RIEMANNIAN OBSERVERS FOR EULER-LAGRANGE SYSTEMS

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Abstract: In this paper, a *geometrically intrinsic* observer for Euler-Lagrange systems is defined and analyzed. This observer is an generalization of the observer recently proposed by Aghannan and Rouchon. Their contractivity result is reproduced and complemented by a proof that the region of contractivity is infinitely thin. However, assuming *a priori* bounds on the velocities, convergence of the observer is shown by means of Lyapunov's direct method in the case of configuration manifolds with constant curvature. The convergence properties of the observer are illustrated by an example where the configuration manifold is the three-dimensional sphere, S^3.

Keywords: Nonlinear Observers, Intrinsic Observers, Differential Geometric Methods, Euler-Lagrange Systems, Contraction Analysis, Nonlinear Systems Theory

1. INTRODUCTION

For a dynamical system, an *observer* is another dynamical system whose task is to reconstruct missing state information, while only using available measurements. The input to the observer is the output of the original system (which may include its input), and the observer is expected to produce as output an estimate of some state function of the original system.
Consider the nonlinear dynamical system

$$\Sigma : \begin{cases} \dot{z} = \mathcal{F}(z,u) \\ y = h(z,u) \end{cases}$$

with state $z \in \mathcal{Z}$, control $u \in \mathcal{U}$ and output $y \in \mathcal{Y}$. Here, $\mathcal{Z}, \mathcal{U}$ and $\mathcal{Y}$ are smooth manifolds. All mappings in this paper, are assumed to be smooth.

Definition 1. (Observer). A dynamical system with state manifold $\mathcal{W}$, input manifold $\mathcal{Y}$, together with a mapping $\hat{\mathcal{F}} : (\mathcal{W} \times \mathcal{Y}) \to T\mathcal{W}$ is an *observer* for the system Σ, if there exists a smooth mapping $\Psi : \mathcal{Z} \to \mathcal{W}$, such that the diagram shown in figure 1 (the dashed arrow excluded), commutes. The observer gives a *full state reconstruction* if there is a mapping $Z : (\mathcal{W} \times \mathcal{Y}) \to \mathcal{Z}$ such that the full diagram in figure 1 is commutative (cf. (van der Schaft, 1985) and (Thau, 1973)).

$$\begin{tikzcd}
 & T\mathcal{Z} \arrow[rr, "\Psi_*"] \arrow[dl, "\tau_{\mathcal{Z}}"'] & & T\mathcal{W} \arrow[dr, "\tau_{\mathcal{W}}"] & \\
\mathcal{Z} & & & & \mathcal{W} \\
 & \mathcal{Z} \times \mathcal{U} \arrow[ul, "\pi_{\mathcal{Z}}"] \arrow[uu, "\mathcal{F}"'] \arrow[rr, "(\Psi \circ \pi_{\mathcal{Z}}, h)"] & & \mathcal{W} \times \mathcal{Y} \arrow[uu, "\hat{\mathcal{F}}"'] \arrow[ur, "\pi_{\mathcal{W}}"'] \arrow[llllu, dashed, bend left, "Z"] &
\end{tikzcd}$$

Fig. 1. Commutative diagram defining an observer.

In diagram 1, Ψ_* denotes the tangent mapping, π is projection upon a cartesian factor, while τ denotes the projection of the tangent bundle.
According to definition 1, the objective when design-

ing a general observer, is to track $\Psi(z)$, rather than z itself. The special case when Ψ equals the identity mapping and $\mathcal{W} = \mathcal{Z}$, is often referred to as an *identity observer*. Also, note that the same observer dynamics, $\hat{\mathcal{F}}$, may allow several *different* full observer mappings, Z, and that in general, a full state observer

$$\hat{\Sigma} : \begin{cases} \dot{w} = \hat{\mathcal{F}}(w, y) \\ \hat{z} = Z(w, y) \end{cases}$$

may *not* be put in the form $\dot{\hat{z}} = \Xi(\hat{z}, y)$.
As a consequence of this definition, an observer has the following property:

Property 1. $w(t_0) = \Psi(z(t_0))$ at some time instance, t_0, yields $w(t) = \Psi(z(t))$ for all $t \geq t_0$.

It is also reasonable to require the stronger property:

Property 2. As time proceeds, the trajectories $w(t)$ and $\Psi(z(t))$ should converge[1] for every input.

This property, *i.e.* the convergence properties of the observer, may be demonstrated in different ways. If G is a Riemannian metric on $\mathcal{W}$, whose Lie derivative along the vector field $\hat{\mathcal{F}}$, is negative for every input, y, ($\mathcal{L}_{\hat{\mathcal{F}}} G < 0$), then the Riemannian distance between any two trajectories tends to zero (*c.f.* (Lohmiller and Slotine, 1998)). This is a property of the control system $\mathcal{W}$ alone. In conjunction with property 1, this implies property 2. More precisely, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Phi^t_{\hat{\mathcal{F}}}\rho_0} \mathrm{d}s = \int_{\Phi^t_{\hat{\mathcal{F}}}\rho_0} \frac{1}{2}(\mathcal{L}_{\hat{\mathcal{F}}} G)(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}x}{\mathrm{d}s})\mathrm{d}s,$$

so if $\mathcal{L}_{\hat{\mathcal{F}}} G < 0$, then

$$\inf \int_{w_1(t)}^{w_2(t)} \mathrm{d}s \leq \int_{\Phi^t_{\hat{\mathcal{F}}}\rho_0} \mathrm{d}s \leq \int_{\rho_0} \mathrm{d}s \triangleq \inf \int_{w_1(0)}^{w_2(0)} \mathrm{d}s.$$

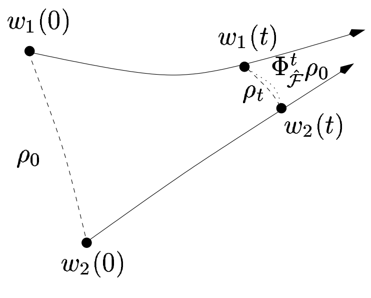

Fig. 2. The length of the geodesic curve ρ_t, between two trajectories is decreasing.

However, the assumption that the observer dynamics is contractive, is very restrictive and in most cases, property 2 has to be shown by means of a common Lyapunov function for $(\mathcal{Z} \times \mathcal{W} \times \mathcal{U})$.

In this paper, we study the observer design problem for a class of nonlinear systems, *viz.* Euler-Lagrange systems, where we assume that the output of the system is the generalized position and force, and that we want to reconstruct the generalized velocities.
The Euler-Lagrange equations are *intrinsic*, *i.e.* geometrically defined in terms of g and U only, and may be written in a coordinate-free way (Hamberg, 2000). It is natural to keep this coordinate independence in the observer design as well. The Riemannian geometric point of view has influenced part of control theory, *e.g.* optimal control and control design. However, the impact on observer design, have been modest.

In (Aghannan and Rouchon, 2003), the authors successfully adopt the formerly mentioned contraction analysis approach to address convergence of an intrinsic observer for Euler-Lagrange systems with position measurements. These results have been specialized to the case of left invariant systems on Lie groups in (Maithripala *et al.*, 2004). In the present paper, we extend the results of (Aghannan and Rouchon, 2003) by using Lyapunov theory to show convergence in the constant curvature case, whenever we have *a priori* given bounds on the state variables. In the case of physical (mechanical or electrical) Euler-Lagrange systems, this assumption is a realistic one.

The organization of this paper is as follows. Section 2 is devoted to introducing some preliminary concepts of tangent bundle geometry (section 2.1) and Euler-Lagrange systems (section 2.2). The design of the observer is the subject of section 3, while section 4 is devoted to the convergence properties of it. Finally, these properties are illustrated in section 5, where we present some simulation results.

2. PRELIMINARIES

2.1 Tangent Bundle Geometry

This paper assumes a previous knowledge of classical tensor analysis as well as familiarity with coordinate-free concepts like tangent bundle, Lie derivatives and affine connections (consult *e.g.* (Lovelock and Rund, 1989) and (Abraham and Marsden, 1978)). Throughout the paper, Einstein summation convention is used, partial derivatives are indicated with a comma, $U_{,i} = \frac{\partial U}{\partial x^i}$, while covariant derivatives are indicated with a bar, $F^i{}_{|j} = F^i{}_{,j} + \Gamma^i_{kj} F^k$. If g_{ij} are the components of a Riemannian metric, g^{ij} denotes the components of the dual ("inverse") metric, and the components of the Levi-Civita connections (the Christoffel symbols) are given by $\Gamma^i_{jk}(x) = \frac{1}{2} g^{il}(g_{lj,k} + g_{kl,j} - g_{jk,l})$. By grad U, we mean the vector field $g^{ij} U_{,j} \frac{\partial}{\partial x^i}$. The curvature tensor, R, is defined by

$$R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z.$$

With the index ordering conventions from (Lovelock and Rund, 1989), R has components $R_j{}^i{}_{kl} \frac{\partial}{\partial x^i} = R(\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^j}$, so that

[1] "Convergence" in some metric sense, or – for relatively compact trajectories – in a purely topological sense.

$$R_m{}^i{}_{jk} = \Gamma^i_{mj,k} - \Gamma^i_{mk,j} + \Gamma^i_{nk}\Gamma^n_{mj} - \Gamma^i_{nj}\Gamma^n_{mk},$$

and the Ricci identity,

$$Y^i{}_{|j|k} - Y^i{}_{|k|j} = R_m{}^i{}_{jk} Y^m, \qquad (1)$$

holds. (It holds whenever the connection is torsion-free.)

We now review some less well-known constructions, namely lifting geometrical structures on a manifold $\mathcal{X}$ to geometrical structures on its tangent bundle, $T\mathcal{X}$ (*c.f.* (Yano and Ishihara, 1973)). Let x be local coordinates on $\mathcal{X}$ and (x, v) the corresponding induced coordinates on $T\mathcal{X}$.
The *vertical lift* of a vector field $Y = Y^i \frac{\partial}{\partial x^i}$ on $\mathcal{X}$, is the vector field on $T\mathcal{X}$ given by $Y^{\mathbb{V}} = Y^i \frac{\partial}{\partial v^i}$.
The *horizontal lift* of Y depends on the choice of a connection and is the vector field on $T\mathcal{X}$ given by $Y^{\mathbb{H}} = Y^i(\frac{\partial}{\partial x^i} - \Gamma^m_{li} v^l \frac{\partial}{\partial v^m})$.
The *geodesic spray* is a vector field $\overset{\nabla}{Z}$ on $T\mathcal{X}$, uniquely constructed from a connection ∇ on $\mathcal{X}$ as $\overset{\nabla}{Z} = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}$.

If ϕ is a differential form on $\mathcal{X}$, $\tau^*\phi$ denotes its pull-back to $T\mathcal{X}$. A differential 1-form ϕ on $\mathcal{X}$, also defines a scalar function $\mathbb{I}(\phi)$ on $T\mathcal{X}$ given by $\mathbb{I}(\mathrm{d}x^i) = v^i$ (This notation is not standard. The letter $\mathbb{I}$ stands for *identification*, since a covector ϕ, in a sense, already *is* a function on the tangent vectors). The $\mathbb{I}$ construction extends to higher order tensors.

The vertical and horizontal lifts, as well as the geodesic spray are in fact characterized by their actions on special functions on $T\mathcal{X}$:

$\cdot(\cdot)$	$\tau^* f$	$\mathbb{I}(\mathrm{d}f)$
$X^{\mathbb{H}}$	$\tau^*(X(f))$	$\mathbb{I}(\nabla_X \mathrm{d}f)$
$X^{\mathbb{V}}$	0	$\tau^*(X(f))$
$\overset{\nabla}{Z}$	$\mathbb{I}(\mathrm{d}f)$	$\mathbb{I}(\nabla \mathrm{d}f)$

(here $\mathbb{I}(\nabla \mathrm{d}f) = f_{,i|j} v^i v^j$). Expressions for the bracket between the vector fields are listed in table 1.

$[\cdot,\cdot]$	$Y^{\mathbb{H}}$	$Y^{\mathbb{V}}$	$\overset{\nabla}{Z}$
$X^{\mathbb{H}}$	$[X,Y]^{\mathbb{H}}$	$-(\nabla_X Y)^{\mathbb{V}}$	$\mathbb{I}((R(\cdot,X) - \nabla X))^{\mathbb{V}}$
$X^{\mathbb{V}}$	$-[Y^{\mathbb{H}}, X^{\mathbb{V}}]$	0	$X^{\mathbb{H}} - \mathbb{I}(\nabla X)^{\mathbb{V}}$
$\overset{\nabla}{Z}$	$-[Y^{\mathbb{H}}, \overset{\nabla}{Z}]$	$-[Y^{\mathbb{V}}, \overset{\nabla}{Z}]$	0

Table 1. Brackets of the lifted vector fields.

Given a Riemannian metric, g, on $\mathcal{X}$, there is a family of natural metrics on $T\mathcal{X}$ given by

$$G(X^{\mathbb{H}} + Y^{\mathbb{V}}, X^{\mathbb{H}} + Y^{\mathbb{V}}) = \tau^*(ag(X,X) + bg(Y,Y) + 2cg(X,Y)),$$

where a, b and c are constants, or in general, functions of $g_{ij}v^i v^j$. The case $a = b = 1$ and $c = 0$, was studied in (Sasaki, 1958). The generalized Sasaki metric reads

$$G = \begin{bmatrix} \mathrm{d}x^i \\ \mathrm{D}v^i \end{bmatrix}^T \otimes \begin{bmatrix} a & c \\ c & b \end{bmatrix} g_{ij} \begin{bmatrix} \mathrm{d}x^j \\ \mathrm{D}v^j \end{bmatrix},$$

where $\mathrm{D}v^i = \mathrm{d}v^i + \Gamma^i_{jk} v^j \mathrm{d}x^k$. Here, $[\mathrm{d}x^i, \mathrm{D}v^i]$ is the coframe dual to the frame $[\frac{\partial}{\partial x^i}^{\mathbb{H}}, \frac{\partial}{\partial x^i}^{\mathbb{V}}]$.
At the origin of a geodesic normal coordinate system, the Lie derivatives of the coframe, equal

$$\mathcal{L}_{Y^{\mathbb{V}}} \begin{bmatrix} \mathrm{d}x^i \\ \mathrm{D}v^i \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline Y^i_{|j} & 0 \end{array}\right] \begin{bmatrix} \mathrm{d}x^j \\ \mathrm{D}v^j \end{bmatrix} \qquad (2)$$

$$\mathcal{L}_{Y^{\mathbb{H}}} \begin{bmatrix} \mathrm{d}x^i \\ \mathrm{D}v^i \end{bmatrix} = \left[\begin{array}{c|c} Y^i_{|j} & 0 \\ \hline R_k{}^i{}_{jl}\, v^k\, Y^l & 0 \end{array}\right] \begin{bmatrix} \mathrm{d}x^j \\ \mathrm{D}v^j \end{bmatrix}$$

$$\mathcal{L}_{\overset{\nabla}{Z}} \begin{bmatrix} \mathrm{d}x^i \\ \mathrm{D}v^i \end{bmatrix} = \left[\begin{array}{c|c} 0 & \delta^i_j \\ \hline R_k{}^i{}_{jl}\, v^k\, v^l & 0 \end{array}\right] \begin{bmatrix} \mathrm{d}x^j \\ \mathrm{D}v^j \end{bmatrix}$$

$$\mathcal{L}_{\mathbb{I}(R(\cdot,Y))^{\mathbb{V}}} \begin{bmatrix} \mathrm{d}x^i \\ \mathrm{D}v^i \end{bmatrix} =$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline (R_m{}^i{}_{kl} Y^l)_{|j}\, v^k\, v^m & (R_j{}^i{}_{kl} + R_k{}^i{}_{jl}) Y^l v^k \end{array}\right] \begin{bmatrix} \mathrm{d}x^j \\ \mathrm{D}v^j \end{bmatrix}.$$

2.2 Euler-Lagrange Systems

An Euler-Lagrange system is a dynamical system with state space $\mathcal{Z} = T\mathcal{X}$, the tangent bundle of a configuration manifold, $\mathcal{X}$. The dynamics is given by

$$\dot{x}^i = v^i, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial x^i} = g_{ij} F^j,$$

where $L(x, v) = \frac{1}{2} g_{ij}(x) v^i v^j - U(x)$, is the *Lagrangian*. Here, g_{ij} is a Riemannian metric on $\mathcal{X}$ and the external forces *may* be interpreted as the input. We further assume that we have direct measurements on the position variables and forces. Combining this with the expression for the Lagrangian, the system can be written, in local coordinates, as

$$\Sigma_0 : \begin{cases} \dot{x}^i = v^i, & i = 1, \ldots, n \\ \dot{v}^i = -\Gamma^i_{jk}(x) v^j v^k - g^{ij} U_{,j} + F^i \\ y = h(x, v, F) = (x, F). \end{cases}$$

In terms of the absolute time-derivative, $\mathrm{D}_t v^i = \frac{\mathrm{d}v^i}{\mathrm{d}t} + \Gamma^i_{jk} v^j \frac{\mathrm{d}x^k}{\mathrm{d}t}$, system Σ_0 can equivalently be written as

$$\boxed{\Sigma : \begin{cases} \dot{x}^i = v^i \\ \mathrm{D}_t v^i = -g^{ij} U_{,j} + F^i \\ y = (x, F). \end{cases}}$$

Using the introduced lifting operations, the dynamics of system Σ, is given by the vector field

$$\boxed{\mathcal{F} = \overset{\nabla}{Z} - (\mathrm{grad}\, U)^{\mathbb{V}} + F^{\mathbb{V}}}$$

3. OBSERVER DESIGN

For the class of systems, Σ, described in section 2.2, we now introduce a full state identity observer, $\hat{\Sigma}$.
Referring to figure 3, we let (ξ, η) denote the state of the observer, $S(x, \xi) = \frac{1}{2}\mathrm{dist}(x, \xi)^2$, $S_\beta = \frac{\partial S}{\partial \xi^\beta}$

and $R^\alpha = \mathbb{I}(R(\cdot, \operatorname{grad} S))^\alpha = R_\beta{}^\alpha{}_{\gamma\iota}\eta^\beta S^\gamma \eta^\iota$, where $R_\beta{}^\alpha{}_{\gamma\iota}$ is the curvature tensor. In addition, Φ^α denotes the *parallel transport* of F^i along the geodesic curve, ρ, from x to ξ. The parallel transport operator, K^i_α, has the following properties, which are easily verified in Fermi coordinates:

$$K^i_{\alpha|\beta} S^\beta = 0. \tag{3}$$

$$K^i_\alpha S^\alpha = -S^i \tag{4}$$

$$K^i_\alpha K^j_\beta g^{\alpha\beta} = g^{ij} \tag{5}$$

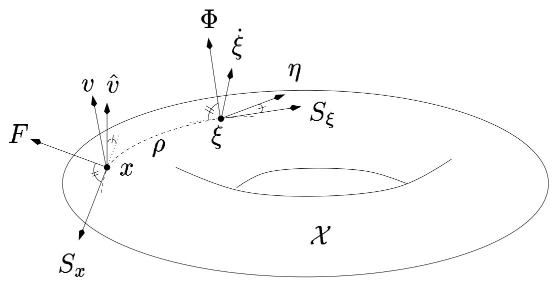

Fig. 3. The system- and observer variables, are denoted by latin and greek letters respectively.

Upon introducing this notation, the following observer dynamics is suggested for Σ:

$$\begin{aligned} \dot\xi^\alpha &= \eta^\alpha - A g^{\alpha\beta} S_\beta, \qquad\qquad \alpha = 1, \dots, n \\ \mathrm{D}_t \eta^\alpha &= -B g^{\alpha\beta} S_\beta - g^{\alpha\beta} U_{,\beta} + C R^\alpha + \Phi^\alpha, \end{aligned} \tag{6}$$

where A, B and C are observer gains, possibly depending on S and $|\eta|_g$. Note that when $\xi = x$, then $S_\beta = 0$ and $K^i_\alpha = \delta^i_\alpha$ (the Kronecker delta), hence (6) satisfies the diagram property of definition 1.

As observer output mapping, Z, we may for instance use $Z_1 = (\xi^\alpha, \eta^\alpha)$, or $Z_2 = (x^i, K^i_\alpha \eta^\alpha)$. Choosing the latter approach, the velocities, v^i, are estimated as

$$\hat v^i = K^i_\alpha \eta^\alpha. \tag{7}$$

Thus, putting (6) and (7) together, the following observer, $\hat\Sigma$, is suggested for Σ

$$\boxed{\hat\Sigma : \begin{cases} \dot\xi^\alpha &= \eta^\alpha - A g^{\alpha\beta} S_\beta \\ \mathrm{D}_t \eta^\alpha &= -B g^{\alpha\beta} S_\beta - g^{\alpha\beta} U_{,\beta} + C R^\alpha + \Phi^\alpha \\ \hat v^i &= K^i_\alpha \eta^\alpha \end{cases}}$$

Using the introduced lifting operators, the dynamics of the observer is governed by the vector field

$$\boxed{\begin{aligned} \hat{\mathcal{F}} = \overset{\nabla}{Z} - A(\operatorname{grad} S)^{\mathbb{H}} - B(\operatorname{grad} S)^{\mathbb{V}} \\ -(\operatorname{grad} U)^{\mathbb{V}} + C\mathcal{R} + \Phi^{\mathbb{V}} \end{aligned}}$$

where $\mathcal{R} = R^\alpha \frac{\partial}{\partial v^\alpha} = [(\operatorname{grad} S)^{\mathbb{H}}, \overset{\nabla}{Z}] + \mathbb{I}(\nabla(\operatorname{grad} S)^{\mathbb{H}})^{\mathbb{V}}$, in accordance with table 1. In the case of flat metric, $\hat\Sigma$ reduces to the well-known Luenberger observer.

The observer $\hat\Sigma$, is essentially the same as the one introduced in (Aghannan and Rouchon, 2003), see also (Maithripala *et al.*, 2004). We here allow the observer gains to vary and have a choice of moving force terms between U and F, which are treated differently in our observer. This latter freedom will however not be exploited in the present paper. In section 4.3, we follow (Aghannan and Rouchon, 2003), by choosing $C = 1$ and the output mapping Z_1, while in section 4.4 we use a general C and Z_2.

4. CONVERGENCE ANALYSIS

In the section, convergence issues are treated by means of contraction analysis (section 4.3) and, in the case of constant curvature, by means of a conventional Lyapunov method (section 4.4). To this end however, we devote section 4.1 and 4.2 to deriving expressions for the variation of some quantities along a geodesic.

4.1 Transport Equations

Letting $S^\alpha = g^{\alpha\beta} S_\beta$, the Hamilton-Jacobi equation $\sigma_{|\alpha}\sigma_{|\beta} g^{\alpha\beta} = 1$, for $\sigma = \sqrt{2S}$, implies

$$S_\alpha S^\alpha - 2S = 0, \tag{8}$$

Taking the covariant derivative of (8), utilizing the fact that the connection is torsion-free ($S_{\alpha|\beta} = S_{\beta|\alpha}$) and raising the first index, we have

$$S^\beta{}_{|\alpha} S^\alpha - S^\beta = 0. \tag{9}$$

Then, combining (4), (3), (8) and (9), it follows that

$$S^i{}_{|\beta} S^\beta - S^i = 0. \tag{10}$$

By taking the covariant derivative of (9) and utilizing Ricci's identity (1), we get [2]

$$S^\beta{}_{|\gamma|\alpha} S^\alpha = S^\beta{}_{|\gamma} - R_\iota{}^\beta{}_{\alpha\gamma} S^\iota S^\alpha - S^\beta{}_{|\alpha} S^\alpha{}_{|\gamma}. \tag{11}$$

In a similar fashion, we obtain

$$K^i_{\beta|\gamma|\alpha} S^\alpha = R_\beta{}^\epsilon{}_{\alpha\gamma} K^i_\epsilon S^\alpha + K^i_{\beta|\alpha} S^\alpha{}_{|\gamma}. \tag{12}$$

It should be possible to derive Grönwall-like estimates of $S^\beta{}_{|\gamma}$ and $K^i_{\beta|\gamma}$ from (11) and (12). In the present paper, however, we focus on spaces of constant curvature.

4.2 Constant Curvature

In the case when $\mathcal{X}$ has constant curvature, *i.e.* when

$$R_\iota{}^\beta{}_{\alpha\gamma} = \kappa(\delta^\beta_\gamma g_{\iota\alpha} - \delta^\beta_\alpha g_{\iota\gamma}), \tag{13}$$

[2] Equation (11) is equivalent to "the radial curvature equation" in (Petersen, 1998).

equation (11) may be explicitly solved for ${S^\beta}_{|\gamma}$ by means of the Ansatz [3]

$$ {S^\beta}_{|\gamma} = \Upsilon_1(S)\delta^\beta_\gamma + \Upsilon_2(S) S^\beta S_\gamma. \quad (14) $$

Equation (9) then immediately yields that $\Upsilon_1 + 2S\Upsilon_2 = 1$. Substituting this back into (11), it reads

$$ (2S\Upsilon_1' + 2S\kappa + \Upsilon_1^2 - \Upsilon_1)(\delta^\beta_\gamma - \frac{1}{2S} S^\beta S_\gamma) = 0, $$

from which we obtain

$$ \Upsilon_1(S) = \begin{cases} \sqrt{2\kappa S} \cot \sqrt{2\kappa S} & \text{if } \kappa > 0 \\ 1 & \text{if } \kappa = 0 \\ \sqrt{2|\kappa| S} \coth \sqrt{2|\kappa| S} & \text{if } \kappa < 0. \end{cases} $$

The formulas when $\kappa \le 0$, are the analytical continuation of the formula when $\kappa > 0$. In the sequel, only the $\kappa > 0$ form is given.

Considering the parallel transport operator, we make the Ansatz (*c.f.* footnote 3)

$$ K^i_\alpha = \Upsilon_3(S) {S^i}_{|\alpha} + \Upsilon_4(S) S^i S_\alpha. \quad (15) $$

From (4), (10) and (15), we obtain $\Upsilon_3 + 2S\Upsilon_4 = 1$. Substituting this back in (15) and utilizing (3), we get

$$ K^i_{\alpha|\beta} S^\beta \big(2S\Upsilon_3' + \Upsilon_3 - \Upsilon_3\Upsilon_1\big)({S^i}_{|\alpha} - \frac{1}{2S} S^i S_\alpha) = 0. $$

Solving for $\Upsilon_3(S)$, we arrive at the final expression

$$ K^i_\alpha = -\frac{\sin\sqrt{2\kappa S}}{\sqrt{2\kappa S}} {S^i}_{|\alpha} - \kappa \frac{\sqrt{2\kappa S} - \sin\sqrt{2\kappa S}}{(\sqrt{2\kappa S})^3} S^i S_\alpha. $$

We differentiate this expression w.r.t. ξ^β, and by using the earlier expressions (14) and (15), we obtain

$$ \begin{cases} K^i_{\alpha|\beta} = \Upsilon(S)(S^i g_{\alpha\beta} + K^i_\beta S_\alpha), \\ \Upsilon(S) = \kappa \dfrac{\tan \frac{1}{2}\sqrt{2\kappa S}}{\sqrt{2\kappa S}}. \end{cases} \quad (16) $$

By manipulating (16), with the roles of x and ξ reversed, we also obtain

$$ K^j_{\alpha|\beta} K^\alpha_m K^\beta_k = K^j_{\alpha|k} K^\alpha_m = \Upsilon(S)(S^j g_{km} - \delta^j_m S_k). \quad (17) $$

4.3 Contraction Theory

Let A, B and C be constants. Then the computations, when examining whether $\mathcal{L}_{\hat{\mathcal{F}}} G$ is negative definite or not, can be done component-wise, that is

$$ \mathcal{L}_{\hat{\mathcal{F}}} G = \mathcal{L}_{\overset{\mathrm{v}}{Z}} G - A\mathcal{L}_{(\operatorname{grad} S)^{\mathrm{H}}} G - B\mathcal{L}_{(\operatorname{grad} S)^{\mathrm{V}}} G - \dots $$

Using the formulas (2), we arrive at

$$ \mathcal{L}_{\hat{\mathcal{F}}} G = \begin{bmatrix} \mathrm{d}\xi^\alpha \\ \mathrm{D}\eta^\alpha \end{bmatrix}^T \otimes \mathcal{M} \begin{bmatrix} \mathrm{d}\xi^\beta \\ \mathrm{D}\eta^\beta \end{bmatrix}, \quad (18) $$

where the matrix

$$ \mathcal{M} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} M_{\alpha\beta} & N_{\alpha\beta} \\ P_{\alpha\beta} & Q_{\alpha\beta} \end{pmatrix} + \begin{pmatrix} M_{\beta\alpha} & P_{\beta\alpha} \\ N_{\beta\alpha} & Q_{\beta\alpha} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix}, $$

has components given by

$$ \begin{aligned} M_{\alpha\beta} &= -AS_{\alpha|\beta} \\ N_{\alpha\beta} &= g_{\alpha\beta} \\ Q_{\alpha\beta} &= C(R_{\beta\alpha\gamma\iota} + R_{\gamma\alpha\beta\iota}) S^\iota \eta^\gamma \\ P_{\alpha\beta} &= Y_{\alpha\beta\gamma\iota}\eta^\gamma\eta^\iota + AR_{\gamma\alpha\beta\iota}\eta^\gamma S^\iota - BS_{\alpha|\beta} \\ &\quad - U_{\alpha|\beta} + g_{mn} F^m K^n_{\alpha|\beta}, \end{aligned} $$

with $Y_{\alpha\beta\gamma\iota} = (R_{\gamma\alpha\beta\iota} - C(R_{\gamma\alpha\epsilon\iota} S^\epsilon)_{|\beta})$.
In the case when we set $C = 1$ and $S = 0$, we have, $S_{\alpha|\beta} = g_{\alpha\beta}$, $S^\iota = 0$ and $K^n_{\alpha|\beta} = 0$, and $\mathcal{M}$ becomes

$$ \mathcal{M} = \begin{bmatrix} -2(aA + cB) g_{\alpha\beta} - 2cU_{\alpha|\beta} & D_{\alpha\beta} \\ D_{\alpha\beta} & 2cg_{\alpha\beta} \end{bmatrix}, $$

where $D_{\alpha\beta} = (a - bB - cA) g_{\alpha\beta} - bU_{\alpha|\beta}$. From this it is possible to derive conditions for contractivity. When $U \equiv 0$ [4], the observer dynamics is contractive for suitable a, b and c. This is in accordance with the results in (Aghannan and Rouchon, 2003). However, whenever $S > 0$ and $Y_{\alpha\beta\gamma\iota}\eta^\gamma\eta^\iota \neq 0$ for some η, then

$$ \mathcal{M} = \begin{bmatrix} 2a & b \\ b & 0 \end{bmatrix} Y_{\alpha\beta\gamma\iota}\eta^\gamma\eta^\iota + \mathcal{O}(\eta), $$

so, for η large enough, $\mathcal{L}_{\hat{\mathcal{F}}} G$ is *indefinite* since the matrix preceding $Y_{\alpha\beta\gamma\iota}\eta^\gamma\eta^\iota$ is. Hence, the contracting neighborhood of the set $S = 0$ shown in (Aghannan and Rouchon, 2003), is *infinitely thin* as $|\eta|_g \to \infty$.

4.4 Lyapunov Approach

We now investigate the convergence of $\hat{\Sigma}$, in the case of constant curvature. We also put $U \equiv 0$ and let B be a constant. For the Lyapunov function candidate

$$ V(x, v, \xi, \hat{v}) = \frac{1}{2} g_{ij} \Delta v^i \Delta v^j + BS(x, \xi), $$

where $\Delta v^i = (v^i - \hat{v}^i)$, the total derivative becomes

$$ \dot{V} = g_{ij}\Delta v^i (\mathrm{D}_t v^j - \mathrm{D}_t \hat{v}^j) + BS_i \dot{x}^i + BS_\alpha \dot{\xi}^\alpha, $$

along the system dynamics of Σ and $\hat{\Sigma}$. Here,

$$ \begin{aligned} \mathrm{D}_t \hat{v}^j &= K^j_{\alpha|k} v^k \eta^\alpha + K^j_{\alpha|\beta} \dot{\xi}^\beta \eta^\alpha + K^j_\alpha \mathrm{D}_t \eta^\alpha \\ &= K^j_{\alpha|k} v^k \eta^\alpha + K^j_{\alpha|\beta} \dot{\xi}^\beta \eta^\alpha + BS^j + F^j + CK^j_\alpha R^\alpha. \end{aligned} $$

From (13) and (17) it follows that

$$ K^j_\alpha R^\alpha = \kappa \Upsilon^{-1}(S) K^j_{\alpha|k} K^\alpha_m \hat{v}^k \hat{v}^m. $$

With $C = -2\kappa^{-1}\Upsilon(S)$, the total derivative becomes

$$ \dot{V} = -2ABS - \Upsilon(S)(g_{km} S_i - g_{jm} S_k) \hat{v}^m \Delta v^i \Delta v^k, $$

where we have used (3), (8) and (17).

Theorem 1. If it is known that $\sup_t |v(t)|_g \le v_{\max}$, and $A > \sqrt{2} B^{-1} S^{-\frac{1}{2}} |\Upsilon(S)| (v_{\max} + |\eta|_g)^2 |\eta|_g$, $B > 4\pi^{-2} |\kappa| v^2_{\max}$ and $C = -2\kappa^{-1}\Upsilon(S)$, then the observer $\hat{\Sigma}$ initiated at $\xi(0) = x(0)$, $\eta(0) = 0$ converges.

[3] This form also follows from a symmetry argument.

[4] It is always possible to move terms between U and F.

5. EXAMPLE

Let $\mathcal{X}$ be the unit 3-sphere parameterized by $x_1, x_2 \in [0,\pi]$ and $x_3 \in [0, 2\pi]$. This is a space of constant curvature $\kappa = 1$. The metric is given by

$$g_{\cdot,\cdot} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2 x_1 & 0 \\ 0 & 0 & \sin^2 x_1 \sin^2 x_2 \end{bmatrix}$$

which implicitly gives the distance function, S, as

$$\cos\sqrt{2S} = \cos x_1 \cos\xi_1 + \sin x_1 \sin\xi_1 [\cos x_2 \cos\xi_2 + \cos(x_3 - \xi_3)\sin x_2 \sin\xi_2].$$

The exterior forces, F, are given by $-\operatorname{grad} W$, where $W = \sin x_1 \cos x_2 \cos x_3$ and $U \equiv 0$. We define an observer $\hat{\Sigma}$ by the choices $A = 3\frac{1+S}{\sqrt{S+10^{-7}}}$, $B = 3$ and $C = -1$. Figure 4 show the convergence of the observer when the initial data are

$$\begin{cases} x_1(0) = \xi_1(0) = 1 \\ x_2(0) = \xi_2(0) = 0.7 \\ x_3(0) = \xi_3(0) = 2 \end{cases} \begin{cases} v_1(0) = 2.25 \\ v_2(0) = 1.25 \\ v_3(0) = 4 \end{cases} \begin{cases} \hat{v}_1(0) = 0 \\ \hat{v}_2(0) = 0 \\ \hat{v}_3(0) = 0. \end{cases}$$

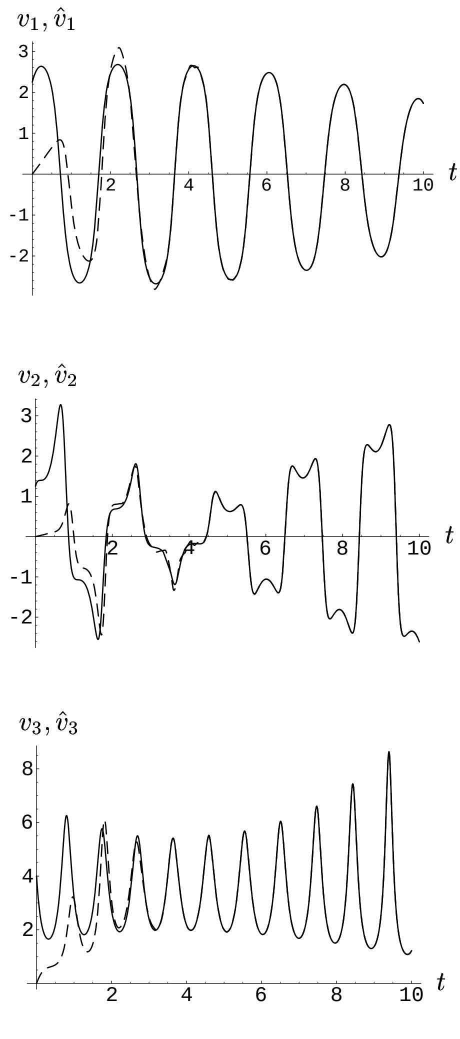

Fig. 4. The solid line refers to the original system, while the dashed line represents the observer.

Similar simulation results have also been obtained in the cases of the hyperbolic plane (constant negative curvature) and the inverted pendulum on a cart (zero curvature).

CONCLUDING REMARKS

The observer presented in this paper, requires the explicit computation of the distance function, S, as well as the parallel transport operator, K, which is prohibitive unless the configuration manifold is extremely simple, *e.g.* manifolds of constant curvature, Lie groups (*c.f.* (Maithripala *et al.*, 2004)) *etc*. For more general spaces, schemes of approximation are called for (*c.f.* (Aghannan and Rouchon, 2003)). This is a topic of current research.

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