

FILTERED GEOMETRIC LATTICES AND LEFSCHETZ SECTION THEOREMS OVER THE TROPICAL SEMIRING

KARIM A. ADIPRASITO AND ANDERS BJÖRNER

ABSTRACT. The purpose of this paper is to establish analogues of the classical Lefschetz Section Theorem for smooth tropical varieties. More precisely, we prove tropical analogues of the Section Theorems of Lefschetz, Andreotti–Frankel, Bott–Milnor–Thom, Hamm–Lê and Kodaira–Spencer, and the vanishing theorems of Andreotti–Frankel and Akizuki–Kodaira–Nakano.

We start the paper by resolving a conjecture of Mikhalkin and Ziegler (2008) concerning positive sum systems of geometric lattices, which generalizes earlier work of Rota, Folkman and Björner. This translates to a crucial index estimate for the stratified Morse data at critical points of the tropical variety.

Tropical geometry is a relatively new field in mathematics, based on early work of Bergman [Ber71] and Bieri–Groves [BG84]. Figuratively speaking, it arises by attempting to do algebraic geometry over the tropical max-plus semiring $\mathbb{T} = ([-\infty, \infty), \max, +)$. Since tropical varieties are, in essence, polyhedral spaces obtained as limits of complex algebraic varieties [Ber71, GKZ94, Vir84], tropical geometry naturally connects the fields of algebraic geometry and combinatorics.

Since its origins in the seventies, tropical geometry has been developed extensively [Gat06, RGST05, Spe05, SS09]. It has been applied to classical algebraic geometry [Gub07, Kat09], enumerative algebraic geometry [KT02, Mik05, Mik06, Shu05], mirror symmetry [Gro11, KS01], integrable systems [AMS12], and to several branches of applied mathematics, such as signals processing, mathematical biology, control theory and optimization, theoretical computer science and mathematical physics, cf. [Gro95, NGVR12, Pin98]. Several classical results and theories in algebraic geometry have natural analogues in tropical geometry, such as Brill–Noether theory and the Riemann–Roch, Torelli and Bézout Theorems, compare [CDPR12, RGST05].

Further motivated by tropical intersection theory and its relation to classical intersection theory of algebraic varieties [Kat12, Mik06], we here want to consider tropical analogues of one of the most central results in algebraic intersection theory, the Lefschetz Section Theorem (or Lefschetz Hyperplane Theorem). We attempt to give an almost complete picture of the Lefschetz Section Theorem in tropical geometry, and give tropical analogues of many of the classical Lefschetz theorems (and associated vanishing theorems). Along the way, we build on and generalize significant results in the topological theory of geometric lattices and matroids of Rota, Folkman, Björner and others.

The Tropical Lefschetz Section Theorems. The classical Lefschetz Section Theorem comes in many different guises. Intuitively, Lefschetz theorems relate the topology of a complex algebraic variety X to the topology of the intersection of X with a hyperplane H transversal to X (or, alternatively, to an ample divisor D of X).

Theorem (The classical Lefschetz Section theorem, [Lef50, AF59]). *Let X denote any smooth projective algebraic n -dimensional variety in $\mathbb{C}\mathbb{P}^d$, and let H denote a generic hyperplane in $\mathbb{C}\mathbb{P}^d$. Then the inclusion*

Date: January 28, 2014.

2010 Mathematics Subject Classification. **todo.**

Key words and phrases. geometric lattices, poset topology, smooth tropical varieties, Lefschetz Section Theorem, stratified Morse Theory, Hodge Theory.

K. A. Adiprasito acknowledges support by an EPDI postdoctoral fellowship and by the Romanian NASR, CNCS — UEFISCDI, project PN-II-ID-PCE-2011-3-0533.

A. Björner acknowledges support from the Swedish National Research Council (VR) under grant No. 2011-11677-88409-18, and also partial support from the National Science Foundation under Grant No. 0932078 000, while the author was in residence at the MSRI, Berkeley, Cal., during the month of November, 2013.

$H \cap X \hookrightarrow X$ induces an isomorphism of integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.

Variants of this important theorem exist for affine varieties and projective varieties, for homology, homotopy, for Hodge and Picard groups, for constructible sheaves and several more; compare [GM88, Laz04, Voi02]. Via duality (Lefschetz duality, Serre duality etc.), Lefschetz theorems go hand-in-hand with so-called vanishing theorems, such as the Andreotti–Frankel [AF59], Akizuki–Kodaira–Nakano [AN54] and Grothendieck–Artin [Laz04] Vanishing Theorems.

In this paper, we shall establish analogues of several of the classical Lefschetz theorems in tropical geometry. More precisely, we shall provide, in this order, tropical analogues of

- the Andreotti–Frankel Vanishing Theorem for affine varieties.
- the Bott–Milnor–Thom Lefschetz Section Theorem for homotopy groups and CW models of projective varieties [Bot59, Mil63].
- the classical Lefschetz Section Theorem for homology groups of projective varieties due to Lefschetz and Andreotti–Frankel [AF59, Lef50].
- the Hamm–Lê Lefschetz Section Theorem for complements affine varieties [Ham83, HL71].
- the Akizuki–Kodaira–Nakano Lefschetz Vanishing Theorem for Hodge groups [AN54, Voi02].
- the Kodaira–Spencer Lefschetz Section Theorem for Hodge groups [KS53].

Contrary to the case of algebraic varieties, the Section Theorems we prove apply more generally to arbitrary hypersurface sections, and not only hyperplane sections.

Tropical Lefschetz Section Theorems for CW models, homotopy and homology. The first main result of this paper is an analogue of the Andreotti–Frankel Vanishing Theorem [AF59] for smooth affine tropical varieties.

Theorem 2.2.4. *Let $X \subset \mathbb{T}^d$ be a smooth, affine n -dimensional tropical variety, and let H denote a tropical hypersurface in \mathbb{T}^d . Then X is obtained from $H \cap X$ by successively attaching n -dimensional cells.*

In particular, the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.

Here, we use the standard notation for attaching cells, compare Section A.1. We also deduce a Lefschetz Section Theorem for projective tropical varieties. Contrary to the original treatment of Andreotti–Frankel, this result does not follow immediately from Lefschetz duality and the affine theorem, but rather from a common generalization of the affine and projective cases (Lemma 2.2.6).

Theorem 2.2.7. *Let X denote any n -dimensional smooth projective tropical variety in \mathbb{TP}^d , and let H denote a tropical hypersurface in \mathbb{TP}^d . Then X is, up to homotopy equivalence, obtained from $H \cap X$ by successively attaching cells of dimension n .*

In particular, the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of homotopy and integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.

Tropical Lefschetz Section Theorems for complements of tropical varieties. Our reasoning extends to the complement of a tropical varieties as well. This is analogous to the Hamm–Lê Lefschetz theorems [Ham83, HL71, Lê87] for complements of algebraic hypersurfaces.

Theorem 2.3.2. *Let X denote a smooth n -dimensional tropical variety in \mathbb{T}^d , and let $C = C(X)$ denote the complement of X in \mathbb{T}^d . Let furthermore H denote an almost totally sedentary hyperplane in \mathbb{T}^d transversal to X . Then C is, up to homotopy equivalence, obtained from $C \cap H$ by successively attaching $(d - n - 1)$ -dimensional cells.*

In particular, the inclusion of $C \cap H$ into C induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $d - n - 3$, and a surjection in dimension $d - n - 2$.

The main tool to prove this result is the construction of an efficient Salvetti complex for complements of Bergman fans. In particular, we also obtain:

Corollary 1.5.3. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let \mathcal{B} denote the proper part of the Boolean lattice on the ground set of M . Then $\mathcal{B} - \mathcal{L}$ is homotopy equivalent to a wedge of spheres of dimension $|M| - r - 1$.*

Tropical Lefschetz Section Theorems for tropical Hodge groups. Finally, we provide a Lefschetz theorem for tropical Hodge groups, or tropical (p, q) -homology, as defined by Itenberg–Katzarov–Mikhalkin–Zharkov [IKMZ]. This is nontrivial: While the classical Lefschetz Section Theorem for Hodge groups of smooth algebraic projective varieties (due to Kodaira–Spencer [KS53]) does follow from the Lefschetz Section Theorem for complex coefficients, and the functoriality of the Hodge decomposition, this approach does not apply here, since the Hodge Index Theorem does not hold for tropical (p, q) -homology, as observed by Shaw [Sha11]. Nevertheless, the Lefschetz Section Theorem holds true for tropical Hodge groups.

Theorem 2.4.5. *Let X denote any n -dimensional smooth projective tropical variety in \mathbb{TP}^d , and let $H \subset \mathbb{TP}^d$ denote a tropical hypersurface transversal to X . Then the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of (p, q) -homology as long as $p + q \leq n - 2$, and a surjection if $p + q = n - 1$.*

An analogous theorem holds for affine tropical varieties, see Theorem 2.4.4. For the proof we, instead of invoking a tropical Hodge Index Theorem and Theorems 2.2.4 and 2.2.7, establish a tropical analogue of the Akizuki–Kodaira–Nakano Vanishing Theorem.

Theorem 2.4.6. *Let X denote any n -dimensional smooth affine tropical variety in \mathbb{T}^d (or \mathbb{TP}^d), and let P denote any rational d -polyhedron in \mathbb{T}^d (resp. \mathbb{TP}^d). Then, every chain in $c \in C_q(X \cap P; \mathcal{F}_p X)$ is homologous to a chain $\tilde{c} \in C_q(X \cap \partial P; \mathcal{F}_p X)$ provided $p + q < n$.*

In particular, we have a quasi-isomorphism $C_q(X \cap P; \mathcal{F}_p X) \rightarrow C_q(X \cap P; \mathcal{F}_p X)$, so that

$$H_q(X \cap P, X \cap \partial P; \mathcal{F}_p X) = 0 \quad \text{for all } p + q < n.$$

The proof of the tropical Kodaira–Spencer Theorem can now be finished in a manner similar to the classical proof, compare also [AN54, Voi02].

Filtered geometric lattices. For the proofs of the tropical Lefschetz theorems, we shall critically use stratified Morse theory ([GM88], see Section A.5): A crucial ingredient of the Morse-theoretic approach to classical Lefschetz Theorems are estimates on Morse indices at critical points, which follow easily from the hermitian structure of the associated smooth complex variety, cf. [AF59, Laz04, Mil63].

In our setting, we analogously need to estimate the topological changes in the sublevel sets with respect to some smooth Morse function, interpreting the tropical variety as a Whitney stratified space. This estimate requires a solution of a conjecture of Mikhalkin and Ziegler [MZ08] about the *lattice of flats of matroids*, also known as *geometric lattices*.

Theorem 1.1.1. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t denote any real number with $t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{L}^{>t}$ is homotopy Cohen–Macaulay of dimension $r - 2$, and in particular $(r - 3)$ -connected.*

Geometrically, the result characterizes the topology of the restriction of the Bergman fan to a generic halfspace. Equivalently, it characterizes the topological type of half-links of the tropical variety at critical points.

This result is interesting in itself; it generalizes earlier results concerning the topological type of the full geometric lattice \mathcal{L} . The homology version of this result goes back to work of Folkman [Fol66], inspired by work of Rota [Rot64] on the Möbius function of geometric lattices. A stronger version concerning shellability (and therefore homotopy equivalence) was later proved by Björner [Bjö80]. Other than for full geometric lattices, the theorem is previously known for the case of matroids of rank 3 [MZ08], and also for the case when the weight ω has only one negative entry (this is implied by a result of Wachs and Walker [WW86]).

Plan for the paper. In the next Section 1, we prove our main theorem on the topology of filtered geometric lattices, using methods from poset fiber theory. We will also sketch an alternative proof based on the methods of [Adi12].

In Section 2, we proceed to apply our results to deriving Lefschetz Theorems for tropical varieties. In each subsection, we first review a classical Lefschetz theorem, then proceed to give a tropical analogue, explaining the differences between the results (if any).

Comments on unexplained concepts and notation with references are collected in the appendix. In particular, we there sketch some required background information on combinatorial, cellular and poset topology, geometry and combinatorics of polyhedral spaces, tropical geometry, tropical Hodge theory, and stratified Morse theory.

1. FILTERED GEOMETRIC LATTICES

1.1. Main result. The first main result concerns a resolution of a conjecture of Mikhalkin and Ziegler [MZ08] about the lattice of flats of a weighted matroid. We assume familiarity with the basic properties of matroids and geometric lattices, see e.g. [Oxl11]. For the homological aspects see [Bjö92].

Let M denote a matroid on the ground set $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$. A **weight** $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ on M is any vector in $\mathbb{R}^{[n]}$. If σ is any subset of $[n]$, and $\mathbf{1}_\sigma$ is its characteristic vector, then we set

$$\omega \cdot \sigma \stackrel{\text{def}}{=} \omega \cdot \mathbf{1}_\sigma = \sum_{e \in \sigma} \omega_e.$$

A weight is **generic** if $\omega \cdot \sigma \neq 0$ for all $\emptyset \subsetneq \sigma \subsetneq [n]$. If $\mathcal{L} = \mathcal{L}[M] \stackrel{\text{def}}{=} \widehat{\mathcal{L}} \setminus \{\widehat{0}, \widehat{1}\}$ is the proper part of the lattice of flats $\widehat{\mathcal{L}}$ of M , and t is any real number, then we use $\mathcal{L}^{>t}$ to denote the subset of \mathcal{L} consisting of elements $\sigma \in \mathcal{L}$ with $\omega \cdot \sigma > t$. We will refer to the posets (partially ordered sets) of the form $\mathcal{L}^{>t}$ as **filtered geometric lattices**. Note that these posets are not lattices in general, let alone geometric lattices.

With this, we can state the main result of this section.

Theorem 1.1.1. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t denote any real number with $t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{L}^{>t}$ is homotopy Cohen–Macaulay of dimension $r - 2$, and in particular $(r - 3)$ -connected.*

The proof of this result is articulated in a few steps. We start from homotopy information available for free matroids, and from this we deduce information concerning $\mathcal{L}^{>t}$, using a generalization of Quillen’s “Theorem A” given in the Appendix (Lemma A.1.4).

As an immediate corollary, we obtain a Lefschetz Theorem for matroids:

Corollary 1.1.2. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t, t' be any pair of real numbers with $t' < t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{L}^{>t'}$ is obtained from $\mathcal{L}^{>t}$ by attaching cells of dimension $r - 2$.*

In particular, $(\mathcal{L}^{>t'}, \mathcal{L}^{>t})$ is homotopy Cohen–Macaulay of dimension $r - 2$.

Proof. This follows directly from Theorem 1.1.1, the long exact sequence of relative homotopy groups, and Lemma A.1.3. □

Let us mention that for full geometric lattices it is known from the work of Rota [Rot64] on the Möbius function that the number of $(r - 2)$ -spheres in the wedge is strictly positive. This is not true for filtered geometric lattices. For example, if there is exactly one positive weight $\omega_i > 0$ then $\mathcal{L}^{>0}$ is contractible. However, the following relative information is immediate in the general case,

Corollary 1.1.3. *If $t' < t \leq \min\{0, \omega \cdot [n]\}$, then*

$$\dim(H_{r-2}(\mathcal{L}^{>t})) \leq \dim(H_{r-2}(\mathcal{L}^{>t'})).$$

1.2. Preliminaries. Let us first observe a general heredity property of filtered geometric lattices that we will use repeatedly for purposes of induction without always mentioning it. The fact that all maximal chains in $\mathcal{L}^{>t}$ have equal length $r - 2$ is a direct consequence.

Lemma 1.2.1. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 1.1.1. Let $(\mathcal{L}^{>t; \omega})_{(\sigma, \tau)}$ be any open interval in $\mathcal{L}^{>t}$. Then there exists a weight ω' on the atoms of $\mathcal{L}_{(\sigma, \tau)}$ and a real number t' such that*

$$(\mathcal{L}^{>t; \omega})_{(\sigma, \tau)} = (\mathcal{L}_{(\sigma, \tau)})^{>t', \omega'}.$$

For this, take $t' = t - \omega \cdot \sigma$ and $\omega' = \omega|_{(\tau - \sigma)}$.

Proof. Consider first the case when $\sigma = \emptyset$. Then $\mathcal{L}_{(\sigma, \tau)} = \mathcal{L}_{<\tau}$ is the lattice of flats of the matroid M' of rank $\varrho(\tau)$ obtained as the restriction of M to τ . Therefore, $\mathcal{L}_{<\tau}^{>t} \cong \mathcal{L}[M']^{>t}$, where M' is endowed with the weight given by the restriction $\omega|_{\tau}$ of ω to the set τ .

Next, suppose that $\tau = [n]$. Then $\mathcal{L}_{(\sigma, \tau)} = \mathcal{L}_{>\sigma}$ is the lattice of flats of the rank $(r - \varrho(\sigma))$ matroid M'' obtained as the contraction of σ in M . Moreover, if M'' is endowed with weight $\omega|_{[n] \setminus \sigma}$, we have

$$\mathcal{L}_{>\sigma}^{>t} \cong \mathcal{L}[M'']^{>(t - \omega \cdot \sigma)}$$

Since $\mathcal{L}_{(\sigma, \tau)} = (\mathcal{L}_{<\tau})_{>\sigma}$, the general result is obtained from these two special cases. \square

Lemma 1.2.2. *$\mathcal{L}^{>t}$ is pure and $(r - 2)$ -dimensional.*

Proof. For rank $r = 2$ the statement boils down to saying that $\mathcal{L}^{>t}$ is nonempty. Suppose that this were not the case. Then $\omega_i \leq t$ for all i , implying that $t \leq \omega \cdot [n] \leq tn$, which is impossible since $t < 0$.

A proof by induction on rank now follows easily from Lemma 1.2.1 by considering intervals $\mathcal{L}_{>\sigma}^{>t}$ where σ is an atom. \square

1.3. Free matroids. We begin with the following strengthening of Theorem 1.1.1 for the special case of free matroids, that is, matroids where all sets are independent.

We reserve the notation $\mathcal{B} = \mathcal{B}[n]$ for the proper part of the lattice of flats of the free matroid on n elements. It coincides with the proper part of the Boolean lattice $\widehat{\mathcal{B}} = 2^{[n]}$ of subsets of $[n] = \{1, \dots, n\}$, that is, $\mathcal{B} = 2^{[n]} \setminus \{\emptyset, [n]\}$.

Theorem 1.3.1. *Let ω denote any generic weight on $[n]$, and suppose that $t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{B}^{>t}$ is shellable and $(n - 2)$ -dimensional.*

In particular, it is homotopy Cohen–Macaulay.

We remark that the conclusion of the theorem can be sharpened to state that $\mathcal{B}^{>t}$ is homeomorphic to a ball or a sphere. Some aspects of this additional information is discussed in [Bjö13], it will not be needed here.

Proof. We use the method of lexicographic shellability [Bjö80, Bjö13]. We may assume that $\omega_i \neq \omega_j$ for $i \neq j$. This can always be achieved by a small perturbation of the weight vector ω that does not change $\mathcal{B}^{>t}$.

To each covering edge (σ, τ) of $\widehat{\mathcal{B}}$ we assign the real number $\lambda(\sigma, \tau) = \omega \cdot (\tau \setminus \sigma)$. This edge labeling induces a labeling of the maximal chains of $\widehat{\mathcal{B}}^{>t}$. We know from Lemma 1.2.2 that these chains are all of cardinality $n + 1$ (including the top and bottom elements \emptyset and $[n]$). The label $\lambda(m)$ of a maximal chain m is simply the induced permutation of the coordinates of the weight vector ω .

There is a unique maximal chain \overline{m} in $\widehat{\mathcal{B}}$ with the property that the labels form a decreasing sequence. After relabeling this is

$$\lambda(\overline{m}) = (\omega_1 > \omega_2 > \dots > \omega_n)$$

We have that

$$\emptyset \in \widehat{\mathcal{B}}^{>t} \Leftrightarrow t < 0 \quad \text{and} \quad [n] \in \widehat{\mathcal{B}}^{>t} \Leftrightarrow t < \omega \cdot [n],$$

so the hypothesis $t \leq \min\{0, \omega \cdot [n]\}$ implies that both endpoints of the chain \overline{m} belong to $\widehat{\mathcal{B}}^{>t}$. From this follows that the entire chain \overline{m} is in $\widehat{\mathcal{B}}^{>t}$, as is easy to see. Also, this chain is lexicographically first among the maximal chains in $\widehat{\mathcal{B}}$, and so also in $\widehat{\mathcal{B}}^{>t}$.

Similar reasoning can be performed locally at each interval (μ, ν) to prove the existence of a unique decreasingly labeled maximal chain in (μ, ν) which lexicographically precedes all the other maximal chains in that interval. This completes the verification of the conditions for lexicographic shellability. \square

1.4. Connectivity and the Cohen–Macaulay property. To finish the proof of Theorem 1.1.1 we need to establish the degree of connectivity for $\mathcal{L}^{>t}$.

Theorem 1.4.1. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in the statement of Theorem 1.1.1. Then $\mathcal{L}^{>t}$ is $(r - 3)$ -connected.*

Proof. We prove the theorem by induction on the cardinality $n = |M|$, the case $n = 1$ being trivial.

In hope of applying Quillen’s Fiber Lemma, let us consider the inclusion map $\varphi : \mathcal{L}^{>t} \hookrightarrow \mathcal{B}^{>t}$. Let us analyze the fibers $\varphi^{-1}(\mathcal{L}_{\geq x}^{>t})$ and the lower ideals $\mathcal{B}_{< x}^{>t}$, for all $x \in \mathcal{B}^{>t}$.

We have that $t < \omega \cdot x$, since $x \in \mathcal{B}^{>t}$, and $t \leq \min\{0, \omega \cdot [n]\}$. Hence, $t \leq \min\{0, \omega \cdot x\}$, and it follows from Lemma 1.2.1 and Theorem 1.3.1 that the posets $\mathcal{B}_{< x}^{>t} \cong \mathcal{B}[x]^{>t}$ are $(|x| - 3)$ -connected.

It remains to consider the fibers $\varphi^{-1}(\mathcal{L}_{\geq x}^{>t})$. Let $\kappa : \mathcal{B} \rightarrow \mathcal{L}$ denote the matroid closure map $S \mapsto \bigvee_{e \in S} e$, and let x be any element in $\mathcal{B}^{>t}$. Then,

$$\varphi^{-1}(\mathcal{B}_{\geq x}^{>t}) = \mathcal{L}_{\geq x}^{>t} = \mathcal{L}_{\geq \kappa(x)}^{>t}.$$

If $\kappa(x) \in \mathcal{L}^{>t}$, the fiber is a cone, and hence contractible.

If $\kappa(x) \notin \mathcal{L}^{>t}$, then by the induction assumption and Lemma 1.2.1, $\mathcal{L}_{\geq \kappa(x)}^{>t}$ is $(\dim \mathcal{L}_{\geq \kappa(x)} - 1)$ -connected.

Hence, by Quillen’s Lemma A.1.4, the inclusion map φ yields an isomorphism of homotopy groups up to dimension k , (and a surjection in dimension $k + 1$), where

$$k := \min_{\substack{x \in \mathcal{B}^{>t} \\ \kappa(x) \notin \mathcal{L}^{>t}}} (\dim(\mathcal{L}_{\geq \kappa(x)}^{>t}) + |x|) - 2.$$

Now,

$$\begin{aligned} & \dim \mathcal{L}_{\geq \kappa(x)}^{>t} + |x| - 2 \\ & \geq \dim \mathcal{L}_{\geq \kappa(x)}^{>t} + \dim(\mathcal{L}_{\leq \kappa(x)}^{>t}) - 1 && \text{(since } \kappa(x) \notin \mathcal{L}^{>t}\text{)} \\ & = \dim \mathcal{L}^{>t} - 1 \\ & = r - 3 \end{aligned}$$

Hence $\mathcal{L}^{>t}$ is $(r - 3)$ -connected, since $\mathcal{B}^{>t}$ is $(r - 3)$ -connected. \square

We can now finish and prove the homotopy Cohen–Macaulay property, which demands that we show that each interval is connected up to its dimension minus one.

Proof of Theorem 1.1.1. Let $(\mathcal{L}^{>t})_{(\sigma, \tau)}$ be an open interval. We know from Lemma 1.2.2 that its order complex has dimension $\varrho(\tau) - \varrho(\sigma) - 2$ and from Lemma 1.2.1 and Theorem 1.4.1 that it is $(\varrho(\tau) - \varrho(\sigma) - 3)$ -connected. \square

1.5. The complement of a filtered geometric lattice. We have established that \mathcal{B} is obtained from $\mathcal{B}^{>t}$ (Theorem 1.3.1), and that $\mathcal{B}^{>t}$ is obtained from $\mathcal{L}^{>t}$ (Theorem 1.4.1), by successively attaching cells of dimension $\geq r - 2$. One can reverse the reasoning of these argument to prove the following theorem:

Theorem 1.5.1. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 1.1.1. Then $\mathcal{B} - \mathcal{L}$ is obtained from $\mathcal{B}^{\leq t} - \mathcal{L}^{\leq t}$ by attaching cells of dimension $\leq n - r - 1$.*

The proof is entirely analogous to the proof of Theorem 1.4.1, and can be left out here. We notice, however, two facts: By Lemma A.1.1 and Alexander duality in the $(n - 2)$ -sphere \mathcal{B} , we have an isomorphism

$$H_i(\mathcal{B} - \mathcal{L}, \mathcal{B}^{\leq t} - \mathcal{L}^{\leq t}) \cong H^{n-i-3}(\mathcal{L}^{>t}).$$

Theorem 1.5.1 therefore provides an alternative proof for at least the homology version of Theorem 1.4.1. Furthermore, if $n - r \neq 2$, Theorems 1.4.1 and 1.5.1 are equivalent by well-known homotopy arguments together with the aforementioned Alexander duality.

Let us also remark that it is possible to give a common proof of Theorem 1.4.1 and Theorem 1.5.1, using combinatorial Morse theory and Alexander duality of combinatorial Morse functions, cf. Theorem 1.6.4 in Section 1.6.

Now, let us notice that the pair $(\mathcal{B} - \mathcal{L}, \mathcal{B}^{\leq t} - \mathcal{L}^{\leq t})$, as a complement of a $(r - 2)$ -dimensional complex \mathcal{L} in the $(n - 3)$ -connected, $(n - 2)$ -dimensional pair $(\mathcal{B}, \mathcal{B}^{\leq t})$, is $(n - r - 2)$ -connected by classical general position arguments. Together with the information that the pair is of dimension $\leq n - r - 1$, we immediately obtain the following generalization of Theorem 1.5.1:

Corollary 1.5.2. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 1.1.1. Then $\mathcal{B} - \mathcal{L}$ is obtained from $\mathcal{B}^{\leq t} - \mathcal{L}^{\leq t}$ by attaching cells of dimension $n - r - 1$.*

In particular, we can extend the results on the homotopy type of geometric lattices to their complements.

Corollary 1.5.3. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let \mathcal{B} denote the proper part of the Boolean lattice on the ground set of M . Then $\mathcal{B} - \mathcal{L}$ is homotopy equivalent to a wedge of spheres of dimension $|M| - r - 1$.*

1.6. Remarks and open problems.

The efficient Salvetti complex for complements of geometric lattices. While we now understand, from a homotopical point of view, the complement of a geometric lattice in the Boolean lattice on the same support, it might be desirable to have a more explicit model available. For this purpose, we can use an idea similar to Salvetti [Sal87] and Björner–Ziegler [BZ92], who described models for the complement of subspace arrangements. Throughout this remark, we use $(M, \mathcal{L}, \mathcal{B}, r, n)$ as in the previous sections.

A naive model for the complement $\mathcal{B} \setminus \mathcal{L}$ of \mathcal{L} in \mathcal{B} is clearly given by the complex $\mathcal{B} - \mathcal{L}$. However, the complex $\mathcal{B} - \mathcal{L}$ can be of dimension up to $n - 2$, while $\mathcal{B} \setminus \mathcal{L}$ only has the homotopy type of a complex of dimension $\leq n - r - 1$, so that this model can be considered quite wasteful.

To obtain a more efficient model for a matroid M on the ground set $[n]$, let \mathcal{NS} denote the poset of **non-spanning** proper subsets of M ordered by inclusion. In other words, \mathcal{NS} consists of the subsets σ of $[n]$ with matroid rank $\varrho(\sigma) < \varrho(M)$. Now, as mentioned in Appendix A.1, the matroid closure map $\kappa : \mathcal{NS} \rightarrow \mathcal{L}$, $x \mapsto \bigvee x$ deformation retracts \mathcal{NS} to the geometric lattice \mathcal{L} in \mathcal{B} . We obtain:

Theorem 1.6.1 (Efficient Salvetti Complex). *With $(\mathcal{L}, \mathcal{B}, r, n)$ as above, we have*

$$\mathcal{B} \setminus \mathcal{L} \simeq \mathcal{B} - \mathcal{L} \simeq \mathcal{B} - \mathcal{NS}.$$

Moreover, $\mathcal{B} - \mathcal{NS}$ is an efficient model, in the sense that $\dim(\mathcal{B} - \mathcal{NS}) \leq n - r - 1$.

Proof. It remains only to verify the claim on the dimension; this follows immediately once we notice that every element of \mathcal{B} of cardinality $\leq r - 1$ is non-spanning. \square

Remark 1.6.2 (Matroid duality is Alexander duality). The dimension of $\mathcal{B} - \mathcal{NS}$ is bounded above by $n - r - 1$, which coincides with the rank of the dual matroid M^* of M . This suggests a connection between $\mathcal{B} - \mathcal{NS} \simeq \mathcal{B} - \mathcal{L}$ and $\mathcal{L}[M^*]$. Indeed, this connection is easily provided by combinatorial Alexander duality [Sta82]. We have

$$\begin{aligned} & \mathcal{B} - \mathcal{NS} \\ & \simeq \{[n] \setminus \sigma : \sigma \text{ spanning in } M\} \\ & \cong \{\tau : \tau \text{ independent in } M^*\} \end{aligned}$$

The second complex is precisely the combinatorial Alexander dual of \mathcal{NS} , and the last isomorphism follows from standard matroid duality.

Remark 1.6.3 (Independent sets and the geometric lattice). Using Lemma A.1.4, the restriction of the matroid closure map $\kappa : \mathcal{NS} \rightarrow \mathcal{L}$ to the nonspanning independent sets \mathcal{I}' , and the inclusion map from \mathcal{I}' to the poset of independent sets \mathcal{I} can be seen to induce isomorphisms of homotopy groups in dimension $\leq (r - 3)$. To sum, we have an isomorphism

$$\pi_i(\mathcal{I}) \cong \pi_i(\mathcal{L}) \quad \text{for all } i \leq r - 3$$

provided by the diagram

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathcal{I} \\ \uparrow & & \nearrow \kappa \\ \mathcal{I}' & & \end{array}$$

Combined with the previous remark and Theorem 1.4.1, this in particular provides an alternative proof of Corollary 1.5.3.

Shellability and combinatorial Morse Theory. For an alternative approach to the conjecture of Mikhalkin and Ziegler one can use the combinatorial Morse theory of Forman [For98]. Intuitively speaking, combinatorial Morse theory is an incremental way to decompose a simplicial complex step-by-step that enriches Whitehead's notion of cell collapses [Whi78] by the notion of **critical cells**, which behave analogous to critical points in classical Morse Theory. The result is:

Theorem 1.6.4. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 1.1.1. Then, there is a collection C of critical $(r - 2)$ -cells such that $\mathcal{L}^{>t} - C$ simplicially collapses to a point. In particular, $\mathcal{L}^{>t}$ is $(r - 3)$ -connected.*

In comparison with Theorem 1.1.1, this result requires a stronger assumption (the total weight of ω is 0), but has a stronger conclusion since it describes the combinatorial structure of $\mathcal{L}^{>0}$, and not only the topological type. For the proof, one uses Alexander duality of combinatorial Morse functions as introduced in [Adi12]; this enables us to prove Theorem 1.6.4 and analogous theorem for the complement of $\mathcal{L}^{>t}$ in \mathcal{B} by a common induction.

It remains to be seen whether our understanding of the combinatorial structure of filtered geometric lattices can be improved further. A strengthening of the conjecture of Mikhalkin and Ziegler [MZ08] predicts the stronger property of shellability.

Open Problem 1.6.5. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 1.1.1. Is it true that $\mathcal{L}^{>t}$ is shellable?*

A positive answer would generalize earlier work of Björner [Bjö80] showing that every full geometric lattice is shellable.

General filters. An equally interesting problem is to characterize the topology of filtered geometric lattices when $t > \min\{0, \omega \cdot [n]\}$.

Open Problem 1.6.6. *Characterize the topology of $\mathcal{L}^{>t}$ for general t .*

It would seem natural to conjecture that, $\mathcal{L}^{>t}$ is always sequentially Cohen–Macaulay, a notion introduced by Stanley to generalize Cohen–Macaulayness to nonpure complexes, cf. [BWW09, Sta96]. This, however, is not the case.

Example 1.6.7. Let us consider the matroid M on ground set $[7]$, endowed with lattice of flats

$$\mathcal{L} \stackrel{\text{def}}{=} \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{1, 2\}, \{6, 7\}, \{1, 3, 6\}, \{1, 4, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}\},$$

compare also Figure 1.1. Let us furthermore consider the weight $\omega = (1, 1, -3, -3, -3, 1, 1)$. Then

$$\mathcal{L}^{>0} = \{\{1\}, \{2\}, \{6\}, \{7\}, \{1, 2\}, \{6, 7\}\}.$$

which consists of two disconnected trees. Hence, $\mathcal{L}^{>0}$ is not sequentially connected, and in particular not sequentially Cohen–Macaulay.

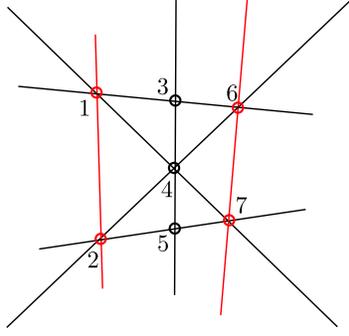


Figure 1.1. The matroid M , its proper flats and the filtered geometric lattice $\mathcal{L}^{>0}$ in red.

2. THE LEFSCHETZ THEOREMS FOR SMOOTH TROPICAL VARIETIES

We now prove several Lefschetz theorems for tropical varieties; in each section we recall classical Lefschetz theorems, and then proceed to prove analogues for tropical varieties. For this purpose, we need to recast our Main Theorem 1.1.1 in the language of Bergman fans:

2.1. The positive side of the Bergman fan. Associated to every matroid M is the **Bergman fan** [AK06, Ber71, Stu02]. We identify the elements of $[n]$ of the base set M with a circuit of integer vectors e_1, \dots, e_n in \mathbb{R}^{n-1} such that $\sum_{i=1}^n e_i = 0$. If F is any subset of $[n]$, we define $e_F = \sum_{e_i \in F} e_i$, and if

$$\mathbf{F} = F < G < H < \dots$$

is any chain of flats in \mathcal{L} , then

$$\text{pos}(\mathbf{F}) \stackrel{\text{def}}{=} \text{pos}\{e_F, e_G, e_H, \dots\},$$

where pos denotes the positive span of a subset of \mathbb{R}^d . The **Bergman fan** $\mathfrak{F}(M)$ of M is the fan

$$\mathfrak{F}(M) \stackrel{\text{def}}{=} \{\text{pos}(\mathbf{F}) : \mathbf{F} < [n] \text{ increasing chain in } \mathcal{L}\}.$$

We then have the following corollary of Theorem 1.1.1.

Lemma 2.1.1. *Let M denote any finite matroid, let $\mathfrak{F} = \mathfrak{F}(M)$ in $\mathbb{R}^{|M|-1}$ denote its Bergman fan, and let H^+ denote a generic halfspace with $\mathbf{0} \in \partial H^+$. Then the geometric link $\text{lk}(\mathbf{0}, \text{R}(\mathfrak{F}, H^+))$ is homotopy Cohen–Macaulay of dimension $r - 2$.*

Proof. Let \mathbf{n} denote the interior normal vector to H . Then

$$\omega = (\omega_1, \dots, \omega_n) = (\mathbf{n} \cdot e_1, \dots, \mathbf{n} \cdot e_n)$$

is a generic weight on the elements $[n]$ of M with $\omega \cdot [n] = 0$. With this we have, for every subset σ of $[n]$, that

$$\sigma \in \mathcal{L}^{>0} \iff \omega \cdot \sigma > 0 \iff \mathbf{n} \cdot e_\sigma > 0 \iff e_\sigma \in H^+$$

so that

$$\text{lk}(\mathbf{0}, \text{R}(\mathfrak{F}, H^+)) \cong \mathcal{L}^{>0}.$$

The claim hence follows from Theorem 1.1.1. □

Similarly, we also have, using Theorem 1.1.1, or the work of Björner [Bjö80].

Lemma 2.1.2. *Let M denote any finite matroid, let $\mathfrak{F} = \mathfrak{F}(M)$ in $\mathbb{R}^{|M|-1}$ denote its Bergman fan. Then the geometric link $\text{lk}(\mathbf{0}, \mathfrak{F})$ is homotopy Cohen–Macaulay of dimension $r - 2$.*

2.2. Lefschetz Section Theorems for cell decompositions, homotopy and homology of tropical varieties. A crucial ingredient of the Lefschetz Section Theorem of Andreotti–Frankel [AF59] is a Vanishing Theorem for affine varieties.

Theorem 2.2.1 (Andreotti–Frankel, [AF59]). *Let X denote a smooth affine n -dimensional variety in \mathbb{C}^d . Then X is homotopy equivalent to a CW complex of dimension $\leq n$. In particular, the homotopy and integral homology groups of X vanish above dimension n .*

The idea for the proof is to use (classical) Morse theory; the Morse function is given by the distance d from a generic point in \mathbb{C}^d . The theorem then follows from the Main Lemma of Morse theory, together with an index estimate for the critical points of d which can be concluded from the hermitian structure of the Hessian. See Milnor’s book [Mil63] for an excellent exposition.

From this affine theorem, one can deduce the classical Lefschetz Section Theorem. For smooth algebraic varieties and homology groups, this theorem was proven first by Lefschetz [Lef50], and later Andreotti–Frankel [AF59].

Theorem 2.2.2 (Lefschetz, Andreotti–Frankel, [Lef50, AF59]). *Let X denote any smooth projective algebraic n -dimensional variety in $\mathbb{C}\mathbb{P}^d$, and let H denote a generic hyperplane in $\mathbb{C}\mathbb{P}^d$. Then the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.*

This follows directly from the fact that $X \setminus H$ is an affine variety, Theorem 2.2.1, and Lefschetz duality. Bott, Thom and Milnor then observed that this theorem extends to homotopy groups, using Thom’s observation that Morse theory gives rise to cell decompositions of the manifold.

Theorem 2.2.3 (Bott, Milnor, Thom, cf. [Bot59, Mil63]). *Let X denote any smooth projective algebraic n -dimensional variety in $\mathbb{C}\mathbb{P}^d$, and let H denote a generic hyperplane in $\mathbb{C}\mathbb{P}^d$. Then X is homotopy equivalent to a space obtained from $H \cap X$ by successively attaching cells of dimension $\geq n$.*

In particular, the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of homotopy and integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.

The tropical case. Similar to the Vanishing Theorem for classical projective varieties, the Lefschetz type theorem for affine tropical varieties proved in this section shall be crucial to derive Lefschetz theorems for projective varieties, although we shall follow a slightly different reasoning due to the absence of Lefschetz duality. Instead, the Andreotti–Frankel Vanishing Theorem takes, in the tropical realm, the form of a Lefschetz Section Theorem for affine varieties. The theorem can be stated as follows:

Theorem 2.2.4. *Let $X \subset \mathbb{T}^d$ be a smooth, affine n -dimensional tropical variety, and let H denote a tropical hyperplane transversal to X . Then X is obtained from $H \cap X$ by successively attaching n -dimensional cells.*

By elementary cellular homology and homotopy theory [Hat02, Whi78], we immediately obtain a Lefschetz Section Theorem for homotopy and homology groups:

Corollary 2.2.5. *Let $X \subset \mathbb{T}^d$ be a smooth, affine n -dimensional tropical variety, and let H denote a tropical hyperplane transversal to X . Then the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.*

For the proof of Theorem 2.2.4, notice that it suffices to consider the finite part $X_{\delta'}$ of X , since the finite part and the original variety are combinatorially equivalent. Therefore, it suffices to prove that $X_{\delta'}$ is obtained from $X_{\delta'} \setminus H$ by successively attaching cells of dimension n .

We may furthermore assume that H is mobile, i.e., that H as a vertex v of sedentarity 0 (that might not lie in $(-\delta', \delta')^d$). However, by passing to a bigger box δ , the vertex v can be assumed to lie in the box $(-\delta, \delta)^d$ and it remains to prove that X_{δ} is obtained from $X_{\delta} \setminus H$ by successively attaching cells of dimension n .

The complement of the tropical hyperplane H in \mathbb{T}^d is divided into $\leq d + 1$ open polyhedral cells C_1, C_2, \dots . To prove Theorem 2.2.4, it therefore suffices to prove the following Lemma:

Lemma 2.2.6. *Let X denote a smooth tropical n -dimensional variety in \mathbb{T}^d , and let C denote any pointed convex d -polyhedron in \mathbb{T}^d (or more generally, any convex body in \mathbb{T}^d). Then $X \cap C$ is obtained from $X \cap \partial_{|\mathfrak{m}} C$ by attaching, successively, cells of dimension n .*

Here $\partial_{|\mathfrak{m}} C$ is the **mobile part** of ∂C , i.e.

$$\partial_{|\mathfrak{m}} C \stackrel{\text{def}}{=} (\partial C)_{|\mathfrak{m}} = \{x \in \partial C : \mathfrak{s}(x) = 0\}$$

Proof. If $\delta > 0$ is chosen large enough so that the box $(-\delta, \delta)^d$ contains all mobile vertices of the common refinement $X \cdot C$, then $X[\delta] \cap C$ and $X \cap C$, and $X[\delta] \cap \partial_{|\mathfrak{m}} C$ and $X \cap \partial_{|\mathfrak{m}} C$, are naturally homeomorphic. Hence, it suffices to prove that $X[\delta] \cap C$ is obtained from $X[\delta] \cap \partial_{|\mathfrak{m}} C$ by iteratively attaching cells of dimension n .

The cell C is bounded by halfspaces H_1, \dots, H_k . Let $d_i(x)$, $1 \leq i \leq k$ denote the distance of a point $x \in \overline{C}$ to the hyperplane H_i , and let α_i , $1 \leq i \leq k$ denote a sequence of positive real numbers. Let us define the function

$$\begin{aligned} \tilde{f} : \overline{C} &\longrightarrow \mathbb{R}^{\geq 0} \\ x &\longmapsto \prod_{i=1}^k d_i^{\alpha_i}(x) \end{aligned}$$

The function \tilde{f} is smooth when restricted to any stratum of $X[\delta]$. Furthermore, the superlevel sets $C_{\geq t} \stackrel{\text{def}}{=} \tilde{f}^{-1}[t, \infty)$ are convex, so that for every stratum σ° , the critical function $\tilde{f}|_{\sigma^\circ} : \sigma^\circ \longrightarrow \mathbb{R}^{\geq 0}$ has at most one critical value (namely, a minimum), cf. Lemma A.5.2. Therefore, the critical points of f are distinct, and finite in number, so that f is a proper Morse function on $X[\delta]$. If the α_i are chosen generically, we may furthermore assume that the critical values of $f \stackrel{\text{def}}{=} \tilde{f}|_{X[\delta]} : X[\delta] \cap \overline{C} \longrightarrow \mathbb{R}^{\geq 0}$ are distinct.

By the Main Theorem of stratified Morse Theory (Theorem A.5.1), it therefore suffices to prove that for every critical point x of \tilde{f} , the link $(X_{[t-\varepsilon, t]}[\delta])$ is obtained from $(X_{[t-\varepsilon]}[\delta])$ by successively attaching n -cells. Here, $\varepsilon > 0$ is chosen small enough, so that $[t - \varepsilon, t]$ contains only one critical value of f .

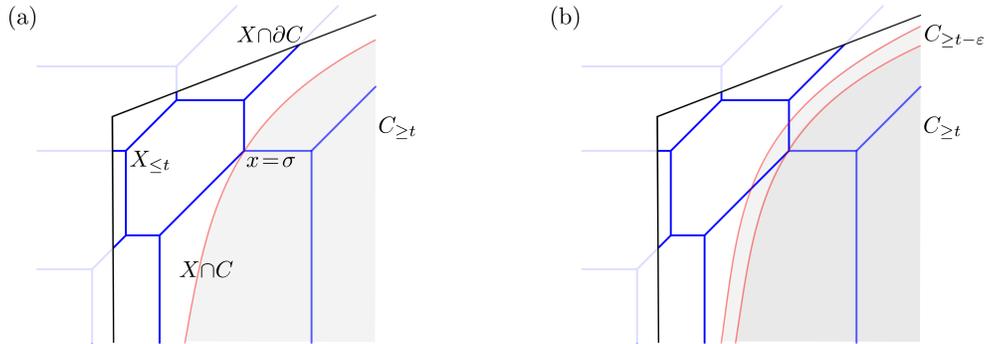


Figure 2.2. Using stratified Morse theory on $X \cap C$, it suffices to consider the Morse data at critical points.

Let σ denote the minimal face of X containing x . The set $C_{\geq t} \stackrel{\text{def}}{=} f^{-1}[t, \infty) \cap [-\delta, \delta]^d$, $t = f(x) \geq 0$ is a convex set with smooth boundary in the box $[-\delta, \delta]^d$. By Lemma A.5.2(a), the tangential Morse data at x is therefore given by $(\sigma, \partial\sigma)$. Furthermore, considering the halfspace $\text{TC}_x C_{\leq t}$, the normal Morse data at x is given by $(\text{CN}_\sigma^1 X_{\leq t}, \text{N}_\sigma^1 X_{\leq t})$, where $\text{N}_\sigma^1 X_{\leq t} = \text{N}_\sigma^1 X \cap \text{N}_\sigma^1 f^{-1}(\infty, t]$ is homotopy equivalent to a wedge of spheres of dimension $(n - \dim \sigma - 1)$ by Lemma 2.1.1.

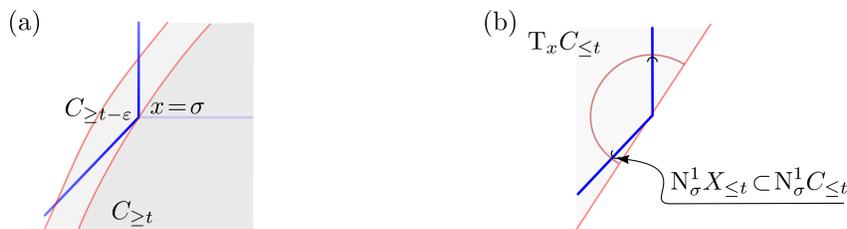


Figure 2.3. The normal Morse data in at a critical point $x \in \sigma^\circ$ is given by restricting $\text{lk}(\sigma, X) \simeq N_\sigma^1$ to the hemisphere $T_x C_{\leq t}$.

Therefore, the Morse data at x is given as

$$\begin{aligned} & (CN_\sigma^1 X_{\leq t}, N_\sigma^1 X_{\leq t}) \times (\sigma, \partial\sigma) \\ & \simeq (C(N_\sigma^1 X_{\leq t} * \partial\sigma), N_\sigma^1 X_{\leq t} * \partial\sigma), \end{aligned}$$

where $N_\sigma^1 X_{\leq t} * \partial\sigma$ is homotopy equivalent to a wedge of $(n-1)$ spheres by Lemma A.1.2. The claim now follows with Theorem A.5.1(b), finishing the proof of Lemma 2.2.6. \square

We can now prove the Lefschetz theorem for projective tropical varieties. As in the classical case, it is an easy consequence of the treatment of affine varieties; however, instead of using Lefschetz duality, we can use a direct argument using Morse theory on the projective variety, based on the fact that tropical projective space is but a union of tropical affine spaces.

Theorem 2.2.7. *Let X denote any n -dimensional smooth projective tropical variety in \mathbb{TP}^d , and let $H \in \mathbb{TP}^d$ denote a tropical hyperplane. Then X is, up to homotopy equivalence, obtained from $H \cap X$ by successively attaching cells of dimension n .*

In particular, the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of homotopy and integral homology groups up to dimension $n-2$, and a surjection in dimension $n-1$.

Proof. We may again assume that H has sedentarity 0. Up to tropical multiplication (i.e. translation), we may furthermore assume H to be given by the hyperplane given by the linear function $\varphi : \mathbb{T}^{d+1} \rightarrow \mathbb{T}$, $x \mapsto \sum x_i$. With this, H induces a partition of $\mathbb{TP}^d = \mathbb{T}^{d+1}/(x \sim \lambda x)$ into d closed polyhedra T_i , $1 \leq i \leq d$, corresponding to copies of $\mathbb{T}_{\leq 0}^d$. Now, for every i , we have that $T_i \cap X$ is obtained from $\partial T_i \cap X$ by attaching cells of dimension n by Lemma 2.2.6. \square

Decomposing the variety, step by step. It is possible to give a more “combinatorial” presentation of the proof of Lemma 2.2.6 by exhibiting how the cells of a slightly refined version of X are attached, one by one, along the sublevel sets of the Morse function.

Let σ denote any face of a tropical variety X in \mathbb{T}^d of fine sedentarity S . Let H^+ denote any halfspace in $\mathbb{R}^{[n] \setminus S}$ containing σ in its boundary, and let $\tilde{H}^+ \stackrel{\text{def}}{=} H^+ \times \mathbb{T}^S$. Then $\text{st}(\sigma, X) \cap \tilde{H}^+$ and $\text{st}(\sigma, R(X, \tilde{H}^+))$ are the **geometric** and **combinatorial half-star** of σ in X w.r.t. \tilde{H}^+ , respectively. With this, we have the following reformulation of Lemma 2.1.1.

Lemma 2.2.8. *Let X denote a smooth tropical n -dimensional variety in \mathbb{T}^d , let σ denote any face of X , and let \tilde{H}^+ be any halfspace in \mathbb{T}^d with σ in its boundary, as above. Then $\partial \text{st}(\sigma, X) \cap \tilde{H}^+ \simeq \partial \text{st}(\sigma, R(X, \tilde{H}^+))$ is a wedge of spheres of dimension $n-1$.* \square

For X and C as in Lemma 2.2.6, let \tilde{X} denote the common refinement of X and C ,

$$\tilde{X} \stackrel{\text{def}}{=} X \cdot C = \{\sigma \cap \tau : \sigma \in X, \tau \in C\}$$

Analogously, for the finite part of $X[\delta]$ of X as in the proof of Theorem 2.2.6, we set $\tilde{X}[\delta] = X[\delta] \cdot C$. Also, we consider again the Morse function $f : C \rightarrow \mathbb{R}$ defined as the weighted product of the distance functions of the hyperplanes defining facets of a given maximal cell C . We then have the following observation:

Proposition 2.2.9. *Let $t \geq 0$, and let notation be the same as above. Then*

$$\tilde{X}[\delta] \cap |f^{-1}(-\infty, t] \simeq \mathbf{R}(\tilde{X}[\delta], f_k^{-1}(-\infty, t]).$$

In particular, if t is a critical value, x the critical point and σ the minimal face of $\tilde{X}[\delta]$ containing it, and $\varepsilon > 0$ chosen so that $(t - \varepsilon, t]$ contains no critical value besides t . Then

$$\mathbf{R}(\tilde{X}[\delta], f_k^{-1}(-\infty, t]) - \sigma = \mathbf{R}(\tilde{X}[\delta], f_k^{-1}(-\infty, t - \varepsilon])$$

Proof. Use Proposition A.2.1. □

If $0 < t_1 < t_2 < t_3 < \dots$ denotes the sequence of critical values of f , then we call the sequence of complexes

$$X'_j \stackrel{\text{def}}{=} (\mathbf{R}(\tilde{X}[\delta], f_k^{-1}(-\infty, t_j]))$$

a **decomposition sequence** for $\mathbf{R}(\tilde{X}, \overline{C_k})$ with **critical faces** σ_j . We then have

$$\tilde{X}_{j-1} \simeq \tilde{X}_j - \sigma_j$$

and the tropical halfstar $\text{st}(\sigma_j, \tilde{X}_j)$ is $(n-1)$ -connected by 2.2.8. Together with Lemma A.1.3, we have proven:

Theorem 2.2.10. *Let X denote a smooth tropical n -dimensional variety in \mathbb{T}^d , and let C denote any convex d -polyhedron in \mathbb{T}^d (or more generally, any convex body in \mathbb{T}^d). Then there is a combinatorial decomposition sequence taking $X \cap C$ to $X \cap \partial C$ by iteratively deleting tropical halfstars.* □

2.3. Lefschetz Section Theorems for complements of tropical varieties. Motivated by the study of complements of subspace arrangements, several Lefschetz theorems were proven that apply to complements of affine varieties, prominently the theorems of Hamm–Lê, cf. [DP03, Ham83, HL71, Lê87]. Motivated by problems concerning the Milnor fiber and subsurface arrangements, they proved using Morse theory for manifolds with boundary applied to the Milnor fiber of the variety:

Theorem 2.3.1 (Hamm–Lê [Ham83, HL71, Lê87], cf. [DP03, Ran02]). *Let φ denote any non-constant complex polynomial in d variables. If H is a generic hyperplane in \mathbb{C}^d , then $C(\varphi) \stackrel{\text{def}}{=} \{x \in \mathbb{C}^d : \varphi(x) \neq 0\}$ is obtained from $C(\varphi) \cap H$ by attaching cells of dimension $d-1$.*

The tropical case. The purpose of this section is to provide a tropical analogue of this influential result.

Theorem 2.3.2. *Let X denote a smooth n -dimensional tropical variety in \mathbb{T}^d , and let $C = C(X)$ denote the complement of X in \mathbb{T}^d . Let furthermore H denote an almost totally sedentary hyperplane in \mathbb{T}^d transversal to X . Then C is, up to homotopy equivalence, obtained from $C \cap H$ by successively attaching $(d-n-1)$ -dimensional cells.*

In particular, the inclusion of $C \cap H$ into C induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $d-n-3$, and a surjection in dimension $d-n-2$.

The central ingredient will be a relative version of Lemma 2.1.1

Lemma 2.3.3. *Let $(M, \mathfrak{F}, H^+, r)$ be as in Lemma 2.1.1, let H^- denote the closure of $\mathbb{R}^{|M|-1} \setminus H^+$, and let $C = \mathbb{R}^{|M|-1} \setminus \mathfrak{F}$. Then C is, up to homotopy equivalence, obtained from $C \cap H^-$ by attaching cells of dimension $|M| - r - 1$.*

Proof. Let \mathcal{B} denote the geometric lattice on groundset of M , and let \mathfrak{F}' denote the associated Bergman fan. Finally, let ω denote the weight associated to H^+ as given in the proof Lemma 2.1.1, so that

$$\text{lk}(\mathbf{0}, \mathbf{R}(\mathfrak{F}, H^-)) \cong \mathcal{L}^{<0}.$$

Then $\mathbb{R}^{|M|-1} \setminus \mathfrak{F} \simeq \mathfrak{F}' \setminus \mathfrak{F}$ radially deformation retracts to $\mathcal{B} \setminus \mathcal{L}$. The same map deformation retracts $H^- \setminus \mathfrak{F}$ to

$$\text{lk}(\mathbf{0}, \mathfrak{F}) \cap H^- \simeq \mathcal{B}^{<0} \setminus \mathcal{L}^{<0}.$$

Hence, the pair $(C, C \cap H^-)$ is homotopy equivalent to the pair $(\mathcal{B} - \mathcal{L}, \mathcal{B}^{<0} - \mathcal{L}^{<0})$. The claim follows by Corollary 1.5.2. □

Proof of Theorem 2.3.2. We may restrict to the bounded part $X[\delta]$ of X , for some $\delta > 0$ large enough, without loss of generality. More specifically, it suffices to prove that $C \cap [-\delta, \delta]^d$ is obtained from $C \cap H \cap [-\delta, \delta]^d$ by attaching cells of dimension $d - n$, where δ is chosen big enough so that the box $(-\delta, \delta)^d$ contains all vertices of X of sedentarity 0 and intersects H , because with such a choice of Δ

$$C \cap [-\delta, \delta]^d \simeq C \quad \text{and} \quad C \cap H \cap [-\delta, \delta]^d \simeq C \cap H$$

Let d_H denote the distance from the hyperplane H . Clearly, the function d_H is smooth (and even linear) on every stratum of $X[\delta]$. Moreover, if we perturb H by a small amount to a generic hyperplane H' ,

$$C \cap H' \cap [-\delta, \delta]^d \simeq C \cap H \cap [-\delta, \delta]^d$$

with the additional benefit that $\tilde{f} \stackrel{\text{def}}{=} d_{H'}$ may be assumed to restrict to a Morse function f on $X[\delta]$.

We may now apply stratified Morse theory; by Theorem A.5.1(c), it suffices to prove that, if x is any critical point of f , and t its value, and $\varepsilon > 0$ chosen small enough such that $[t, t + \varepsilon)$ contains no further critical values of f , then $C_{\leq t + \varepsilon} = C \cap \tilde{f}^{-1}(0, t + \varepsilon]$ is obtained from $C_{\leq t}$ by successively attaching $(d - n - 1)$ -cells.

Now, clearly the minimal stratum of $X[\delta]$ containing x is x itself, so that the tangential Morse data at x is trivial. It remains to estimate the normal Morse data at x , which, if we set

$$H_x^- \stackrel{\text{def}}{=} \tilde{f}^{-1}[t, \infty) \quad \text{and} \quad H_x \stackrel{\text{def}}{=} \tilde{f}^{-1}t = \partial H_x^-,$$

is given by the relative link

$$(\mathbb{T}_x^1 H \cap X_x^-, \mathbb{T}_x^1 H \cap X_x),$$

i.e., $C_{\leq t + \varepsilon}$ is obtained from $C_{\leq t}$ by attaching $\mathbb{T}_x^1 H \cap X_x^-$ along $\mathbb{T}_x^1 H \cap X_x$. Since by Lemma 2.3.3, $\mathbb{T}_x^1 H \cap X_x^-$ is obtained from $\mathbb{T}_x^1 H \cap X_x$ by successively attaching n -cells, the claim follows by Theorem A.5.1(d). \square

2.4. Lefschetz Section Theorems and Vanishing Theorems for tropical Hodge groups. The Lefschetz Section Theorem for Hodge groups was first established by Kodaira and Spencer.

Theorem 2.4.1 (Kodaira–Spencer, c.f. [KS53]). *Let X denote any smooth projective algebraic n -dimensional variety in $\mathbb{C}\mathbb{P}^d$, and let H denote a generic hyperplane in $\mathbb{C}\mathbb{P}^d$. Then the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.*

For a modern proof of this result, recall that by the Hodge Index Theorem, we have

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega^q(X)).$$

Together with the Dolbeault operators, this decomposition is functorial; the result now follows from Theorem 2.2.2 for complex coefficients.

Alternatively, one can prove the theorem directly and algebraically, using the Vanishing Theorem of Akizuki–Kodaira–Nakano:

Theorem 2.4.2 (Akizuki–Kodaira–Nakano Vanishing Theorem, cf. [AN54]). *Let $X \in \mathbb{R}\mathbb{P}^d$ denote a smooth, compact projective n -dimensional variety, and let $L \rightarrow X$ be a positive line bundle. Then*

$$H^q(X, \Omega^p(L)) = 0 \quad \text{for all } p + q > n.$$

Equivalently (by Serre duality), we have

$$H^q(X, \Omega^p(-L)) = 0 \quad \text{for all } p + q < n.$$

Theorem 2.4.1 then follows from the long exact sequence of Hodge groups, cf. [AN54, Voi02].

Remark 2.4.3. Notice that applying the observed functoriality of the Hodge Index Theorem again, we can conclude Theorem 2.2.2 for the complex field of coefficients, and by the universal coefficient theorem for rational coefficients. In other words, in the situation of smooth complex algebraic varieties, the Lefschetz theorem for Hodge groups is *weaker* than the Lefschetz theorems of Lefschetz, Andreotti–Frankel and Bott–Milnor–Thom.

The tropical case. Contrary to the smooth case, the analogous theorems for tropical varieties are not as easily derived, since the Hodge Index Theorem does not hold for smooth tropical varieties [Sha11, Thm. 3.3.5]. Our Lefschetz Section Theorem for Hodge groups is stated as follows:

Theorem 2.4.4. *Let X denote any n -dimensional smooth affine tropical variety in \mathbb{T}^d , and let $H \subset \mathbb{T}^d$ denote a hypersurface transversal to X . Then the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of (p, q) -homology up to dimension $p + q \leq n - 2$, and a surjection in dimension $p + q = n - 1$.*

Analogously, we have a similar theorem for projective tropical varieties.

Theorem 2.4.5. *Let X denote any n -dimensional smooth projective tropical variety in \mathbb{TP}^d , and let $H \subset \mathbb{TP}^d$ denote a hypersurface transversal to X . Then the inclusion $H \cap X \hookrightarrow X$ induces an isomorphism of (p, q) -homology up to dimension $p + q \leq n - 2$, and a surjection in dimension $p + q = n - 1$.*

For the proof, we shall have to work around the fact that the Hodge Index Theorem is, in its strongest form, not available for tropical varieties. Instead, we follow the classical, direct proof of the Kodaira–Spencer Lefschetz Section Theorem, and use a weak version of the Hodge Index Theorem. Using this and Lemma 2.2.8, we first prove an analogue of the Akizuki–Kodaira–Nakano Vanishing Theorem.

Theorem 2.4.6. *Let X denote any n -dimensional smooth affine tropical variety in \mathbb{T}^d (or \mathbb{TP}^d), and let P denote any rational d -polyhedron in \mathbb{T}^d (resp. \mathbb{TP}^d). Then, every chain in $c \in C_q(X \cap P; \mathcal{F}_p X)$ is homologous to a chain $\tilde{c} \in C_q(X \cap \partial P; \mathcal{F}_p X)$ provided that $p + q < n$.*

In particular, we have a quasi-isomorphism $C_q(X \cap P; \mathcal{F}_p X) \rightarrow C_q(X \cap \partial P; \mathcal{F}_p X)$, so that

$$H_q(X \cap P, X \cap \partial P; \mathcal{F}_p X) = 0 \text{ for all } p + q < n.$$

The Lefschetz Section Theorem for Hodge groups then swiftly follows.

Pushing Chains and the tropical AKN Theorem 2.4.6. The idea for the proof is to “push” (p, q) -chains in $X \cap P$ towards $X \cap \partial P$. This in particular gives us a procedural view on the deformation of chains, and quickly implies the tropical AKN Theorem 2.4.6. The main ingredient to this “Pushing Lemma” will be Main Lemma 2.1.1; to apply the latter, we will need a simple, weak version of the Hodge Index Theorem for tropical varieties and manifolds.

Lemma 2.4.7 (Hodge Index Vanishing). *Let X denote any k -connected smooth tropical manifold. Then*

$$H_q(X; \mathcal{F}_p X) = 0 \text{ for all } p + q \leq k$$

For a simple proof, one can use the universal coefficient spectral sequence for generalized homology theories [Ada69, Ada74]. We give a more elementary, combinatorial argument here.

Proof. We work by induction on the dimension d of the manifold X ; the case $d = 0$ is a triviality. Furthermore, $H_q(X; \mathcal{F}_p X) = 0$ if $p + q > d$ by the stabilization Lemma A.4.5. We may therefore assume that $p + q \leq d$.

The central argument is to show that we can push every chain homologously to a chain supported in a terminal single q -face (say τ). Since the link of τ is a $(d - 2)$ -connected smooth tropical manifold (it is itself the link of a joins of Bergman fans), we can then conclude that if the chain is, in fact a cycle, then it is a cycle in said link, and therefore a boundary by induction on the dimension d .

Now, let c be a (p, q) -chain supported in a single q -face σ of X , and assume that

$$c = v_1 \wedge v_2 \wedge \cdots \wedge v_p \subset \bigwedge^p \text{lin } T_\sigma \hat{\sigma},$$

where $\hat{\sigma} \supset \sigma$ is a face of σ of dimension $\geq p$. Now, let σ' denote any q -face of X adjacent to σ , and let F denote the minimal face of X containing σ and σ' . By induction on the dimension, we push c to an homologous chain c' supported in σ , but with the additional property that

$$c' = \sum_{\tau \in \text{st}(F, X)} v_1^\tau \wedge v_2^\tau \wedge \cdots \wedge v_p^\tau, v_i^\tau \in \text{lin } T_\sigma \tau.$$

But then c' is homologous to a chain \tilde{c} supported in σ' by pushing c through F .

Now, we can use the connectedness of X to push c to any face we choose. The claim then follows as above. \square

Lemma 2.4.8 (Pushing chains). *Let X and P be chosen as in Theorem 2.4.6. Let $c \in C_q(X; \mathcal{F}_p X)$ denote a (p, q) -chain of X such that for some face σ of X , c is supported in a tropical half-star $\text{st}(\sigma, \mathbf{R}(X, \tilde{H}^+))$. If $p + q \leq n - 1$, then there is a (p, q) -chain $\tilde{c} \in C_q(X; \mathcal{F}_p X)$ homologous to c that is supported in $(\text{supp } c) - \sigma$.*

Proof. We abbreviate $\Sigma \stackrel{\text{def}}{=} \text{st}(\sigma, \mathbf{R}(X, \tilde{H}^+))$ for the duration of the proof. Notice that by Lemmas 2.2.8 and A.4.4, together with Hodge index vanishing, the relative tropical Hodge groups $H_q(\Sigma, \partial\Sigma; \mathcal{F}_p X)$ vanish.

Now, let us first assume that for the restriction c' of c to Σ , we have $\text{supp } (\partial c') \subset \partial\Sigma$. Then, by Lemma 2.2.8 and Lemma 2.4.7, c' is a boundary in $C_q(\Sigma, \partial\Sigma; \mathcal{F}_p X)$, so that there is a cycle $b \in C_{q+1}(\Sigma, \partial\Sigma; \mathcal{F}_p X)$ with

$$c' + \tilde{c} = \partial b, \quad \text{where } \tilde{c} \in \partial\Sigma.$$

Hence, $c - c' + \tilde{c}$ is homologous to c modulo b , and supported in $\text{supp } c - \sigma$.

If c is not a cycle, then we may assume that c' is a cycle modulo $\partial\Sigma$ by pushing ∂c to $\partial\Sigma$ as in the previous argument. Applying the argument above again, we can see that c can be pushed to $\partial\Sigma$ as well. \square

Together with the combinatorial decomposition sequence of Theorem 2.2.10, this finishes the proof of Theorem 2.4.6.

Proof of the tropical Kodaira–Spencer Theorems 2.4.4 and 2.4.5. We assume throughout that $p \geq 1$, since the case when $p = 0$ was dealt with already (since $\mathcal{F}_0 \equiv \mathbb{Z}$). We divide the proof into two parts by showing that the maps

$$H_q(H \cap X; \mathcal{F}_p X) \longrightarrow H_q(X; \mathcal{F}_p X)$$

and

$$H_q(H \cap X; \mathcal{F}_p H \cap X) \longrightarrow H_q(H \cap X; \mathcal{F}_p X)$$

induced by inclusion are isomorphisms for $p + q < n - 1$, and onto for $p + q \leq n - 1$. For this, we show that

- (I) every chain $c \in C_q(X; \mathcal{F}_p X)$ is homologous to a chain $\tilde{c} \in C_q(H \cap X; \mathcal{F}_p X)$ and
- (II) every chain $c \in C_q(H \cap X; \mathcal{F}_p X)$ is homologous to a chain $\tilde{c} \in C_q(H \cap X; \mathcal{F}_p H \cap X)$

as long as $p + q < n$.

Now, Claim (I) is immediate from Theorem 2.4.6, since H divides \mathbb{T}^d resp. \mathbb{TP}^d into polyhedra to which we can apply Theorem 2.4.6 separately.

For Claim (II), let us notice first that by Lemma A.4.5, we can assume every chain to be stable. We may assume that c is supported in a single face σ and use h to denote the minimal face of H that contains σ . Let $\hat{\sigma}$ denote the face of X that intersects H such that

$$c = v_1 \wedge v_2 \wedge \cdots \wedge v_p, \quad v_i \in \text{lin } \mathbf{T}_\sigma \hat{\sigma} \subset \text{lin } \mathbf{T}_\sigma X.$$

Now, there are three situations to consider:

- if $c \in C_q(H \cap X; \mathcal{F}_p H \cap X)$, there is nothing to prove.
- If h is of codimension at least p in H , let τ_i denote the cofacets of σ in $H \cap X$, then the primitive integral vectors in $n_i \stackrel{\text{def}}{=} N_\sigma \tau_i$ span $\mathbf{T}_\sigma \hat{\sigma}$. Therefore, c may be written as a linear combination of the exterior products of the vectors n_i . It therefore lies in $C_q(H \cap X; \mathcal{F}_p H \cap X)$.
- Finally, assume that h is of codimension $\ell < p$ in H . Then we can write

$$c = w_1 \wedge w_2 \wedge \cdots \wedge w_{p-\ell-1} \wedge v_\ell \wedge v_p,$$

where the w_i , $1 \leq i \leq p - \ell - 1$, lie in $\mathbf{T}_\sigma \sigma$, so that

$$c' \stackrel{\text{def}}{=} w_1 \wedge w_2 \wedge \cdots \wedge w_{p-\ell-1} \in C_q(H \cap X; \mathcal{F}_{p-\ell-1} H \cap X).$$

Now, c' is homologous to a chain \tilde{c}' in $C_q(\partial h; \mathcal{F}_{p-\ell-1} H \cap X)$ as long as

$$p - \ell - 1 + q \leq \dim X \cap h = \dim X - \ell - 1 = n - \ell - 1.$$

We hence conclude that there exists a chain \tilde{c} in $C_q(\partial h; \mathcal{F}_p X)$ which is homologous to c . Iterating this argument, we see that we can find a chain homologous to c in a face of codimension at least p , and the desired conclusion follows from the previous step. \square

2.5. Remarks and open problems.

Lefschetz for abstract varieties. Classical Lefschetz Theorems are often phrased abstractly, using the notions of ample divisors and positive line bundles, cf. [Laz04, Voi02]. There seems to be no such notion in tropical geometry that is generally agreed upon, although several attempts have been made [Car13].

Open Problem 2.5.1. *Define tropical line bundles and divisors. Does the notion give rise to Lefschetz Theorems for abstract smooth tropical varieties?*

Constructible sheaves in tropical geometry. Another instance of a Lefschetz Section Theorem is the Artin–Grothendieck Vanishing Theorem [Laz04, Thm. 3.1.13] for constructible sheaves. Again, no analogous notion seems to exist for tropical varieties.

Open Problem 2.5.2 (Tropical Lefschetz Section Theorem for constructible sheaves). *What is the tropical analogue of the Vanishing Theorem of Artin–Grothendieck.*

Tropical Lefschetz manifolds. Again, the problem seems mainly to come up with a good and working notion. Finally, a worthwhile goal is obviously to understand the Hard Lefschetz Theorem for tropical varieties, cf. [Del80].

Open Problem 2.5.3. *What is a tropical Kähler manifold? Does it satisfy a tropical analogue of the Hard Lefschetz Theorem.*

Tropical subspace arrangements. Theorem 2.3.2 is, to our knowledge, the first time the “complement” of a (smooth) tropical variety was studied explicitly. It might be interesting to study this further.

Open Problem 2.5.4. *Find interesting properties of tropical subspace arrangements.*

APPENDIX A. BASIC NOTIONS

A.1. Some basic combinatorial topology. We recall some basic facts from algebraic topology and the topology of posets. We refer the reader to [Mun84], [Bjö95] and [Whi78] for more details. All topological spaces have the homotopy type of simplicial complexes and in particular always have a CW composition.

Acyclicity and Connectivity. A topological space X is said to be **k -connected** if either of the following equivalent conditions hold:

- $\pi_i(X) = 0$ for all $i \leq k$, i.e., every embedding of the sphere S^i , $i \leq k$, into Δ is null-homotopic,
- X is homotopy equivalent to a CW complex that, except for the basepoint, has no cells of dimension $\leq k$.

Similarly, a pair of topological spaces (X, Y) is **k -connected** if $\pi_i(X, Y) = 0$ for all $i \leq k$.

A space X is **k -acyclic** if $\tilde{H}_i(X; \mathbb{Z}) = 0$ for all $i \leq k$, and a pair of spaces (X, Y) is **k -acyclic** if $\tilde{H}_i(X, Y; \mathbb{Z}) = 0$ for all $i \leq k$. Every k -connected space is k -acyclic, by elementary cellular homology. We will repeatedly make use of the fact that by the Theorems of Whitehead and Hurewicz (see for instance [Hat02, Sec. 4]), a k -acyclic space (or pair of spaces), $k \geq 1$, is k -connected if and only if it is 1-connected.

The Cohen–Macaulay property. A pure simplicial complex Δ of dimension $d - 1$ is **homotopy Cohen–Macaulay** if any of the following equivalent conditions holds

- for all faces σ in Δ , the link $\text{lk}(\sigma, \Delta)$ is $(d - \dim \sigma - 3)$ -connected.
- for all faces σ in Δ , the link $\text{lk}(\sigma, \Delta)$ is homotopy equivalent to a wedge of $(d - \dim \sigma - 2)$ -dimensional spheres.

Here the empty set is considered to be a (-1) -dimensional face, and $\text{lk}(\emptyset, \Delta) = \Delta$.

A pure $(d-1)$ -dimensional simplicial complex Δ is **Cohen–Macaulay over** \mathbb{Z} if $\tilde{H}_i(\text{lk}(\sigma, \Delta); \mathbb{Z}) = 0$ for all faces $\sigma \in \Delta$ and all $i < \dim \text{lk}(\sigma, \Delta) = d - \dim \sigma - 2$. Being C-M over R is similarly defined for other coefficient rings R . See [Sta96] for some of the algebraic ramifications of this concept.

Elementary cellular topology. We use $A * B$ to denote the join of two topological spaces (CW-complexes, simplicial complexes) A, B , and $CX \stackrel{\text{def}}{=} \text{point} * X$ to denote the (abstract) cone over a topological space X .

If X is any topological space, and $Y \subset X$ is any subspace, then we say that X is **obtained from** Y by attaching an i -cell if X is homotopy equivalent to a subspace X' that can be decomposed as the union

$$Y \cup e / \alpha(x) \sim x$$

where e is an i -cell and α is a continuous map $\partial e \rightarrow Y$.

We now recall three classical results in combinatorial topology.

Lemma A.1.1. *Let Δ and $\Gamma \subset \Delta$ denote a pair of simplicial complexes. Then $\Delta \setminus \Gamma$ deformation retracts to $\Gamma - \Delta$. \square*

Lemma A.1.2. *Let Δ, Γ denote two topological spaces that are k -connected and ℓ -connected, respectively. Then $\Delta * \Gamma$ is $(k + \ell + 2)$ -connected.*

Proof. Let us consider the spaces $\Delta' = \Delta * \Gamma \setminus \Gamma$ and $\Gamma' = \Delta * \Gamma \setminus \Delta$. Then $\Delta' \simeq \Delta$, $\Gamma' \simeq \Gamma$ and $\Delta' \cap \Gamma' \simeq \Delta \times \Gamma$. By considering the Mayer–Vietoris sequence for Δ' and Γ' , together with the Künneth formula, we see that $\Delta * \Gamma$ is $(k + \ell)$ -acyclic. The claim follows with the Whitehead and Hurewicz Theorems, cf. [Hat02, Qui78]. \square

Lemma A.1.3. *Let Δ denote a polytopal complex, and let σ be any ℓ -cell of Δ . If $\text{lk}(\sigma, \Delta)$ is k -connected, then Δ is, up to homotopy equivalence, obtained from $\Delta - \sigma$ by successively attaching $\geq (k + \ell + 2)$ -dimensional cells.*

Proof. By a stellar subdivision at σ and Lemma A.1.2, it suffices to address the case $\ell = 0$, i.e., the case when σ is a vertex. We may furthermore assume that $k \geq 0$, since the claim is trivial otherwise. Let K denote a CW complex homotopy equivalent to $\text{lk}(v, \Delta)$ and constructed so that it has no reduced cells of dimension $\leq k$. Let $f : K \rightarrow \text{lk}(v, \Delta)$ denote a continuous mapping realizing the homotopy equivalence $K \simeq \text{lk}(v, \Delta)$, and let

$$M_f = K \times [0, 1] \cup \text{lk}(v, \Delta) / (x, 0) \sim f(x)$$

denote its mapping cylinder. Then Δ is homotopy equivalent to

$$((\Delta - v) \cup M_f) \cup CK / x \in \partial CK \sim (x, 1)$$

Now if c is any reduced cell of ∂CK , then Cc is a disk in CK of dimension $\geq k + 2$ (since c is a cell of dimension $\geq k + 1$). Since all cells are of this form, the claim follows. \square

Topology of posets. Posets \mathcal{P} are interpreted topologically via their **order complex** $\Delta(\mathcal{P})$, whose faces are the totally ordered subsets (chains) of \mathcal{P} . Here $\Delta(\cdot)$ is usually suppressed from the notation. For instance, for a $(d-1)$ -dimensional homotopy Cohen–Macaulay poset \mathcal{P} as above, we have $\mathcal{P} \simeq \bigvee S^{d-1}$. As a general reference for poset topology, see e.g. [Bjö95].

For a poset \mathcal{P} and two comparable elements $a, b \in \mathcal{P}$, we have the **interval** $\mathcal{P}_{[a,b]} \stackrel{\text{def}}{=} \{y \in \mathcal{P} : a \leq y \leq b\}$ (and similarly for open and half-open intervals). As special cases, we have the **lower** (resp. **upper**) **ideal** $\mathcal{P}_{\leq x} = \{y \in \mathcal{P} : y \leq x\}$ (resp. $\mathcal{P}_{\geq x} = \{y \in \mathcal{P} : y \geq x\}$) of an element $x \in \mathcal{P}$.

An order-preserving map $f : P \rightarrow P$ is called a **closure operator** if $x \leq f(x) = f^2(x)$ for all $x \in P$. One can deduce from Lemma A.1.4 that such a map induces homotopy equivalence of P and its image $f(P)$. But more is true: a closure operator is a strong deformation retract. See e.g. [Bjö95, p. 1852].

A concrete example of a closure operator that plays a role in this paper is the closure map of matroid, sending an arbitrary set of points to the smallest closed set containing it. A homotopy inverse is the identity map, sending a closed set to itself.

A well-known consequence of Lemma A.1.2 (see e.g. [Qui78, Bjö95]) is that a poset is Cohen-Macaulay (resp. homotopy C-M) if and only if its intervals of length k are $(k-1)$ -acyclic (resp. $(k-1)$ -connected) for all k .

A central tool to our line of reasoning is Quillen's "Theorem A", which we now give in a version that is slightly more general than those available in the literature, cf. [Qui78, Bjö03, BWW05].

Lemma A.1.4. *Let \mathcal{P} , \mathcal{Q} be two posets, and $\varphi : \mathcal{P} \rightarrow \mathcal{Q}$ an order-preserving map. Assume that for every $x \in \mathcal{Q}$, the fiber $\varphi^{-1}(\mathcal{Q}_{\leq x})$ is m_x -connected and $\mathcal{Q}_{> x}$ is ℓ_x -connected, and let $k \stackrel{\text{def}}{=} \min_{x \in \mathcal{Q}} (m_x + \ell_x) - 2$. Then*

- \mathcal{Q} is, up to homotopy equivalence, obtained from \mathcal{P} by attaching cells of dimension $\geq k+2$,
- φ induces an isomorphism of homotopy groups up to dimension k , and a surjection in dimension $k+1$,
- \mathcal{P} is k -connected if and only if \mathcal{Q} is k -connected.

Proof. Let us consider the poset M_φ whose ground set is the disjoint union of the elements of \mathcal{P} and \mathcal{Q} , and where we define

- for $q, q' \in \mathcal{Q} \subset M_\varphi$, then $q \leq q'$ in M_φ if and only if $q \leq q'$ in \mathcal{Q} ,
- for $p, p' \in \mathcal{P} \subset M_\varphi$, then $p \leq p'$ in M_φ if and only if $p \leq p'$ in \mathcal{P} , and
- for $p \in \mathcal{P} \subset M_\varphi$ and $q \in \mathcal{Q} \subset M_\varphi$, then $p \leq q$ in M_φ if and only if $\varphi(p) \leq q$ in \mathcal{Q} .

The poset M_φ triangulates the mapping cylinder of φ , and is therefore homotopy equivalent to \mathcal{Q} . More precisely, \mathcal{Q} is a strong deformation retract of M_φ , as can also be seen from the fact that the mapping $c : M_\varphi \rightarrow \mathcal{Q}$ defined by

$$c(x) \stackrel{\text{def}}{=} \begin{cases} \varphi(x) & \text{if } x \in \mathcal{P} \\ x & \text{if } x \in \mathcal{Q} \end{cases}$$

is a closure operator. Moreover, if $\tilde{\varphi}$ denotes the inclusion map $\mathcal{P} \hookrightarrow M_\varphi$, then for every $x \in \mathcal{Q} \subset M_\varphi$, we have the isomorphisms

$$\varphi^{-1}(\mathcal{Q}_{\leq x}) \cong \tilde{\varphi}^{-1}(\mathcal{Q}_{\leq x}) \quad \text{and} \quad \mathcal{Q}_{> x} \cong (M_\varphi)_{> x}.$$

The key observation now is that we can obtain \mathcal{P} from M_φ by removing the elements of $\mathcal{Q} \subset M_\varphi$ one by one, until only \mathcal{P} is left. We do so in an increasing fashion, removing the elements from bottom to top.

To make this precise, let \mathcal{I} denote any poset $\mathcal{P} \subsetneq \mathcal{I} \subset M_\varphi$, and let μ denote a minimal element of $\mathcal{I} \setminus \mathcal{P}$, such that $\mathcal{I}_{\geq \mu} = (M_\varphi)_{\geq \mu} = \mathcal{Q}_{\geq \mu}$, i.e., no element greater than μ has yet been deleted from $(M_\varphi)_{\geq \mu}$.

Now, $\text{lk}(\mu, \mathcal{I}) \cong \mathcal{Q}_{> \mu} * \varphi^{-1}(\mathcal{Q}_{\leq \mu})$, is k -connected by assumption and Lemma A.1.2. Hence \mathcal{I} is obtained from $\mathcal{I} - \mu$ by successively attaching cells of dimension $\geq k+2$ by Lemma A.1.3. By extension, $M_\varphi \simeq \mathcal{Q}$ is obtained from \mathcal{P} by successively attaching cells of dimension $\geq k+2$. The first claim follows, and this implies the other two. \square

A.2. Geometry and combinatorics of polyhedral spaces and polyhedral fans.

Polyhedral spaces. A **(closed) polyhedral space** in \mathbb{R}^d is a finite collection of polyhedra in \mathbb{R}^d such that the intersection of any two polyhedra is a face of both, and that is closed under passing to faces of the polyhedra in the collection. The elements of a polyhedral space are called **faces**, and the inclusion-wise maximal faces are the **facets** of the polyhedral space. A polyhedral space is **bounded** if and only if all polyhedra are bounded, i.e. if they are polytopes.

Let A and B , $B \subset A$, denote two polyhedral spaces such that for every face b of B , and every face a of A containing b , there exists a unique face \tilde{a} of A with $b \prec \tilde{a} \leq a$. Then \tilde{a} is a cofacet of b in A , and $O = A \setminus B$ is an **open polyhedral space**. Such spaces are analogous to precompact, open manifolds and precompact, open Whitney stratified spaces; the faces of O are the faces of A , minus the faces of B . Finally, a **polyhedral fan** is a polyhedral space all whose faces are pointed at $\mathbf{0}$.

A.2.1. *Tangent spaces and normal spaces.* **Geometric links** are defined with a differential-geometric approach, compare [GM88]. Let $X \subset \mathbb{R}^d$ be any Whitney-stratified space (for us, it shall suffice to consider polyhedral spaces, polyhedral fans and smooth submanifolds of \mathbb{R}^n), and let $p \in X$ be any point.

Then $T_p X$ is used to denote the tangent space of X at p , and $T_p^1 X$ is the restriction of $T_p X$ to unit vectors. If Y is any subspace of X , then $N_{(p,Y)} X$ denotes the subspace of the tangent space spanned by vectors orthogonal to $T_p Y \subset T_p X$, and we define $N_{(p,Y)}^1 X \stackrel{\text{def}}{=} N_{(p,Y)} X \cap T_p^1 Y$. Related notions are that of **tangent cone** and **normal cone**: We set

$$\text{TC}_p X \stackrel{\text{def}}{=} T_p X + x \quad \text{and} \quad \text{NC}_p X \stackrel{\text{def}}{=} N_p X + x$$

Underlying spaces, restrictions, deletions and refinements. The **underlying space** $|X|$ of a polyhedral space X is the union of its faces. With abuse of notation, we often speak of the polyhedral space when we actually mean its underlying space. For example, we often do not distinguish in notation between a polytope and the complex formed by its faces. In another instance of abuse of notation, if $M \subset \mathbb{R}^d$ is any set, and X is a polyhedral space, then we write $X \subset M$ to denote the fact that $|X|$ lies in M , and set $X \cap M = |X| \subset M$. We define the **restriction** $R(X, M)$ of a polyhedral space X to a set M to be the inclusion-wise maximal subcomplex D of X such that $D \subset M$. Finally, the **deletion** $X - D$ of a subcomplex D from X is the subcomplex of X given by $R(X, X \setminus D^\circ)$. If X and Y are two polyhedral spaces with the same underlying space, then Y is called a **refinement** or **subdivision** of X if every face y of Y is contained in some face x of X . Similarly, for polyhedral spaces X, Y we define the **common refinement** $X \cdot Y$ as the polyhedral space $\{x \cap y : x \in X, y \in Y\}$.

Topology of restrictions. In general, there is little relation between a polyhedral space and its restrictions. However, the following observation for restrictions of polyhedral complexes is useful to keep in mind for applications of stratified Morse theory.

Proposition A.2.1. *Let X denote any polyhedral space in \mathbb{R}^d , and let C denote the complement of an open, convex set K in \mathbb{R}^d . Then $X \cap C = X \setminus K$ deformation retracts onto $R(X, C)$.*

Proof. If A, B are convex sets in \mathbb{R}^d with a point of intersection x then $A \setminus B$ deformation retracts onto $\partial A \setminus B$ via restriction of the radial projection

$$\begin{aligned} A \setminus \{x\} &\longrightarrow \partial A \\ y &\longmapsto (x + \text{pos}(y - x)) \cap \partial A. \end{aligned}$$

We can now argue by induction on the faces of X : We claim that if σ is any facet of X that intersects K , then σ deformation retracts onto $\partial\sigma \setminus K$, and therefore

$$X \cap C = ((X - \sigma) \cap C) \cup (\sigma \cap C)$$

deformation retracts onto $(X - \sigma) \cap C$. With this procedure, we can iteratively remove all faces of X not in C by deformation retractions. The claim follows. \square

Stars and links. Now, let X be any polyhedral space, and let σ be any face of X . The **star** of σ in X , denoted by $\text{st}(\sigma, X)$, is the minimal subcomplex of X that contains all faces of X containing σ . If X is simplicial and v is a vertex of X such that $\text{st}(v, X) = X$, then X is called a **cone** with apex v over the base $X - v$.

Let τ be any face of a polyhedral space or fan X containing a face σ , and assume that σ is nonempty and p is any interior point of σ . Then the set $N_{(p,\sigma)}^1 \tau$ of unit tangent vectors in $N_{(p,\sigma)}^1 X$ pointing towards τ forms a spherical polytope isometrically embedded in $N_{(p,\sigma)}^1 X$. Again, $N_{(p,\sigma)}^1 \tau$ and its embedding into $N_{(p,\sigma)}^1 X$ are uniquely determined up to ambient isometry, so we abbreviate $N_\sigma^1 \tau \stackrel{\text{def}}{=} N_{(p,\sigma)}^1 \tau$ and $N_\sigma^1 X \stackrel{\text{def}}{=} N_{(p,\sigma)}^1 X$, unless p is relevant in another context. The collection of all polytopes in $N_\sigma^1 X$ obtained this way forms a polyhedral space, denoted by $\text{lk}_p(\sigma, X)$, the **link** of σ in X (cf. [DM99, Sec. 2.2]). Unless p is relevant, we omit it in the notation for the link. This is still well-defined: Up to isometry, $\text{lk}_p(\sigma, X)$ does not depend on p . We set $\text{lk}(\emptyset, X) \stackrel{\text{def}}{=} X$.

If X is a polyhedral space in \mathbb{R}^d , then $\text{lk}_p(\sigma, X)$ is naturally embedded in $N_{(p,\sigma)}^1 \mathbb{R}^d$: $\text{lk}_p(\sigma, X)$ is the collection of spherical polytopes $N_{(p,\sigma)}^1 \tau$ in the $(d - \dim \sigma - 1)$ -sphere $N_{(p,\sigma)}^1 \mathbb{R}^d$, where τ ranges over the faces of X containing σ . Up to ambient isometry, this does not depend on the choice of p ; we shall consequently omit it whenever possible.

A.3. Basics notions in tropical geometry. We shall give a brief overview over the essentials of tropical geometry; for more, we refer to [Gat06, Kat09, MS09, Mik06, RGST05, SS09].

Tropical affine and projective space, sedentarity and mobility. Set $\mathbb{T} \stackrel{\text{def}}{=} [-\infty, \infty) = \mathbb{R} \cup \{-\infty\}$, the tropical numbers. \mathbb{T} is a semiring endowed with the **(tropical) addition** $\oplus : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$ and **(tropical) multiplication** $\odot : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$ defined as

$$a \oplus b \stackrel{\text{def}}{=} \max\{a, b\} \quad \text{and} \quad a \odot b \stackrel{\text{def}}{=} a + b.$$

We will also write “ $a + b$ ” for $a \oplus b$ and “ ab ” = “ $a \cdot b$ ” for $a \odot b$.

The **tropical affine space** \mathbb{T}^n of dimension n is the space $[-\infty, \infty)^n$. The **fine sedentarity** $\mathfrak{S} : \mathbb{T}^n \rightarrow 2^{[n]}$ of a point $x \in \mathbb{T}^n$ is the set $\{i \in [n] : x_i = -\infty\}$. The **sedentarity** $\mathfrak{s} : \mathbb{T}^n \rightarrow \mathbb{N}$ is defined as $\mathfrak{s}(x) = \#\mathfrak{S}(x)$. A point resp. set of sedentarity 0 is also called **mobile**. The **mobile part** of a subset A in \mathbb{T}^d is also denoted by $A|_{\mathfrak{m}}$, and we write $A|_{\mathfrak{S}=I}$ to restrict to the subset of fine sedentarity I . In particular, we have a decomposition

$$\mathbb{T}^n = \bigcup_{I \subset [n]} \mathbb{R}^I \times (-\infty)^{[n] \setminus I}.$$

We define **tropical projective n -space** as

$$\mathbb{TP}^n \stackrel{\text{def}}{=} \mathbb{T}^{n+1} \setminus (-\infty)^{n+1} / \mathbf{x} \sim “\lambda \mathbf{x}”.$$

Tropical projective space \mathbb{TP}^d can be obtained as a union of $d + 1$ copies \mathbb{T}_i , $i \in 1, \dots, d + 1$ of tropical affine space \mathbb{T}^d , restricted to nonpositive coordinates: If $i \in [n]$ is any element, then the set $\tilde{S}_i \stackrel{\text{def}}{=} \{x \in \mathbb{T}^{d+1}\}$ such that $x_i = \max_{j \in [d]} x_j \cong \mathbb{T}_{\leq 0}^d \times \mathbb{T}$ projects to the copy \mathbb{T}_i of $\mathbb{T}_{\leq 0}^d$ spanned by x_j , $j \in [d + 1] \setminus \{i\}$. The notions of sedentarity and mobility therefore naturally extend to tropical projective space. Similarly, we shall silently extend notions for affine tropical geometry to projective tropical geometry using this decomposition, whenever this extension is obvious.

Tropical polynomials. If U is an open connected subset of \mathbb{T}^n , then a function $f : U \rightarrow \mathbb{T}$ is **regular** if there is a finite subset $A \in \mathbb{Z}^n$ such that $\alpha_i \geq 0$, $\alpha \in A$ if $i \in \mathfrak{S}(U)$, and numbers $a_\alpha \in \mathbb{T}$, $\alpha \in A$ such that

$$f(x) = \max_{j \in A} (j \cdot x + a_j) = “\sum_{j \in A} a_j \mathbf{x}^{\alpha_j}”$$

That is, a regular function on \mathbb{T}^n is but a “tropical Laurent polynomial”; the condition $\alpha_i \geq 0$, $\alpha \in A$ merely ensures well-definedness. We say that A is the **index-set** of f , and the numbers a_α are the **coefficients** of f . The regular functions give a presheaf \mathcal{O}_{pre} on \mathbb{T}^n , which in turn gives rise to the **structure sheaf** $\mathcal{O} = \mathcal{O}_{\mathbb{T}^n}$ on \mathbb{T}^n . A **tropical polynomial** is a regular function that is well-defined on an open domain U , $\mathbb{R}^n \subset U \subset \mathbb{T}^n$. We call a tropical polynomial **entire** if it is well-defined on \mathbb{T}^n , or equivalently, all α are nonnegative. A tropical polynomial for which the coefficients α have constant 1-norm $|\alpha|$ is **homogenous**.

Tropicalized varieties. Let $\varphi : \mathbb{T}^n \rightarrow \mathbb{T}$ be any tropical polynomial with index-set A . If $\mathbf{x} \in \mathbb{T}^n$ is any tropical vector, we define

$$\text{in}_{\mathbf{x}}(\varphi) = \{\alpha \in A : f(x) = “a_{\alpha} \mathbf{x}^{\alpha}”\}$$

The **tropicalized hypersurface** $V(\varphi) \subset \mathbb{T}^n$ is defined as

$$V(\varphi) \stackrel{\text{def}}{=} \{x \in \mathbb{T}^n : \#\text{in}_{\mathbf{x}}(\varphi) \geq 2\}.$$

More generally, a **tropicalized variety** is associated to an ideal I of tropical polynomials; we then naturally have

$$V(I) \stackrel{\text{def}}{=} \bigcap_{\varphi \in I} V(\varphi).$$

Even though an ideal of tropical polynomials is infinite in general, a tropical variety can *always* be written as the intersection of a *finite* number of tropical hypersurfaces cf. [MS09]. In particular, tropicalized varieties are naturally polyhedral spaces in \mathbb{T}^d , and may be thought of as such. If I is a *homogenous* ideal of tropical polynomials, then $V(I)/\sim$ is naturally a projective tropical variety. A **tropicalized hyperplane** is the tropical variety associated to a tropical polynomial of degree 1. A **tropical hyperplane** in \mathbb{T}^d is the zero-locus of a tropical affine linear function, i.e. a function

$$\begin{aligned} \varphi : \mathbb{T}^d &\longrightarrow \mathbb{T} \\ x &\longmapsto "a_0 + \sum a_i x_i", \quad (a_0, \dots, a_d) \in \mathbb{T}^{d+1} \end{aligned}$$

The **(fine) sedentarity** of H is the (fine) sedentarity of (a_1, \dots, a_d) , and H is **almost totally sedentary** if the sedentarity of H is $d - 1$. A **projective tropical hyperplane** H in \mathbb{TP}^d , is the image $H = \tilde{H}/\mathbf{x} \sim " \lambda \mathbf{x} "$ under projection of a tropical hyperplane \tilde{H} in \mathbb{T}^{d+1} that is invariant under tropical multiplication with a scalar (or equivalently, the variety of a tropical linear function with vanishing constant term).

Polyhedral spaces in \mathbb{T}^d . Recall that tropical space \mathbb{T}^d is stratified into copies $\mathbb{R}^I \times (-\infty)^{[d] \setminus I}$, $I \subset [d]$ of euclidean vector spaces. A **d -polyhedron** P in \mathbb{T}^d is the closure of a polyhedron in \mathbb{R}^I , $I \subset [d]$ such that for every face Q of P in \mathbb{T}^J , $J \subset [d]$, and every $I \subsetneq J$, we have

$$Q \cap \mathbb{T}^I = \emptyset \quad \text{or} \quad \dim Q \cap \mathbb{T}^I = \dim Q - \#J + \#I.$$

A **closed polyhedral space** Σ in \mathbb{T}^d is a collection of polyhedra in \mathbb{T}^d with the property that the intersection of any two polyhedra is a face of both. Open polyhedral spaces in \mathbb{T}^d are as usual defined, as quotients of closed polyhedral spaces.

Tropical varieties and smooth tropical varieties. A **balanced polyhedral space** is a closed polyhedral space \mathbb{T}^n such that for every codimension-1 face τ of Σ of sedentarity 0, we have

$$\sum_{\substack{F \in \Sigma \\ F \text{ cofacet of } \tau}} p_{\tau, F} = 0,$$

where $p_{\tau, F}$ is the primitive integer vector in $N_{\tau} F \subset T_{\tau} F \subset \mathbb{R}^n$.

A **tropical variety**, or **weighted balanced polyhedral space**, is a closed polyhedral space \mathbb{T}^d with a collection of positive integer weights ω_{τ} on the codimension 1-faces of Σ such that for every codimension-1 face τ of Σ of sedentarity 0, we have

$$\sum_{\substack{F \in \Sigma \\ F \text{ cofacet of } \tau}} \omega_{\tau} p_{\tau, F} = 0$$

where $p_{\tau, F}$ is given as above. As the name suggests, a tropicalized variety is always a tropical variety, cf. [MS09].

A **smooth tropical variety** in \mathbb{T}^d of dimension n is a closed polyhedral space Σ in \mathbb{T}^d such that for every face σ of Σ , $N_{\sigma} \Sigma$ is a Bergman fan. A **tropical hypersurface** is a smooth tropical variety of codimension one in \mathbb{T}^d or \mathbb{TP}^d .

A.3.1. *Tropical manifolds.* Tropical manifolds are an abstraction of smooth tropical varieties, introduced by Mikhalkin [Mik06]. An **integral affine map** $\varphi : \mathbb{T}^n \longrightarrow \mathbb{T}^m$ is a map that arises from a well-defined extension of an **integral affine map** $\tilde{\varphi} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which in turn is defined as the composition of an integral linear map and an *arbitrary* translation. A **(smooth) tropical manifold** in \mathbb{R}^d of dimension n is an abstract polyhedral space X with charts $(U_{\alpha}, \Phi_{\alpha})$; $\Phi_{\alpha} : U_{\alpha} \longrightarrow V_{\alpha} \subset \mathbb{T}^{N_{\alpha}}$ such that

- (1) $\forall \alpha$, V_{α} is an open subset of $\mathfrak{F}(M) \times \mathbb{T}^s(Y_{\alpha})$, where M is a loopless matroid with $\varrho(M) - 1 + s = n$, and the map Φ_{α} is a homeomorphism.
- (2) $\forall \alpha, \alpha'$, $\Phi_{\alpha} \circ \Phi_{\alpha'}^{-1} : \Phi_{\alpha'}(U_{\alpha} \cap U'_{\alpha}) \longrightarrow \Phi_{\alpha}(U_{\alpha} \cap U'_{\alpha}) \subset V_{\alpha}$ can be extended to an integral affine map $\varphi : \mathbb{T}^{N_{\alpha'}} \longrightarrow \mathbb{T}^{N_{\alpha}}$.
- (3) the charts are of finite type, i.e. there exists a finite number of open sets (Q_i) such that $\bigcup Q_i = X$, and such that for every Q_i , there is an α such that $Q_i \subset U_{\alpha}$ and $\Phi_{\alpha}(Q_i) \subset V_{\alpha}$.

A more appropriate name for smooth tropical manifolds could therefore be (abstract) smooth tropical variety.

Bounded support. To study the geometry of tropical varieties, or more generally polyhedral spaces in \mathbb{T}^d , it is sometimes useful to restrict to a “bounded frame” instead of studying the unbounded variety. This is in particular helpful when we want to study tropical manifolds using principles of stratified Morse theory.

For a polyhedral space $X \in \mathbb{T}$ we shall sometimes consider the **finite part** (or **bounded part**) of X . Clearly, there is a positive real number $\delta < \infty$ such that every face of X of sedentarity 0 intersects $(-\delta, \delta)^n$. Then we define $X[\delta] \subset \mathbb{T}_\delta^n$. The polyhedral spaces $X[\delta]$ and X are naturally combinatorially equivalent, via the isomorphism

$$\sigma \in X \longmapsto \begin{cases} \sigma \cap [-\delta, \delta]^n & \text{if } \mathfrak{S}(\sigma) = \emptyset \\ \sigma \times [\mathbb{T}^{[n] \setminus \mathfrak{S}(\sigma)}] \cap (-\delta)^{\mathfrak{S}(\sigma)} \times [-\delta, \delta]^{[n] \setminus \mathfrak{S}(\sigma)} & \text{else.} \end{cases}$$

Up to combinatorial equivalence, the finite part $X[\delta]$ therefore does not depend on δ .

A.4. Tropical Hodge theory and (p, q) -homology. Tropical (p, q) -homology was introduced by Itenberg–Katzarkov–Mikhalkin–Zharkov [IKMZ]. Despite its name, tropical (p, q) -theory is should be thought of as an analogue of Hodge theory in complex algebraic geometry. For more details on (p, q) -homology, we refer the reader to [IKMZ, MZ13, Sha11, Zha12].

Tropical link and tangent space. Let X be a polyhedral space in affine tropical space \mathbb{T}^d , and let σ denote a face of X of fine sedentarity S . Then $t\mathbb{T}_\sigma X \stackrel{\text{def}}{=} \mathbb{R}(T_\sigma X, \mathbb{R}^{[n] \setminus S})$ is the **tropical tangent space** of σ in X . Let $\sigma \subset \tau$ be any pair of faces of a polyhedral space X in \mathbb{T}^d . Then there is a natural map

$$d_{\tau \rightarrow \sigma} : t\mathbb{T}_\tau(X \cap \mathbb{R}^S) \longrightarrow t\mathbb{T}_\sigma(X \cap \mathbb{R}^S).$$

If $\mathfrak{S}(\sigma) = \mathfrak{S}(\tau)$, then $d_{\tau \rightarrow \sigma}$ is given by natural inclusion of tangent spaces. If $\mathfrak{S}(\sigma) \neq \mathfrak{S}(\tau)$, then $\mathfrak{S}(\tau) \subset \mathfrak{S}(\sigma)$ and $d_{\tau \rightarrow \sigma}$ is given by restriction of the orthogonal projection

$$\mathbb{R}^{[n] \setminus \mathfrak{S}(\sigma)} \longrightarrow \mathbb{R}^{[n] \setminus \mathfrak{S}(\tau)}.$$

p -groups. The coefficients of tropical Hodge theory are given by the p -groups, which form analogues to the sheaf of differential forms Ω^k in classical Hodge theory.

Definition A.4.1 (p -groups). Let Σ denote any polyhedral fan, i.e. any collection of rational polyhedral cones in \mathbb{R}^d pointed at $\mathbf{0}$. For $p \geq 0$, we associate to Σ the subgroup $\mathcal{F}_p(\Sigma)$ of $\bigwedge^p \mathbb{Z}^d$ induced by elements $v_1 \wedge v_2 \wedge \cdots \wedge v_p$, where v_1, v_2, \dots, v_p are integer vectors that lie in a common subspace $\text{lin } \sigma, \sigma \in \Sigma$, where lin denotes the linear span of a subset of \mathbb{R}^d . The groups $\mathcal{F}_p(\Sigma)$ are also known as the **p -groups** of Σ . Dually, one can define $\mathcal{F}^p(\Sigma) \stackrel{\text{def}}{=} \text{Hom}(\mathcal{F}_p(\Sigma); \mathbb{Z})$, the **co- p -groups**.

Examples A.4.2. We collect some obvious examples for p -groups:

- (i) The 0-th group $\mathcal{F}_0(\Sigma)$ is isomorphic to \mathbb{Z} , regardless of the subspace arrangement.
- (ii) The 1-th group is isomorphic to a sublattice of $\mathbb{Z}^d \cap \text{lin } \Sigma$.
- (iii) The n -th group $\mathcal{F}_n(\Sigma)$, $n > \dim \text{lin } \Sigma$ vanishes regardless of the arrangement.
- (iv) The co- p -groups form, quite naturally, a graded algebra $\mathcal{F}^\bullet(\cdot)$. For every matroid M , we then have a natural isomorphism between $\mathcal{OS}^\bullet(M)$, the projectivized Orlik–Solomon algebra of Σ , and $\mathcal{F}^\bullet(\mathfrak{F}(M))$, the graded algebra of co- p -groups of the Bergman fan of M [Zha12].

Tropical Hodge theory. We give a very intuitive definition of tropical cellular Hodge theory, for a more thorough treatment we refer the reader to [IKMZ, MZ13]. Tropical Hodge groups can, alternatively, be defined using generalized singular or simplicial homology theories, but we will stick to a construction based on cellular homology with non-constant coefficients.

Let X denote any tropical variety (realized in tropical affine or projective space, or abstract), regarded as a polyhedral space. If σ is a face of X , and p is a nonnegative integer, then we set $\mathcal{F}_p(\sigma) \stackrel{\text{def}}{=} \mathcal{F}_p(T_\sigma X)$. With this, we have the **tropical (p, q) -chains**

$$C_q(X; \mathcal{F}_p) \stackrel{\text{def}}{=} \bigoplus_{\sigma \text{ } q\text{-face of } X} \mathcal{F}_p(\sigma) = \bigoplus_{\sigma \text{ } q\text{-face of } X} \tilde{H}_q(\sigma, \partial\sigma; \mathcal{F}_p(\sigma)).$$

There is a natural boundary map $\partial : C_q(X; \mathcal{F}_p) \rightarrow C_{q-1}(X; \mathcal{F}_p)$ that arises as the composition of the classical cellular boundary map $\tilde{\partial}$, composed with the map $d_{\sigma \rightarrow \tau}^*$ of p -groups induced by the map

$$d_{\sigma \rightarrow \tau} : \mathfrak{t}T_\sigma(X \cap \mathbb{R}^S) \rightarrow \mathfrak{t}T_\tau(X \cap \mathbb{R}^S),$$

where τ is any facet of σ .

This gives us a chain complex $C_\bullet(\mathcal{F}_p)$; the associated homology groups $H_{(p,q)}(X)$ are the **(p, q) -homology groups** [IKMZ, MZ13, Sha11, Zha12].

Some facts in tropical Hodge theory. (p, q) -groups, or tropical Hodge groups are natural analogues of the classical Hodge group in algebraic geometry: in [IKMZ] it is proven that if X is a smooth projective tropical variety obtained as the limit ([Ber71, GKZ94, Mik04] of a 1-parameter family (X_t) of smooth complex projective varieties, then the Hodge numbers of a generic fiber X_t can be computed from the Hodge numbers of X .

Not everything is analogous to the classical situation though: the (p, q) -homologies do not seem to satisfy the intuitive analogue of the Hodge Index Theorem [Sha11, Thm. 3.3.5].

Also, while it is challenging to recover integral homology from the classical Hodge groups, it is easy to do so with the tropical Hodge groups, as $\mathcal{F}_0(\Sigma) \cong \mathbb{Z}$. Let us close this section by mentioning some useful results to keep in mind.

Theorem A.4.3 (cf. [MZ13, IKMZ]). *Let X denote a smooth tropical variety.*

- *The tropical Hodge groups are independent of the cell-structure of the tropical variety chosen, cf. [MZ13, Prp. 2.2].*
- *Let sk_j denote the j -skeleton of a polyhedral space. The inclusion $\text{sk}_j X \hookrightarrow X$ induces a map*

$$H_q(\text{sk}_j X; \mathcal{F}_p X) \rightarrow H_q(X; \mathcal{F}_p X),$$

that is surjective for all $j \geq q$, and an isomorphism if $j > q$.

Being a homology theory with non-constant coefficients, relative Hodge groups have to be handled with care; however, since the ring of coefficients is cell-wise constant, we can in some situations still argue essentially as above. Another instance for such an argument is the following:

Lemma A.4.4. *Let X denote any smooth tropical variety, let Y denote any subcomplex and let v be any vertex of Y . Then we have a natural quasi-isomorphism of chain complexes*

$$C_{q-1}(\partial \text{lk}(v, Y); \mathcal{F}_p X_v) \rightarrow (st(v, Y), \partial st(v, Y); \mathcal{F}_p X).$$

Here, $\partial \text{lk}(v, Y)$ is seen as a subcomplex of the smooth tropical manifold $TT_v^1 X$, so that the local ring of coefficients at a cell $\sigma \in \partial \text{lk}(v, Y)$ is given by $\mathcal{F}_p T_x \sigma * v$, where $\sigma * v$ is the minimal face of $\text{lk}(v, Y)$ containing v and σ . We denote this ring of coefficients by X_v .

Proof. The isomorphism of chain complexes is given by the join operation \mathfrak{j} , which sends a cell $\sigma \in \text{lk}(v, Y)$ to the cell $v * \sigma$. It is easy to check that the induced map

$$\mathfrak{j} : C_{q-1}(\partial \text{lk}(v, Y); \mathcal{F}_p X_v) \rightarrow C_q(st(v, Y), \partial st(v, Y); \mathcal{F}_p X)$$

induces the desired quasi-isomorphism. □

Let X , once again, be a smooth tropical variety. We call a chain $c(C_q(X; \mathcal{F}_p X))$ **stable** if for each cell $\sigma \in X$, the restriction $c|_\sigma$ is of the form

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p,$$

where

$$v_i \in T_\sigma \hat{\sigma} \subset \text{lin } T_\sigma X, \text{ for } 1 \leq i \leq p,$$

and where $\hat{\sigma} \subset X$ is a polyhedron with $T\hat{\sigma} = T\sigma$. The cell $\hat{\sigma}$ can also be characterized as a maximal cell of X that contains σ in the interior, and will also be called the **stabilizing cell**.

Lemma A.4.5. *Every chain is homologous to a stable chain up to passing to a refinement of the variety.*

For the proof, notice that if $\bigwedge v_i$ lies minimally in $\bigwedge^p \text{lin } T_\sigma \tau$, where $\tau \supsetneq \sigma$, but $\bigwedge v_i \notin \bigwedge^p T_\sigma \tau$, then it may be pushed into the cell τ (up to refining X) in direction $N_\sigma \tau$ and along some vector field parallel to τ .

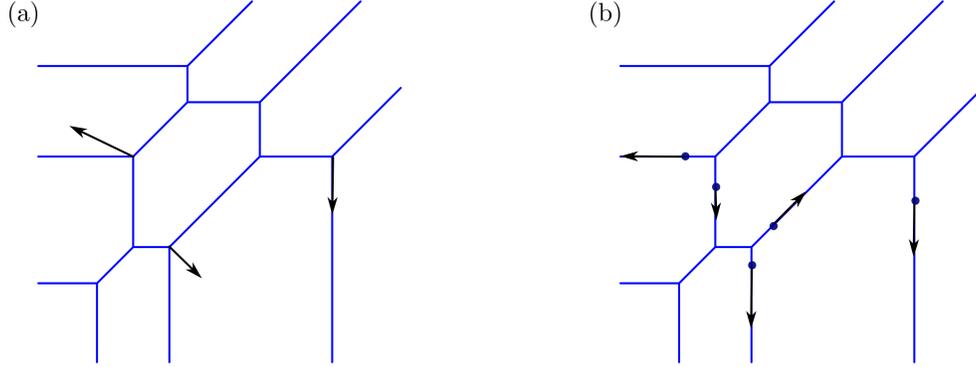


Figure 1.4. (a) An “unstable” tropical $(1,0)$ -chain c in a tropical variety X . (b) To achieve stabilization in the cellular category, the variety might have to be refined.

In Figure 1.4 we exemplify the pushing process by pushing a $(1,0)$ -chain c into the 1-skeleton of a variety X , obtaining a new stable chain \tilde{c} homologous to c . Let now σ be any vertex in the support of \tilde{c} and let $\hat{\sigma}$ denote its stabilization. Then we can define $k|\hat{\sigma} \stackrel{\text{def}}{=} \tilde{c}|_\sigma \cdot e_{\hat{\sigma}}$, where $e_{\hat{\sigma}}$ is a primitive vector in $T_\sigma \hat{\sigma} = T_{\hat{\sigma}} \hat{\sigma}$ that agrees with the orientation of $\hat{\sigma}$.

By linear extension over all faces in the support of σ , we obtain a map γ that takes stable $(1,0)$ -chains in X to $(0,1)$ -chains in X . Notice that γ takes cycles to cycles, and boundaries to boundaries by construction, so that it induces an isomorphism

$$H_0(X, \mathcal{F}_1) \xrightarrow{\gamma^*} H_1(X, \mathcal{F}_0).$$

By extending this reasoning, one can conclude:

Corollary A.4.6 (Conjugation symmetry of the tropical Hodge diamond). *There is a natural isomorphism*

$$H_q(X, \mathcal{F}_p) \cong H_p(X, \mathcal{F}_q).$$

A.5. Stratified Morse theory. We here recall the basic principles and notions of stratified Morse Theory as far as necessary. In particular, we will not introduce Whitney stratified spaces but work with the simpler notion of polyhedral spaces:

Strata and linearized tangent spaces. If σ is any face of a polyhedral space, then we call its relative interior σ° a **stratum**. If x is a point in $\partial\sigma$, then the **generalized**, or **linearized tangent space** of σ at x is defined as

$$\text{lin } T_x \sigma = \lim_{\substack{y \in \sigma^\circ \\ y \rightarrow x}} T_y \sigma,$$

compare also [GM88, Sec. I.1.8].

Morse functions on polyhedral and stratified spaces. Let now $\tilde{f} : S \rightarrow \mathbb{R}$ denote any smooth function whose domain $S \subset \mathbb{R}^d$ is open, and let X denote any (closed) polyhedral space in S . A **critical point** of the restriction $f = \tilde{f}|_X$ of f to X is a critical point in any one of the strata of X , i.e. a point x in the relative interior σ° of some face of σ for which $df_{T_x X}(x) = df_{T_x \sigma^\circ}(x) = 0$. **Critical values** are the values of critical points under f .

We call f a **Morse function** on X if

- $f = \tilde{f}|_X$ is proper, and the critical points of f are finite and distinct.
- All critical points are nondegenerate, i.e. for every face $\sigma \in X$, and every critical point $x \in \sigma^\circ$, the Hessian of $f|_S$ at x is non-singular.
- For every such critical point x of f , and for every generalized tangent space $Q \subset T_x \mathbb{R}^d$ at x , we have $df_{T_x \sigma^\circ}(x) \neq 0$ unless $Q = T_x \sigma^\circ$.

For open polyhedral spaces, we additionally have to require that the gradient field is uniformly oriented at the boundary of the space. I.e., if $O = A \setminus B$ is an open polyhedral space, then the restriction $f = \tilde{f}|_O$ of f to O is a **Morse function** on O if $\tilde{f}|_A$ is a Morse function and for every face $a \in A$ intersecting a , every point $x \in a^\circ \cap B$ and $\nu = \mathbb{T}_{(x, a^\circ \cap B)} a$, we have

$$\langle \nu, \nabla f(x)|_{\text{lin } \mathbb{T}_x a} \rangle > 0.$$

The Main Theorem of stratified Morse Theory. With this, we can state the main theorem of stratified Morse theory, specialized to polyhedral spaces.

Theorem A.5.1 (Goresky–MacPherson [GM88, Sec. I]). *Let X denote a polyhedral space, and let $f = \tilde{f}|_X : X \rightarrow \mathbb{R}$ denote a Morse function on X as above. Then*

- (a) *If $(s, t] \subset \mathbb{R}$ is an interval containing no critical values of f , then $X_{\leq s} = f^{-1}(-\infty, s]$ is a deformation retract of $X_{\leq t}$.*
- (b) *If t is any critical value of f , x the associated critical point and $s < t$ is chosen so that $(s, t]$ contains no further critical values of f . Then, the Morse data at x (and therefore the change in topology from $X_{\leq s}$ to $X_{\leq t}$) is given by the product of tangential and normal Morse data of f at x .*

We may similarly consider the change in topology of the complement of X in S along \tilde{f} .

- (c) *If u is chosen so that the interval $[t, u)$ contains no critical values of \tilde{f} and f , then $(S \setminus X)_{\leq t} = (\tilde{f}^{-1}(-\infty, t]) \setminus X$ is a deformation retract of $(S \setminus X)_{\leq u}$.*
- (d) *If, on the other hand, u is chosen so that the interval $[t, u)$ contains no critical values of \tilde{f} and f apart from t , then the Morse data at an associated critical point x is given as a product of normal Morse data and tangential Morse data at x .*

Convex superlevel sets. In our particular situation, we can easily work out the normal and tangential Morse data.

Lemma A.5.2. *With the notation as in Theorem A.5.1, let us assume that for every critical value t of f , $\tilde{f}^{-1}[t, \infty)$ is closed and convex. Let t be any critical value of f , x the associated critical point and let $s < t$ be chosen so that $(s, t]$ contains no further critical values of f . Then we have the following refinement:*

- (a) *The tangential Morse data at x is given by $(\sigma, \partial\sigma)$.*
- (b) *The normal Morse data at x is given by*

$$(\text{CN}_\sigma^1 X \cap f^{-1}(-\infty, t], \text{N}_\sigma^1 X \cap f^{-1}(-\infty, t]).$$

- (c) *In particular, we have the homotopy equivalence*

$$(X_{\leq t}, X_{\leq s}) \simeq (\text{CN}_\sigma^1 X \cap f^{-1}(-\infty, t], \text{N}_\sigma^1 X \cap f^{-1}(-\infty, t]) * (\sigma, \partial\sigma)$$

Proof. Since $\tilde{f}^{-1}[t, \infty)$ is closed, smooth and convex for every critical value t , the Morse function $f|_{\sigma^\circ}$ takes a minimum at $x \in \sigma^\circ$. Claim (a) follows. Claim (b) holds regardless of the requirement on superlevel sets, cf. [GM88, P. I, Sec. 3.9]. Finally, the homotopy equivalences of claim (c) follow from Theorem A.5.1(b). \square

REFERENCES

- [Ada69] J.F. Adams, *Lectures on generalised cohomology*, Category Theory Homology Theory Appl., Proc. Conf. Seattle Res. Center, Battelle Mem. Inst., 1969, pp. 1–138. [15](#)
- [Ada74] ———, *Stable Homotopy and Generalised Homology*, Chicago Lectures in Mathematics, The University of Chicago Press, 1974. [15](#)
- [Adi12] K. A. Adiprasito, *Combinatorial stratifications and minimality of 2-arrangements*, preprint at [arXiv:1211.1224](#). [3](#), [8](#)
- [AN54] Y. Akizuki and S. Nakano, *Note on Kodaira–Spencer’s proof of Lefschetz theorems*, Proc. Japan Academy, Series A, Mathematical Sciences **30** (1954), no. 4, 266–272. [2](#), [3](#), [14](#)
- [AF59] A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. (2) **69** (1959), 713–717. [1](#), [2](#), [3](#), [10](#)
- [AK06] F. Ardila and C. J. Klivans, *The Bergman complex of a matroid and phylogenetic trees*, J. Combin. Theory Ser. B **96** (2006), 38–49. [9](#)
- [AMS12] C. Athorne, D. Maclagan, and I. Strachan (eds.), *Tropical geometry and integrable systems*, Contemporary Mathematics, vol. 580, Providence, RI, American Mathematical Society, 2012. [1](#)

- [Ber71] G. M. Bergman, *The logarithmic limit-set of an algebraic variety*, Trans. Amer. Math. Soc. **157** (1971), 459–469. [1](#), [9](#), [24](#)
- [BG84] R. Bieri and J. R. J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195. [1](#)
- [Bjö80] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), 159–183. [3](#), [5](#), [8](#), [9](#)
- [Bjö92] ———, *The homology and shellability of matroids and geometric lattices*, Matroid applications, Encyclopedia Math. Appl., vol. 40, Cambridge Univ. Press, Cambridge, 1992, pp. 226–283. MR 1165544 (94a:52030) [4](#)
- [Bjö95] ———, *Topological methods*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872. [17](#), [18](#), [19](#)
- [Bjö03] ———, *Nerves, fibers and homotopy groups*, Journal of Combinatorial Theory, Series A **102** (2003), no. 1, 88 – 93. [19](#)
- [Bjö13] ———, *Positive sum systems*, Extended abstract, to appear in Proc. 2013 INDAM conference "Combinatorial methods in topology and algebra". [5](#)
- [BWW05] A. Björner, M. L. Wachs, and V. Welker, *Poset fiber theorems*, Trans. Amer. Math. Soc. **357** (2005), 1877–1899. [19](#)
- [BWW09] A. Björner, M. L. Wachs, and V. Welker, *On sequentially Cohen-Macaulay complexes and posets.*, Isr. J. Math. **169** (2009), 295–316 (English). [8](#)
- [BZ92] A. Björner and G. M. Ziegler, *Combinatorial stratification of complex arrangements*, J. Amer. Math. Soc. **5** (1992), 105–149. [7](#)
- [Bot59] R. Bott, *On a theorem of Lefschetz*, Michigan Math. J. **6** (1959), 211–216. [2](#), [10](#)
- [Car13] D. Cartwright, *Tropical complexes*, arXiv preprint [arXiv:1308.3813](#) (2013). [17](#)
- [CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva, *A tropical proof of the Brill-Noether theorem*, Adv. Math. **230** (2012), 759–776. [1](#)
- [DM99] M. W. Davis and G. Moussong, *Notes on nonpositively curved polyhedra*, in “Low-dimensional topology” (Eger 1996/Budapest 1998), Bolyai Soc. Math. Stud., vol. 8, János Bolyai Math. Soc., Budapest, 1999, pp. 11–94. [20](#)
- [Del80] P. Deligne, *La conjecture de Weil. II.*, Publ. Math., Inst. Hautes Étud. Sci. **52** (1980), 137–252 (French). [17](#)
- [DP03] A. Dimca and S. Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*, Ann. of Math. (2) **158** (2003), 473–507. [13](#)
- [Fol66] J. Folkman, *The homology groups of a lattice*, J. Math. Mech. **15** (1966), 631–636. [3](#)
- [For98] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), 90–145. [8](#)
- [Gat06] A. Gathmann, *Tropical algebraic geometry*, Jahresber. Deutsch. Math.-Verein. **108** (2006), 3–32. [1](#), [21](#)
- [GKZ94] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994. [1](#), [24](#)
- [GM88] M. Goresky and R. D. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. [2](#), [3](#), [20](#), [25](#), [26](#)
- [Gro11] M. Gross, *Tropical Geometry and Mirror Symmetry*, CBMS Regional Conference Series in Mathematics, vol. 114, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2011. [1](#)
- [Gro95] Max-Plus Working Group, *Max-plus algebra and applications to system theory and optimal control*, Proc. ICM, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 1511–1522. [1](#)
- [Gub07] W. Gubler, *The Bogomolov conjecture for totally degenerate abelian varieties*, Invent. Math. **169** (2007), 377–400. [1](#)
- [HL71] H. Hamm and D. T. Lê, *Un théorème du type de Lefschetz*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A946–A949. [2](#), [13](#)
- [Ham83] H. A. Hamm, *Lefschetz theorems for singular varieties*, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 547–557. [2](#), [13](#)
- [Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002, available at [math.cornell.edu/~hatcher/AT/ATpage.html](#). [10](#), [17](#), [18](#)
- [IKMZ] I. Itenberg, G. Katzarkov, G. Mikhalkin, and I Zharkov, *Tropical homology*, in preparation. [3](#), [23](#), [24](#)
- [Kat09] E. Katz, *A tropical toolkit*, Expo. Math. **27** (2009), no. 1, 1–36. MR 2503041 (2010f:14069) [1](#), [21](#)
- [Kat12] E. Katz, *Tropical intersection theory from toric varieties.*, Collect. Math. **63** (2012), no. 1, 29–44 (English). [1](#)
- [KS53] K. Kodaira and D. C. Spencer, *On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 1273–1278. [2](#), [3](#), [14](#)
- [KS01] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 203–263. [1](#)
- [KT02] M. Kontsevich and Y. Tschinkel, *Non-archimedean Kähler geometry*, IHES Preprints (2002), [ihes.fr/maxim/TEXTS/Non-archimedean_Kahler_geometry.pdf](#). [1](#)
- [Laz04] R. Lazarsfeld, *Positivity in Algebraic Geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. [2](#), [3](#), [17](#)
- [Lê87] D. T. Lê, *Le concept de singularité isolée de fonction analytique*, Complex analytic singularities, Adv. Stud. Pure Math., vol. 8, North-Holland, Amsterdam, 1987, pp. 215–227. [2](#), [13](#)
- [Lef50] S. Lefschetz, *L'analyse situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1950. [1](#), [2](#), [10](#)
- [MS09] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, 2009, Book in preparation. [21](#), [22](#)
- [Mik04] G. Mikhalkin, *Amoebas of algebraic varieties and tropical geometry*, Different faces of geometry **3** (2004), 257–300. [24](#)
- [Mik05] ———, *Enumerative tropical algebraic geometry in \mathbb{R}^2* , J. Amer. Math. Soc. **18** (2005), no. 2, 313–377. [1](#)

- [Mik06] ———, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 827–852. [1](#), [21](#), [22](#)
- [MZ13] G. Mikhalkin and I. Zharkov, *Tropical eigenwave and intermediate jacobians*, arXiv preprint [arXiv:1302.0252](#) (2013). [23](#), [24](#)
- [MZ08] G. Mikhalkin and G.M. Ziegler, *The positive part of a geometric lattice*, personal communication, 2008. [3](#), [4](#), [8](#)
- [Mil63] J. Milnor, *Morse Theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. [2](#), [3](#), [10](#)
- [Mun84] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, CA, 1984. MR 755006 (85m:55001) [17](#)
- [NGVR12] V. Noel, D. Grigoriev, S. Vakulenko, and O. Radulescu, *Tropical geometries and dynamics of biochemical networks application to hybrid cell cycle models*, Proc. 2nd International Workshop on Static Analysis and Systems Biology (SASB 2011), Electron. Notes Theor. Comput. Sci., vol. 284, Elsevier Sci. B. V., Amsterdam, 2012, pp. 75–91. [1](#)
- [Oxl11] J. Oxley, *Matroid Theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. [4](#)
- [Pin98] J.-E. Pin, *Tropical semirings*, Idempotency (Bristol, 1994), Publ. Newton Inst., vol. 11, Cambridge Univ. Press, Cambridge, 1998, pp. 50–69. [1](#)
- [Qui78] D. Quillen, *Homotopy properties of the poset of nontrivial p -subgroups of a group*, Advances in Mathematics **28** (1978), 101–128. [18](#), [19](#)
- [Ran02] R. Randell, *Morse theory, Milnor fibers and minimality of hyperplane arrangements*, Proc. Amer. Math. Soc. **130** (2002), 2737–2743. [13](#)
- [RGST05] J. Richter-Gebert, B. Sturmfels, and T. Theobald, *First steps in tropical geometry*, Idempotent mathematics and mathematical physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 289–317. [1](#), [21](#)
- [Rot64] G.-C. Rota, *On the foundations of combinatorial theory. I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368 (1964). [3](#), [4](#)
- [Sal87] M. Salvetti, *Topology of the complement of real hyperplanes in \mathbb{C}^n* , Invent. Math. **88** (1987), 603–618. [7](#)
- [Sha11] K. M. Shaw, *Tropical intersection theory and surfaces*, 2011, Thesis (Ph.D.)–Université de Genève. [3](#), [15](#), [23](#), [24](#)
- [Shu05] E. Shustin, *A tropical approach to enumerative geometry*, Algebra i Analiz **17** (2005), no. 2, 170–214. [1](#)
- [Spe05] D. E. Speyer, *Tropical Geometry*, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–University of California, Berkeley. [1](#)
- [SS09] D. E. Speyer and B. Sturmfels, *Tropical mathematics*, Math. Mag. **82** (2009), 163–173. [1](#), [21](#)
- [Sta82] R. P. Stanley, *Linear Diophantine equations and local cohomology.*, Invent. Math. **68** (1982), 175–193 (English). [7](#)
- [Sta96] ———, *Combinatorics and Commutative Algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996. [8](#), [18](#)
- [Stu02] B. Sturmfels, *Solving Systems of Polynomial Equations*, CBMS Regional Conference Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2002. [9](#)
- [Vir84] O. Ya. Viro, *Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7*, Topology (Leningrad, 1982), Lecture Notes in Math., vol. 1060, Springer, Berlin, 1984, pp. 187–200. [1](#)
- [Voi02] C. Voisin, *Théorie de Hodge et Géométrie Algébrique Complexe*, Cours Spécialisés [Specialized Courses], vol. 10, Société Mathématique de France, Paris, 2002. [2](#), [3](#), [14](#), [17](#)
- [WW86] M. L. Wachs and J. W. Walker, *On geometric semilattices*, Order **2** (1986), no. 4, 367–385. [3](#)
- [Whi78] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978. [8](#), [10](#), [17](#)
- [Zha12] I. Zharkov, *The Orlik-Solomon algebra and the Bergman fan of a matroid*, arXiv preprint [arXiv:1209.1651](#). [23](#), [24](#)

INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, BURES-SUR-YVETTE, FRANCE

E-mail address: adiprasito@math.fu-berlin.de, adiprasito@ihes.fr

DEPARTMENT OF MATHEMATICS, KUNGLIGA TEKNISKA HÖGSKOLAN, STOCKHOLM, SWEDEN

E-mail address: bjorner@kth.se