an exciting control to maximize the rate of convergence for an observer, namely how to design an *active observer*. Maximizing "observability" has been an important issue in the field of active perception in robotics and computer vision [9]. However, study from the systems and control point of view in terms of observer design still lacks, [10]. This paper considers the problem of active observer design for mobile robotic systems and an alternative design method is presented. The disposition is as follows; In Section II, a brief review on nonlinear observability and observers is given. This would set stage for our study on observability and active observer design for mobile robotic systems in Section III. To illustrate the concepts introduced in Section III-A, a case study is given in Section III-B. The simulation results thereof are presented in Section IV and finally, some concluding remarks are made in Section V.

II. Preliminaries

Consider the nonlinear control system

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{F}(x,u) \quad \text{(system dynamics)} \\ y &= h(x) \quad \text{(system output)} \end{cases}$$

with state $x \in \mathcal{X}$, control $u \in \mathcal{U}$ and output $y \in \mathcal{Y}$. Here $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are smooth manifolds of dimension n, p and m respectively. All mappings in this paper, are assumed to be smooth. If Σ is complete, the composed mapping from $u(\cdot)$ to $y(\cdot)$ is referred to as the input-output map of Σ at x_0 [11]:

$$\mathcal{IO}^{\Sigma}_{x_0} \colon u(\cdot) \mapsto y(\cdot).$$

The most common definitions of the observability properties of Σ then boil down to the injectivity properties of $\mathcal{IO}^{\Sigma}_{x_0}$ with respect to the initial condition, x_0. Consider two states, x_1 and x_2, being equivalent (denoted $x_1 \sim x_2$) if and only if they have the same input-output map for all admissible inputs, *i.e.*

$$x_1 \sim x_2 \Longleftrightarrow \mathcal{IO}^{\Sigma}_{x_1}(u(\cdot)) = \mathcal{IO}^{\Sigma}_{x_2}(u(\cdot)), \quad \forall u(\cdot) \in \mathcal{U}.$$

Further, let $\mathbb{I}(x_0)$ denote the *equivalence class* of x_0, *i.e.* let $\mathbb{I}(x_0) = \{x \in \mathcal{X} \colon x \sim x_0\}$. Based on this, we arrive at the following two definitions [12], [13].

Definition 1 (Indistinguishability): Two states, x_1 and x_2 are said to be *indistinguishable* iff they are equivalent.

Definition 2 (Observability): Σ is said to be *observable at* x_0 if $\mathbb{I}(x_0) = \{x_0\}$. It is further said to be observable if $\mathbb{I}(x) = \{x\}$ for all $x \in \mathcal{X}$.

It is notable that the equivalence relation on $\mathcal{X}$, and hence observability, is a *global concept* in two senses:

Property 1: All states in $\mathcal{X}$ are to be distinguished from each other.

Property 2: The generated trajectories are unrestricted.

Also, observability is an *infinite-horizon concept*, since:

Property 3: There is no upper bound on the time-interval that has to be considered in order to distinguish points.

Consequently it is possible to introduce various restrictions, or relaxations on Definition 2. Some of these modifications are considered below.[1]

Given a system Σ and an open set $\Omega \subseteq \mathcal{X}$, the restriction Σ_Ω refers to a control system with state space Ω, defined

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The authors are with the Division of Optimization and Systems Theory at the Royal Institute of Technology (KTH), Stockholm, Sweden

[1] The observability nomenclature is not standardized. In this article, the terms used by Hermann and Krener in [12] and Respondek in [13] are merged.

by the restriction of $\mathcal{F}$ and h to $\Omega \times \mathcal{U}_\Omega$ and Ω respectively. Here $\mathcal{U}_\Omega$ denotes the subset of all admissible inputs that generates trajectories that lie in Ω.

Definition 3 (Ω-indistinguishability): Two initial states, $x_1, x_2 \in \Omega$ are said to be Ω- *indistinguishable* if

$$\mathcal{IO}^{\Sigma_\Omega}_{x_1}(u(\cdot)) = \mathcal{IO}^{\Sigma_\Omega}_{x_2}(u(\cdot)), \ \ \forall u(\cdot) \in \mathcal{U}_\Omega.$$

This relation will be denoted $x_1 \tilde{_\Omega}\, x_2$ and $\mathbb{I}_\Omega(x)$.

Definition 4 (Strong observability): The system Σ is said to be strongly observable at x_0 if for *every* open neighborhood Ω of x_0, $\mathbb{I}_\Omega(x_0) = \{x_0\}$. Σ is called *strongly observable* if it is strongly observable for all $x \in \mathcal{X}$.

Note that strong observability implies observability since $\mathbb{I}_\Omega(x) = \{x\}$ for all $\Omega \subseteq \mathcal{X}$ gives $\mathbb{I}(x) = \{x\}$ for the special choice of $\Omega = \mathcal{X}$.

Definition 5 (Weak observability): The system Σ is called weakly observable at x_0 if there exist a neighborhood of x_0, $N(x_0)$, such that $\mathbb{I}(x_0) \cap N(x_0) = \{x_0\}$. Σ is *weakly observable* if it is weakly observable at every $x \in \mathcal{X}$.

Definition 6 (Instant observability): The system Σ is said to be instantaneously observable at x_0 if there exist a neighborhood $N(x_0)$, such that for every open neighborhood Ω of x_0 contained in N, $\mathbb{I}_\Omega(x_0) = \{x_0\}$. Σ is called *instantaneously observable* if it is so at every $x_0 \in \mathcal{X}$.

For the dynamical system, Σ, an observer may be defined as follows (*cf.* [1], [4], [14]).

Definition 7 (Observer): A dynamical system with state manifold $\mathcal{Z}$, input manifold $\mathcal{U} \times \mathcal{Y}$, together with a mapping $\hat{\mathcal{F}} : (\mathcal{Z} \times \mathcal{U} \times \mathcal{Y}) \to T\mathcal{Z}$ is an *observer* for the system Σ, if there exists a smooth mapping $\Psi : \mathcal{X} \to \mathcal{Z}$, such that the diagram shown in Figure 1, commutes and the error trajectory $x(t) - \hat{x}(t)$ converges to zero as $t \to \infty$.

$$\begin{tikzcd}
 & T\mathcal{X} \arrow[r, "\Psi_*"] \arrow[dl, "\tau_\mathcal{X}"'] & T\mathcal{Z} \arrow[dr, "\tau_\mathcal{Z}"] & \\
\mathcal{X} & & & \mathcal{Z} \\
 & \mathcal{X} \times \mathcal{U} \arrow[ul, "\pi_\mathcal{X}"] \arrow[u, "\mathcal{F}"'] \arrow[r, "(\Psi \circ \pi_\mathcal{X}, \pi_\mathcal{U}, h \circ \pi_\mathcal{X})"] & \mathcal{Z} \times \mathcal{U} \times \mathcal{Y} \arrow[u, "\hat{\mathcal{F}}"'] \arrow[ur, "\pi_\mathcal{Z}"'] \arrow[ull, dashed, bend left, "\Phi"] &
\end{tikzcd}$$

Fig. 1. Commutative diagram defining an observer.

In diagram 1, Ψ_* denotes the tangent mapping, π is projection upon a cartesian factor, while τ denotes the projection of the tangent bundle.

According to Definition 7, the objective when designing a general observer, is to track $\Psi(x)$, rather than x itself. Note that the same observer dynamics, $\hat{\mathcal{F}}$, may allow several *different* full observer mappings, Φ, and that in general, a full state observer

$$\hat{\Sigma} : \begin{cases} \dot{z} &= \hat{\mathcal{F}}(z, u, y) \\ \hat{x} &= \Phi(z, u, y) \end{cases}$$

may *not* be put in the form $\dot{\hat{x}} = \Xi(\hat{x}, u, y)$.

III. Mobile robotic systems

One distinguishing feature of mobile robots is the use of *exteroceptive* sensors for sensing the environment and aid localization. The output of Σ is next extended to more explicitly incorporate exteroceptive sensor readings.

Bearing in mind the particular applications encountered in the robotics community, it seems convenient to split the state vector, $x \in \mathcal{X}$, into two parts; one defining the state of the platform in its *work-space*, $\mathcal{W}$, and the other only in its *configuration-space*, $\mathcal{C}$, so that $x = (x_w, x_c) \in \mathcal{W} \times \mathcal{C} = \mathcal{X}$. The work-space of the robot, $\mathcal{W}$, is assumed to be a smooth and connected manifold of dimension $n_w \in \{1, 2, 3\}$. However, the configuration-space, $\mathcal{C}$, might have arbitrary dimension, n_c, and includes typically the description of the internal states of the platform.

Consider control-affine dynamic systems of form:

$$\Sigma_{rob} : \begin{cases} \dot{x}_w &= f_w(x) + g_w(x)u \\ \dot{x}_c &= f_c(x) + g_c(x)u \\ y &= \tilde{h}(x, s_\theta(x)) \\ q &= \theta(s_\theta), \end{cases}$$

where $x_w \in \mathcal{W}$, $x_c \in \mathcal{C}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. We use $s_\theta(x)$ to indicate the interaction of the sensors with the environment but also to emphasize the dependence of the output on the *environmental map*, θ. In this paper, the case where the components of the environment (*e.g.* surrounding terrain, obstacles or walls) can be modeled as a single, connected, $(n_w - 1)$-dimensional smooth manifold (hyper-surface) in $\mathcal{W}$ will be in focus. It is further assumed that this hyper-surface can be parametrized as

$$q = \theta(s_\theta), \qquad s_\theta \in \mathcal{S} \subseteq \mathbb{R}^{(n_w - 1)},$$

where θ is known. This last assumption relates to one of the fundamental problems in robotics, namely the simultaneous localization and map building problem (SLAM), where one tries to reconstruct the environmental map, θ, and the full state vector, x, at the same time. By assuming the map to be given, we focus on a subproblem in SLAM, namely the re-localization problem where the state vector, x, is to be reconstructed based on a combination of exteroceptive and introceptive sensor-readings.

Example 1: Consider a nonholonomic vehicle equipped with a range sensor mounted along its direction of orientation, ϕ. It moves inside an elliptic field, with half-axes c_1 and c_2, centered at the origin of $\mathcal{W}$. Then the hyper-surface

$$\left(\frac{q_1}{c_1}\right)^2 + \left(\frac{q_2}{c_2}\right)^2 - 1 = 0,$$

models the surrounding in $\mathbb{R}^2$. It can be parameterized by the angle $\tau \in S^1$, so that

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos(\tau) \\ c_2 \sin(\tau) \end{bmatrix} = \theta(s_\theta),$$

with $s_\theta = [\tau \ c_1 \ c_2]$. The control system can be modeled as

$$\begin{aligned}\dot{x}_1 &= u_1 \cos(\phi)\\ \dot{x}_2 &= u_1 \sin(\phi)\\ \dot{\phi} &= u_2\\ y &= (c_1\cos(\tau) - x_1)\cos(\phi) + (c_2 \sin(\tau) - x_2)\sin(\phi),\end{aligned}$$

where $(x_1, x_2) \in \mathbb{R}^2$ is the position of the reference point on the robot, $\phi \in S^1$ denotes its orientation and the two control inputs, u_1 and u_2 are the robot's linear- and lateral velocities respectively. In addition, τ as a function of the state is implicitly defined by

$$\frac{c_2 \sin(\tau) - x_2}{c_1 \cos(\tau) - x_1} = \tan(\phi).$$

Example 2: Consider a nonholonomic mobile robot equipped with a centrally mounted video camera. The environment consists of the goal flag and the start flag. The task for the robot is to map the environment while localizing itself in the map. Naturally, one of the easiest ways to construct a coordinate system is to set the goal flag as the origin and set the start flag on the x_1-axis, *i.e.* with coordinates $(d_0, 0)$. If we assume that on the image plane what we can identify is the distance of the vertical line feature to the center and the focal length of the camera is one, then the output of the system can be expressed as

$$\begin{aligned} y_1 &= \tan(\phi - \text{atan}(-x_2, d_0 - x_1))\\ y_2 &= \tan(\phi - \text{atan}(-x_2, -x_1)).\end{aligned}$$

A. Observability and active observers

As pointed out in Section II, observability is an *infinite-horizon* concept (Property 3). To adapt this for the area in mind, the following is suggested.

Definition 8 (Small-time observability): A nonlinear system, Σ_{rob}, is said to be *small-time observable* at x_1, if for any $x_2 \in \mathcal{X}$ and $T > 0$, there exists a control, $u(\cdot) \in \mathcal{U}$ and $t_0 \leq T$, such that

$$\tilde{h}(x(t_0, x_1, u(\cdot)), s_\theta) \neq \tilde{h}(x(t_0, x_2, u(\cdot)), s_\theta).$$

It is further said to be small-time observable if it is so at every $x_1 \in \mathcal{X}$.

Although not made explicit due to space limitation, modified versions of Definition 8 (*i.e.* weakly/strongly small time observability) can be obtained in apparent manners.

To stress the distinction between the newly introduced definition and those of Section II, recall that Ω-distinguishability, the underlying concept of Definition 4, only involves separation of points in the restricting Ω. In extension, the term "instantaneously" in Definition 6 has to be interpreted in two senses; namely that a point can be *instantly distinguished* from its *instant neighbors*. Therefore there is no natural setting for solely modifying Property 3, without necessarily modifying Property 1 and/or 2. In contrast, small-time observability requires instant distinction of x_1 from *all* other states $x_2 \in \mathcal{X}$, or in the case of weakly small-time observability, instant distinction of x_1 from all x_2 in some open neighborhood of x_1. Hence, they *only* restrict the time-interval that have to be considered in order to find deviating output.

Given the environmental map $\theta(s_\theta)$, the sensor measurements are considered as a mapping, $\tilde{h} : \mathcal{X} \to \mathcal{Y}$. For a given measurement, $y \in \mathcal{Y}$, the inverse image of y under $\tilde{h}$ is the set of all $x \in \mathcal{X}$ such that $\tilde{h}(x, s_\theta) = y$. In general, $\mathcal{X}$ and $\mathcal{Y}$ do not have the same cardinal number so that a measurement might correspond to more that one state in $\mathcal{X}$.

Definition 9 (Set of feasible states): The *set of feasible states* with respect to y, denoted $\mathcal{FS}_y$, is defined as the inverse image of y under $\tilde{h}$ in the state-space, *i.e.*

$$\mathcal{FS}_y = \{x \in \mathcal{X} : \tilde{h}(x, s_\theta) = y\}.$$

To introduce a measure of how well a certain point in the state-space matches a given measurement, a functional or *value-function* is needed:

Definition 10 (Value-function): A non-negative functional,

$$V_y \colon \mathcal{X} \to \mathbb{R}^+,$$

such that,

$$x \in \mathcal{FS}_y \Longleftrightarrow V_y(x) = 0,$$

is called a value-function.

It is notable that Definition 10 is well-suited for scenarios where one might have noisy measurements. In such cases, the feasible states may consist of all x, such that $V_y(x) \leq \varepsilon$, for some $\varepsilon \in \mathbb{R}^+$.

By utilizing the value-function, it is possible to drive the state estimation within the set of feasible states, $\mathcal{FS}_y$. This will be shown in greater detail in Section III-B. Next, we focus on the problem of localizing the actual state *within* this set. In order to distinguish the states in $\mathcal{FS}_y$, it is necessary that the system output do not remain constant, *i.e.* the exciting control has to be designed such that $\dot{y} \neq 0$. For each point $x_0 \in \mathcal{X}$, it is possible to associate another set to it consisting of all points that have the same output flow.

Definition 11 (Set of output flow equivalent states): Given any admissible control, $u(\cdot) \in \mathcal{U}$, for each state $x_0 \in \mathcal{X}$, the set of states that are *output flow equivalent to* x_0 *under* $u(\cdot)$, denoted $\mathcal{OF}^u_{x_0}$, is defined as all states $x_1 \in \mathcal{X}$, such that there exists $T > 0$ such that for all $t \in [0 \quad T]$,

$$\begin{aligned}&\tilde{h}(x(t, x_1, u(\cdot)), s_\theta) - \tilde{h}(x_1, s_\theta) \equiv\\ &\tilde{h}(x(t, x_0, u(\cdot)), s_\theta) - \tilde{h}(x_0, s_\theta).\end{aligned}$$

By means of the two sets defined in this section, it is possible to put constraints on the exciting control.

Proposition 1: Given $x_0 \in \mathcal{X}$, if there exists an exciting control, $u_0(\cdot) \in \mathcal{U}$, and a neighborhood, $N(x_0)$ such that

$$\mathcal{FS}_y \cap \mathcal{OF}^{u_0}_{x_0} \cap N(x_0) = \{x_0\}, \tag{1}$$

then the system is weakly small-time observable at x_0.

Proof: We prove by contradiction. Suppose the system is *not* weakly small-time observable at x_0, *i.e.*

$$\begin{aligned}&\exists\, x_1 \in N(x_0)\backslash x_0 \text{ and } T > 0 : \forall t \in [0 \quad T] \text{ and } \forall u(\cdot) \in \mathcal{U},\\ &\tilde{h}(x(t, x_1, u(\cdot)), s_\theta) \equiv \tilde{h}(x(t, x_0, u(\cdot)), s_\theta). \qquad (2)\end{aligned}$$

For the special choise of $t = 0$, Equation (2) gives

$$\tilde{h}(x_1, s_\theta) = \tilde{h}(x_0, s_\theta) = y, \tag{3}$$

meaning that $x_1 \in \mathcal{FS}_y$. Consider then the special choice of $u(\cdot) = u_0(\cdot)$, which together with Equation (3) and Definition 11 implies that $x_1 \in \mathcal{OF}_{x_0}^{u_0}$. Hence we have shown that assuming (2) implies the existence of x_1 such that

$$(x_1 \in N(x_0) \backslash x_0) \wedge (x_1 \in \mathcal{FS}_y) \wedge (x_1 \in \mathcal{OF}_{x_0}^{u_0}), \quad \Leftrightarrow$$
$$\mathcal{FS}_y \cap \mathcal{OF}_{x_0}^{u_0} \cap N(x_0) \neq \{x_0\}.$$

■

Constraint (1) serves as the basis for design of active observers.

B. Design study

In this section, we revisit the robot model from Example 1. The sensor readings however will differ. It is now assumed that the robot is equipped with l range-measuring sensors, oriented at angles $\alpha_i, i = 1, \cdots, l$ with respect to ϕ. Referring to Figure 2, sensor i measures distance ρ_i against some smooth closed curve, $\theta : S^1 \to \mathbb{R}^2$, that models the terrain. Each sensor is directed along a ray making an angle of $\phi + \alpha_i$ with the x_1-axis. Thus the outputs for the system are

$$y_i = \rho_i, \quad i = 1, \cdots, l.$$

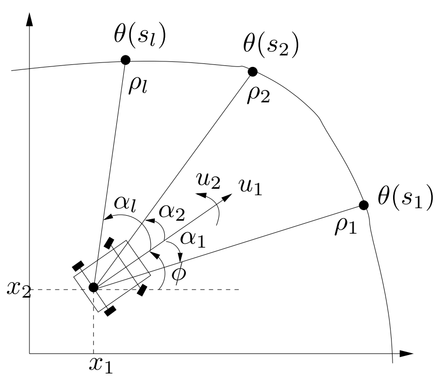

Fig. 2. The unicycle robot equipped with l range-measuring sensors.

The goal is to reconstruct the full state, x, based on sensor readings $\rho = (\rho_1, \cdots, \rho_l)$. As remarked in conjunction with Definition 7, the objective when designing a general observer is to track $z = \Psi(x)$, rather than x itself. To this end, it is noted that the problem of reconstructing x in this particular problem, is equivalent to the reconstruction of vehicle orientation, ϕ, and the parameter values, $s_i \in S^1$, $i = 1, \cdots, l$, corresponding to the points on the curve measured against. It is so since the relative orientation angle of each sensor, α_i, is known. Setting the observer state

$$z = \Psi(x) = [s_1, \cdots, s_l, \phi]^T \in \mathcal{Z},$$

the following geometrical relationship between x and z holds:

$$\theta(s_i) = [x_1, x_2]^T + \rho_i R_{(\phi+\alpha_i)} e_1, \quad i = 1, \cdots, l, \tag{4}$$

where $e_1 = [1 \quad 0]^T$ and R_α denotes the rotation matrix,

$$R_\alpha = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}^T.$$

Aiming at constructing an appropriate value-function that can aid the observer design, for each sensor i, define a mapping $v_i : \mathcal{Z} \to \mathbb{R}^2$ according to

$$v_i(z) = \theta(s_i) - \rho_i R_{(\phi+\alpha_i)} e_1.$$

Intuitively, $v_i(z)$ points out where measurement i indicates that the vehicle is located in $\mathbb{R}^2$. Next, define $v_{ij} : \mathcal{Z} \to \mathbb{R}^2$ as

$$v_{ij}(z) = v_i(z) - v_j(z),$$

which indicates the difference between the vehicle location estimated by measurements ρ_i and ρ_j. Finally, the value-function is defined as

$$V_\rho(z) = \sum_{i=1}^{l-1} \sum_{j>i} v_{ij}(z)^T v_{ij}(z).$$

The non-negative value-function, $V_\rho(z)$, serves as a measure of how well z, matches a set of measurements, ρ. To see this, it is noticed that $V_\rho(z) = 0$ implies that in the observer state, z (which naturally corresponds to a state, $x \in \mathcal{X}$, by relation (4)), the vehicle precisely measures the distances ρ_i against the points $\theta(s_i), i = 1, \cdots, l$. In the other direction, clearly if ρ are the measured distances and z is the actual observer state, then $v_i(z) = [x_1, x_2]^T$, for all i, and hence $V_\rho(z) = 0$. This allows us to specify the set of feasible states by means of the value-function, as discussed earlier. In addition, the value-function can be used in the observer design as follows: the time derivative of $V_\rho(z) \in \mathbb{R}$ equals

$$\dot{V}_\rho(z) = \frac{\partial V_\rho(z)}{\partial z} \dot{z}.$$

Then by choosing the steepest descent direction, it is clear that the gradient flow

$$\dot{z}_V = -k_V \left[\frac{\partial V_\rho(z)}{\partial z}\right]^T,$$

should be included in the observer design. It serves to drive the state estimation within the set of feasible states. As for the set of output flow equivalent states, from (4) we obtain

$$0 = \theta'(s_i)^T R_\phi (u_1 e_2 + \dot{\rho}_i R_{\alpha_i} e_2 - \rho_i u_2 R_{\alpha_i} e_1) \triangleq Q_i(z),$$

by first differentiating with respect to time and the multiplying by $\theta'(s_i)^T R_{\frac{\pi}{2}}$ from the left. Then, the mapping $Q : \mathcal{Z} \to \mathbb{R}^l$, defined by

$$Q(z) = [Q_1(z), \cdots, Q_l(z)]^T = 0,$$

characterizes the set of output flow equivalent states for this system. Under suitable assumptions on the exciting control, the sensor orientations and the environmental map (see [15] for details), it can be shown that this set and the set of feasible states together fulfill the condition of Proposition 1, which implies that we are bound to have weakly small-time observability.

Setting $V_Q(z) = Q^T(z)Q(z) \in \mathbb{R}^+$, gives

$$\dot{V}_Q(z) = 2Q^T(z)\frac{\partial Q(z)}{\partial z}\dot{z}.$$

Again, with the choice of the steepest descent direction, the following term is to be included in the observer dynamics,

$$\dot{z}_Q = -k_Q\left[\frac{\partial Q(z)}{\partial z}\right]^T Q(z).$$

Putting it all together, the following observer dynamics is proposed for this particular problem:

$$\dot{z} = \dot{z}_V + \dot{z}_Q = -k_V\left[\frac{\partial V_\rho(z)}{\partial z}\right]^T - k_Q\left[\frac{\partial Q(z)}{\partial z}\right]^T Q(z),$$

where $k_V, k_Q > 0$ are suitably chosen observer gains.

To complete the observer design, the full observer mapping, Φ, is to be decided (*cf.* Figure 1). By relation (4), any parameter value, s_i, together with ϕ, suffice for reconstructing x. Thus there are several choices for Φ. However, in the case of faulty measurements, different parameter values might give inconsistent state estimation, why for instance a simple vector average can be chosen.

IV. Simulations

In this section we consider the case when the robot is equipped with two range-measuring sensors ($l = 2$) and moves inside the same elliptic field as considered in Example 1. In what follows, x and z will denote the true states while $\hat{x}$ and $\hat{z}$ will denote the estimations of them. All true states will be plotted with blue/dashed lines, while estimations will be graphed in red/solid. The robot starts off from $x(0) = [1, \; -1, \; \frac{\pi}{2}]^T$, which corresponds to $z(0) = [\frac{23\pi}{4}, \; \frac{242\pi}{1101}, \; \frac{\pi}{2}]^T$ in the $\mathcal{Z}-$space. The observer is initialized at $\hat{z}(0)$, a *randomly* generated point in the vicinity of $z(0)$. The observer gains are set to $k_V = 5, k_Q = 1$.

Figure 3 shows the trajectory of the components of $z(t)$ (in dashed/blue) and $\hat{z}(t)$ (in solid/red). This, together with Figure 4, where the relative errors have been plotted, shows the convergence of the observer in the $\mathcal{Z}$-space.

Of more practical importance however is the convergence of $\hat{x}(t)$ to $x(t)$ in the state-space, $\mathcal{X}$. Figures 5 and 6 show the observation and relative errors as measured after mapping $\hat{z}$ into $\hat{x}$ by means of the full observer mapping, Φ.

Noisy measurements

Next, attention is paid to the case when the presence of measurement noise is recognized. The noise parameter has been chosen such that the relative measurement errors amount to approximately 5%. Referring to Figure 7, it can be noted how the observer rejects the disturbance and tracks the true observer state quite well even in the presence of measurement errors. This statement is verified when considering the time history of $x(t)$ and $\hat{x}(t)$ in the state-space (Figure 8). In cases when (4) is inconsistent for $i = 1$ and 2, a simple vector average has served as the estimated position. One desirable property of this choice is that a true measurement from one sensor can be used constructively to compensate for the faulty measurement of the other one. Thus we notice in Figure 8 that, in the presence of measurement noise, the position estimation is

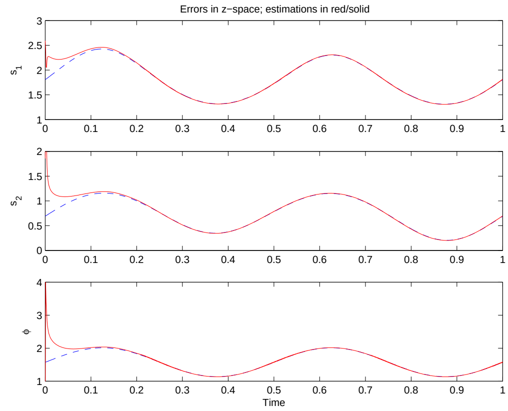

Fig. 3. Observation error in $\mathcal{Z}$-space.

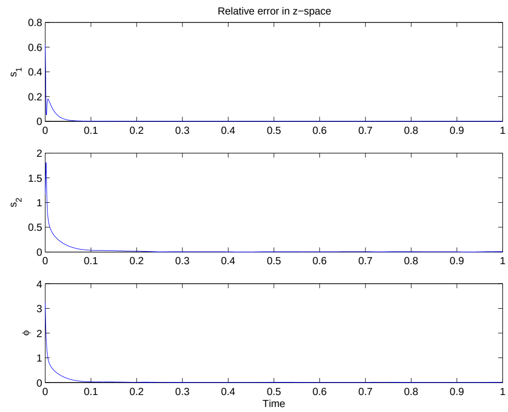

Fig. 4. Relative error in $\mathcal{Z}$-space.

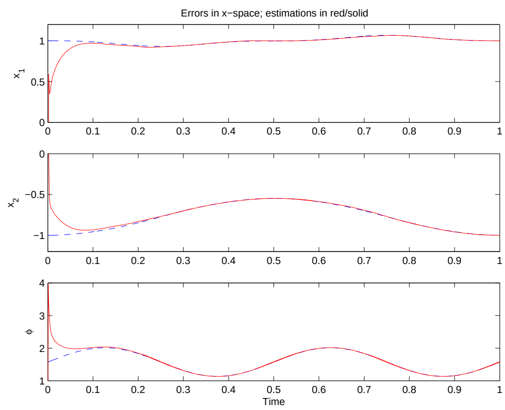

Fig. 5. Observation error in $\mathcal{X}$-space.

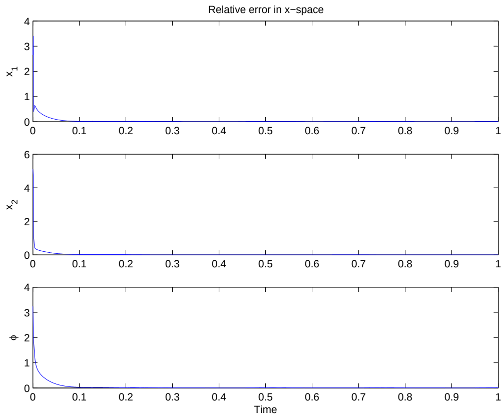

Fig. 6. Relative error in $\mathcal{X}$-space.

much better than the estimation of the orientation angle, ϕ.

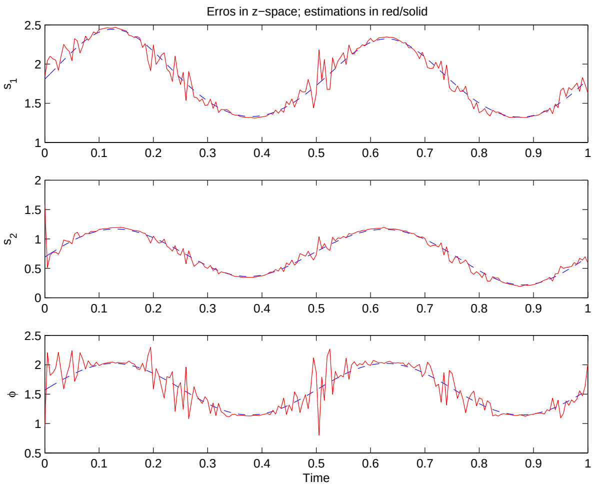

Fig. 7. Observation error in $\mathcal{Z}$-space with noise.

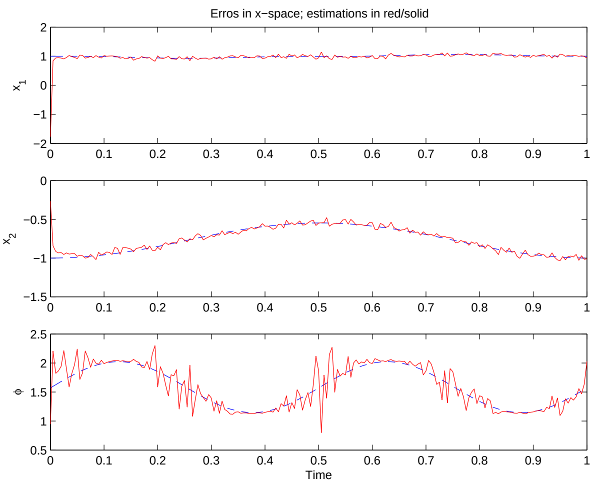

Fig. 8. Observation error in $\mathcal{X}$-space with noise.

V. Concluding remarks

In this paper, the extension of the concepts of observability and observer design to the field of mobile robotics is considered. Such systems have several distinguishing features. Firstly, mobile robots are typically non-uniformly observable systems so that the observer gains, as well as its convergence properties will depend on the system input. In addition, beacuse of the interaction of the exteropective sensors with the environment, the convergence of the observer typically will also depend on the environment. Therefore, in order to succeed in reconstructing the state, the exciting control has to be chosen in a deliberate manner, *i.e.* an *active observer* has to be designed. Finally, since most existing observer design techniques are only applicable to uniformly observable nonlinear systems, alternative approaches that aid the observer design are needed. The set of feasible configurations, its relation with the value-function, the set of output flow equivalent states, and the inter-relation between these two sets, provide such a setting. The design study presented here-within, serves to illustrate the use of these concepts in the observer design process.

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