Mathematics Department
KTH, Spring 2012
A. Björner

Algebraic Combinatorics: Problem set #1

(1) Let Q_π be the right tableau to $\pi \in S_n$ under Robinson-Schensted. Show that the following are equivalent for $1 \leq i \leq n-1$:

(a) $\pi(i) > \pi(i+1)$,

(b) The row of $i+1$ in Q_π is lower than the row of i,

(c) The column of $i+1$ in Q_π is the same or to the left of the column of i.

(2) Show that the number of odd hook lengths minus the number of even hook lengths for a partition λ is a triangular number (i.e., of the form $\binom{k}{2}$).

(3) "The domino game" is played on the set of all partitions λ. If along the diagram boundary of λ there is a "domino", that is, a pair of adjacent boxes, horizontally ($\square\!\square$) or vertically (rotate $90°$), whose removal leaves a new partition μ, then $\lambda \to \mu$ is a legal move.

(a) Show that the final position of the game is the same, irrespective of how the game is played.

(b) Show that the final position is $\varnothing$ (the empty diagram) $\Longleftrightarrow$ λ has the same number of odd and even hook lengths.

(c) The number of moves in a complete game from λ can be predicted, as well as the final position, if one only knows the number of odd and of even hook lengths in λ. How?

(4) Let D_λ be a Ferrers diagram (of shape $\lambda \vdash n$) filled with integers so that all rows are weakly increasing ($\leq$). Now, sort separately each column in D_λ so that it is weakly increasing. Show that after the resorting, all rows are still weakly increasing.

(5) Let $E(\mathrm{is}_n) = \frac{1}{n!}\sum_{\pi\in S_n} \mathrm{is}(\pi)$, where $\mathrm{is}(\pi)$ is the maximal length of an increasing subsequence of π. (In other words, $E(\mathrm{is}_n)$ is the expected value of $\mathrm{is}(\cdot)$ for a random permutation.)

Show that $E(\mathrm{is}_n) \geq \sqrt{n}$.

(6) Let $I(\pi) = \{(\pi_i, \pi_j) \mid i < j, \pi_i > \pi_j\}$ for $\pi \in S_n$, and let $2 \leq m \leq n$.

Show that the number of $\pi \in S_n$ such that $|I(\pi)| \equiv k \pmod{m}$ is $\frac{n!}{m}$, independently of $k \in \mathbb{N}$.

(7) Let t_n be the number of involutions in S_n. Show that:

(a) $t_n = t_{n-1} + (n-1)t_{n-2}, \quad t_0 = t_1 = 1$,

(b) $\sqrt{n!} \leq t_n \leq \sqrt{p_n \cdot n!}$, where p_n denotes the number of partitions of n,

(c) $\sum_{n\geq 0} t_n \frac{x^n}{n!} = e^{x+\frac{x^2}{2}}$.

(8) Let $\pi \in S_n$ be an involution with k fixed points. Show that the tableau P_π associated to π by the RS algorithm has exactly k columns of odd length.

Schedule for classes until end of March:

All Tuesdays 13.15 - 15.00, *except March 13*

Also two Thursdays 15.15 - 17.00, *March 1 and March 22*