SF1544

Övning 1

Who am I

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- Övningar (English) och datorlabbar (Svenska)

Structure of the övning

- Beamer presentation
- Matlab demo
- Blackboard (when is needed)

Today

- Fixed point and fixed point iteration method
- Roots of a function / Rot (eller lösning) till ekvation
- Newton method
- Sensitivity analysis / Tillförlitlighetsbedömning

Fixed point method

Definition

Let $g(x)$ a function, $\bar{x}$ is a fixed point if $g(\bar{x}) = \bar{x}$

Examples

- $g(x) = x^2$ has fixed points $\bar{x} = 0$ and $\bar{x} = 1$
- $g(x) = x^2 + x - 2$ has the fixed points $\bar{x} = \sqrt{2}$ and $\bar{x} = -\sqrt{2}$
- $g(x) = x$, all the real numbers $\bar{x}$ are fixed points

Fixed point method

Fixed point method

Let x_0 an approximation of the fixed point $\bar{x}$, let us define

$$x_{n+1} = g(x_n)$$

if $|g'(\bar{x})| < 1$ then

$$x_n \to \bar{x}$$

More precisely (liner convergence)

$$|x_{n+1} - \bar{x}| \approx |g'(\bar{x})||x_n - \bar{x}|$$

- If $|g'(\bar{x})| \geq 1$ the fixed point method does not converges
- The smaller $|g'(\bar{x})|$ the faster is the convergence of the fixed point method

Exercise from the book

Which of the following fixed point iterations converges to $\sqrt{5}$? Which is faster?

$$x_{n+1} = \frac{4}{5}x_n + \frac{1}{x_n}, \qquad x_{n+1} = \frac{1}{2}x_n + \frac{5}{2x_n}, \qquad x_{n+1} = \frac{x_n + 5}{x_n + 1}$$

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$$g(x) = \frac{4}{5}x + \frac{1}{x}, \qquad g(x) = \frac{1}{2}x + \frac{5}{2x}, \qquad g(x) = \frac{x + 5}{x + 1}$$

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$$g(\sqrt{5}) = \sqrt{5}, \qquad ?????? \qquad g(\sqrt{5}) = \sqrt{5}$$

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$$g(\sqrt{5}) = \sqrt{5}, \qquad ?????? \qquad g(\sqrt{5}) = \sqrt{5}$$

$$g'(x) = \frac{4}{5} - \frac{1}{x^2}, \qquad ?????? \qquad g'(x) = -\frac{4}{(x+1)^2}$$

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$$|g'(\sqrt{5})| = 0.8, \qquad ?????? \qquad |g'(x)| = 0.38197$$

MATLAB DEMO

Roots of a function / Rot (eller lösning) till ekvation

Definition

Let $f(x)$ a function, $\bar{x}$ is a root if

$$f(\bar{x}) = 0$$

Roots of a function / Rot (eller lösning) till ekvation

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Examples

- $f(x) = x$ has root $\bar{x} = 0$
- $f(x) = x^2 - 2$ has roots $\bar{x} = \sqrt{2}$ and $\bar{x} = -\sqrt{2}$
- $f(x) = \sin(x)$ has roots $\bar{x} = 0$, $\bar{x} = \pi$, $\bar{x} = 2\pi$, $\bar{x} = 3\pi$, ...

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- $f(x) = \sin(x)$ has roots $\bar{x} = 0$, $\bar{x} = \pi$, $\bar{x} = 2\pi$, $\bar{x} = 3\pi$, ...
- We can compute the roots of f using fixed point method

$$f(\bar{x}) = 0 \iff \underbrace{f(\bar{x}) + \bar{x}}_{:=g(x)} = \bar{x} \iff g(\bar{x}) = \bar{x}$$

Roots of a function / Rot (eller lösning) till ekvation

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- We can compute the roots of f using fixed point method

$$f(\bar{x}) = 0 \iff \underbrace{f(\bar{x}) + \bar{x}}_{:=g(x)} = \bar{x} \iff g(\bar{x}) = \bar{x}$$

- A fixed point $g(\bar{x}) = \bar{x}$ is a root of a function

$$g(\bar{x}) = \bar{x} \iff \underbrace{g(\bar{x}) - \bar{x}}_{=:f(\bar{x})} = 0 \iff f(\bar{x}) = 0$$

Newton method

Newton method

Let x_0 an approximation of $\bar{x}$, let us define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

then $x_n \to \bar{x}$. More precisely (quadratic convergence)

$$|x_{n+1} - \bar{x}| \approx \frac{|f''(\bar{x})|}{2|f'(\bar{x})|} |x_n - \bar{x}|^2$$

Stopping criteria:

$$e_{n+1} := |x_{n+1} - x_n| = \frac{|f(x_n)|}{|f'(x_n)|}$$

Exercise

Solve the equation with the Newton method

$$\sin(x) = -6x^2 - 10$$

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Solution:

- $\sin(x) = 6x^2 - 10 \iff \sin(x) - 6x^2 + 10 = 0$

Exercise

Solve the equation with the Newton method

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Solution:

- $\sin(x) = 6x^2 - 10 \iff \sin(x) - 6x^2 + 10 = 0$
- $f(x) = \sin(x) - 6x^2 + 10$

Exercise

Solve the equation with the Newton method

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Solution:

- $\sin(x) = 6x^2 - 10 \iff \sin(x) - 6x^2 + 10 = 0$
- $f(x) = \sin(x) - 6x^2 + 10$
- $f'(x) = \cos(x) - 12x$

Exercise

Solve the equation with the Newton method

$$\sin(x) = -6x^2 - 10$$

Solution:

- $\sin(x) = 6x^2 - 10 \iff \sin(x) - 6x^2 + 10 = 0$
- $f(x) = \sin(x) - 6x^2 + 10$
- $f'(x) = \cos(x) - 12x$
- Replace in Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin(x_n) - 6x_n^2 + 10}{\cos(x_n) - 12x_n}$$

Exercise

Solve the equation with the Newton method

$$\sin(x) = -6x^2 - 10$$

Solution:

- $\sin(x) = 6x^2 - 10 \iff \sin(x) - 6x^2 + 10 = 0$
- $f(x) = \sin(x) - 6x^2 + 10$
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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin(x_n) - 6x_n^2 + 10}{\cos(x_n) - 12x_n}$$

- Conclusion

$$x_{n+1} = x_n - \frac{\sin(x_n) - 6x_n^2 + 10}{\cos(x_n) - 12x_n}$$

MATLAB DEMO

Sensitivity analysis/Tillförlitlighetsbedömning by examples

Question: What's important to calculate measures of uncertainty?

Sensitivity analysis/Tillförlitlighetsbedömning by examples

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- Example: area of a circle

$$A = \pi r^2$$

Sensitivity analysis/Tillförlitlighetsbedömning by examples

Question: What's important to calculate measures of uncertainty?

- Example: area of a circle

$$A = \pi r^2$$

Assume we can measure the radius r with an error Δr

$$\begin{aligned} A + \Delta A &= \pi(r + \Delta r)^2 \\ &= \pi(r^2 + 2r\Delta r + \Delta r^2) \\ &= \pi(r^2 + 2r\Delta r + \cancel{\Delta r^2}) \end{aligned}$$

Sensitivity analysis/Tillförlitlighetsbedömning by examples

Question: What's important to calculate measures of uncertainty?

- Example: area of a circle

$$A = \pi r^2$$

Assume we can measure the radius r with an error Δr

$$\begin{aligned} A + \Delta A &= \pi (r + \Delta r)^2 \\ &= \pi (r^2 + 2r\Delta r + \Delta r^2) \\ &= \pi (r^2 + 2r\Delta r + \cancel{\Delta r^2}) \end{aligned}$$

Conclusion

$$\Delta A = 2\pi r \Delta r \qquad \text{absolute error}$$

$$\frac{\Delta A}{A} = \frac{2\pi r \Delta r}{\pi r^2} = 2\frac{\Delta r}{r} \qquad \text{relative error}$$

Sensitivity analysis/Tillförlitlighetsbedömning by examples

Practical example

$$\Delta A = 2\pi r \Delta r \qquad \text{absolute error}$$

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Sensitivity analysis/Tillförlitlighetsbedömning by examples

Practical example

$$\Delta A = 2\pi r \Delta r \qquad \text{absolute error}$$

$$\frac{\Delta A}{A} = 2\frac{\Delta r}{r} \qquad \text{relative error}$$

- If $r = 2$ and $\Delta r = 0.01$ then $\Delta A = 4\pi 0.01 \approx 0.12566$

Sensitivity analysis/Tillförlitlighetsbedömning by examples

Practical example

$$\Delta A = 2\pi r \Delta r \qquad \text{absolute error}$$

$$\frac{\Delta A}{A} = 2\frac{\Delta r}{r} \qquad \text{relative error}$$

- If $r = 2$ and $\Delta r = 0.01$ then $\Delta A = 4\pi 0.01 \approx 0.12566$
- We know that the relative error in measure r is 2%, i.e.,

$$\frac{\Delta r}{r} = 0.02$$

Then the relative error in measuring the Area is 4%, i.e.,

$$\frac{\Delta A}{A} = 2\frac{\Delta r}{r} = 0.04$$

Sensitivity analysis: Evaluation of a function

Let $f(x)$ a function

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$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \mathcal{O}(\Delta x^2)$$

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Sensitivity analysis: Evaluation of a function

Let $f(x)$ a function

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$$f(x) + \Delta f(x) = f(x) + \Delta x f'(x) + \cancel{\mathcal{O}(\Delta x^2)}$$

Then

$$\Delta f(x) = \Delta x f'(x) \qquad \text{absolute error}$$

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta x f'(x)}{f(x)} \qquad \text{relative error}$$

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

.

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

. The absolute error in the function evaluation is

$$\Delta f(x) = \Delta x f'(x) = \frac{\Delta x}{2\sqrt{x}}$$

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

. The absolute error in the function evaluation is

$$\Delta f(x) = \Delta x f'(x) = \frac{\Delta x}{2\sqrt{x}}$$

If, for example, If $x = 2$ and $\Delta x = 0.01$ then Δf, then

$$\Delta f(x) = \frac{0.01}{2\sqrt{2}} \approx 0.0035$$

Sensitivity analysis: Evaluation of a function, practical example

Let

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OBSERVE: The error in the output $\Delta f(x)$ is smaller then the error in the input Δx.

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

.

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

. The relative error in the function evaluation is

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta x f'(x)}{f(x)} = \frac{\Delta x}{2\sqrt{x}\sqrt{x}} = \frac{1}{2}\frac{\Delta x}{x}$$

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

. The relative error in the function evaluation is

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta x f'(x)}{f(x)} = \frac{\Delta x}{2\sqrt{x}\sqrt{x}} = \frac{1}{2}\frac{\Delta x}{x}$$

If,for example, the relative error is x is 6%, i.e.,

$$\frac{\Delta x}{x} = 0.06$$

Sensitivity analysis: Evaluation of a function, practical example

Let

$$f(x) = \sqrt{x}$$

. The relative error in the function evaluation is

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta x f'(x)}{f(x)} = \frac{\Delta x}{2\sqrt{x}\sqrt{x}} = \frac{1}{2}\frac{\Delta x}{x}$$

If,for example, the relative error is x is 6%, i.e.,

$$\frac{\Delta x}{x} = 0.06$$

then the relative error in $f(x)$ is 3%, i.e.,

$$\frac{\Delta f(x)}{f(x)} = 0.03$$

Example from an old exam

2. (2p) Felgränsen för $z = 3x^2y^3$ där $x = 1.00 \pm 0.02$ och $y = 1.00 \pm 0.03$ ges approximativt av

☐ 0.01	☐ 0.06
☐ 0.02	☐ 0.09
☐ 0.03	☐ 0.22
☐ 0.05	☒ 0.4

Example from an old exam

$$z = 3x^2y^3$$

Example from an old exam

$$z = 3x^2y^3$$

$$\begin{aligned} z + \Delta z &= 3(x + \Delta x)^2(y + \Delta y)^3 \\ &= 3(x^2 + 2x\Delta x)(y^3 + 3y^2\Delta y) \\ &= 3(x^2y^3 + 2xy^3\Delta x + 3x^2y^2\Delta y) \\ &= \underbrace{3x^2y^3}_{=z} + 3(2xy^3\Delta x + 3x^2y^2\Delta y) \end{aligned}$$

Example from an old exam

$$z = 3x^2y^3$$

$$\begin{aligned} z + \Delta z &= 3(x + \Delta x)^2(y + \Delta y)^3 \\ &= 3(x^2 + 2x\Delta x)(y^3 + 3y^2\Delta y) \\ &= 3(x^2y^3 + 2xy^3\Delta x + 3x^2y^2\Delta y) \\ &= \underbrace{3x^2y^3}_{=z} + 3(2xy^3\Delta x + 3x^2y^2\Delta y) \end{aligned}$$

Then

$$\Delta z = 3(2xy^3\Delta x + 3x^2y^2\Delta y) = 0.39$$

Example from an old exam

Another way (If you don't have time at the exam)

$$z = 3x^2y^3$$

Example from an old exam

Another way (If you don't have time at the exam)

$$z = 3x^2y^3$$

$$
\begin{aligned}
z &= 3 \\
z_{\text{max}} &= 3(1+0.02)^2(1+0.03)^3 \approx 3.4106 \\
z_{\text{min}} &= 3(1-0.02)^2(1-0.03)^3 \approx 2.6296
\end{aligned}
$$

Example from an old exam

Another way (If you don't have time at the exam)

$$z = 3x^2y^3$$

$$\begin{aligned} z &= 3 \\ z_{\max} &= 3(1+0.02)^2(1+0.03)^3 \approx 3.4106 \\ z_{\min} &= 3(1-0.02)^2(1-0.03)^3 \approx 2.6296 \end{aligned}$$

Then we notice

$$\tilde{z} := \frac{z_{\min} + z_{\max}}{2} = 3.0201$$

Example from an old exam

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Then we notice

$$\tilde{z} := \frac{z_{\min} + z_{\max}}{2} = 3.0201$$

and

$$|z_{\max} - \tilde{z}| = |z_{\min} - \tilde{z}| = 0.3905$$

Example from an old exam

Another way (If you don't have time at the exam)

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$$\begin{aligned} z &= 3 \\ z_{\text{max}} &= 3(1+0.02)^2(1+0.03)^3 \approx 3.4106 \\ z_{\text{min}} &= 3(1-0.02)^2(1-0.03)^3 \approx 2.6296 \end{aligned}$$

Then we notice

$$\tilde{z} := \frac{z_{\text{min}} + z_{\text{max}}}{2} = 3.0201$$

and

$$|z_{\text{max}} - \tilde{z}| = |z_{\text{min}} - \tilde{z}| = 0.3905$$

$$\Delta z = 0.3905$$

Example from an old exam

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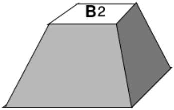
☐ 0.01	☐ 0.06
☐ 0.02	☐ 0.09
☐ 0.03	☐ 0.22
☐ 0.05	☒ 0.4

Example from the book

8.3 Råttor har gnagt på de gamla pyramiderna, så att de numera är rejält stympade. Volymen V hos en sådan stympad pyramid ges av formeln



$$V = \frac{h}{3}\left(B_1 + \sqrt{B_1 B_2} + B_2\right)$$

där h är höjden, B_1 är bottenytan och B_2 den parallella övre ytan. Efter att råttorna jagats bort har följande värden uppmätts: $h = 6 \pm 0.3$, $B_1 = 8 \pm 0.2$ och $B_2 = 3 \pm 0.1$ (angivna i pe – pyramidabla enheten). Bestäm volymen med felgränser.

Example from the book

The short way (If you don't have time)

$$h = 6 \pm 0.3, \qquad B_1 = 8 \pm 0.2, \qquad B_2 = 3 \pm 0.1$$

then

$$V = \frac{h}{3}(B_1 + B_2 + \sqrt{B1B2})$$

then

$$V_{\max} = \frac{6+0.3}{3}((8+0.2) + (3+0.1) + \sqrt{(8+0.2)(3+0.1)}) \approx 34.32$$
$$V_{\min} = \frac{6-0.3}{3}((8-0.2) + (3-0.1) + \sqrt{(8-0.2)(3-0.1)}) \approx 29.37$$

Example from the book

The short way (If you don't have time)

$$h = 6 \pm 0.3, \qquad B_1 = 8 \pm 0.2, \qquad B_2 = 3 \pm 0.1$$

then

$$V = \frac{h}{3}(B_1 + B_2 + \sqrt{B1B2})$$

then

$$V_{\max} = \frac{6+0.3}{3}((8+0.2) + (3+0.1) + \sqrt{(8+0.2)(3+0.1)}) \approx 34.32$$

$$V_{\min} = \frac{6-0.3}{3}((8-0.2) + (3-0.1) + \sqrt{(8-0.2)(3-0.1)}) \approx 29.37$$

and

$$\tilde{V} \approx \frac{V_{\max} + V_{\min}}{2} = 31.85$$

Example from the book

The short way (If you don't have time)

$$h = 6 \pm 0.3, \qquad B_1 = 8 \pm 0.2, \qquad B_2 = 3 \pm 0.1$$

then

$$V = \frac{h}{3}(B_1 + B_2 + \sqrt{B1B2})$$

then

$$V_{\max} = \frac{6+0.3}{3}((8+0.2)+(3+0.1)+\sqrt{(8+0.2)(3+0.1)}) \approx 34.32$$
$$V_{\min} = \frac{6-0.3}{3}((8-0.2)+(3-0.1)+\sqrt{(8-0.2)(3-0.1)}) \approx 29.37$$

and

$$\tilde{V} \approx \frac{V_{\max} + V_{\min}}{2} = 31.85$$

$$\Delta V = |\tilde{V} - V_{\max}| = |\tilde{V} - V_{\min}| = 2.48$$

Example from the book

$$V = \frac{h}{3}(B_1 + B_2 + \sqrt{B1B2})$$

The long way

Example from the book

$$V = \frac{h}{3}(B_1 + B_2 + \sqrt{B1B2})$$

The long way

$$\begin{aligned}
V + \Delta V &= \frac{h + \Delta h}{3}(B_1 + \Delta B1 + B_2 + \Delta B_2 + \sqrt{(B1 + \Delta B1)(B2 + \Delta B2)}) \\
&= \frac{h + \Delta h}{3}(B_1 + \Delta B1 + B_2 + \Delta B_2 + \sqrt{B1B2 + B2\Delta B1 + B1\Delta B2}) \\
&= \frac{h + \Delta h}{3}\left(B_1 + \Delta B1 + B_2 + \Delta B_2 + \frac{B2\Delta B1 + B1\Delta B2}{2\sqrt{B1B2}}\right) \\
&= \frac{h + \Delta h}{3}\left(B_1 + B_2 + \Delta B_1 + \Delta B_2 + \frac{B2\Delta B1 + B1\Delta B2}{2\sqrt{B1B2}}\right) \\
&= \frac{h}{3}\left(B_1 + B_2 + \Delta B_1 + \Delta B_2 + \frac{B2\Delta B1 + B1\Delta B2}{2\sqrt{B1B2}}\right) + \frac{\Delta h}{3}(B_1 + B_2)
\end{aligned}$$