SF1544

Övning 2

Who am I

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- Övningar (English) och datorlabbar (Svenska)

Structure of the övning

- Beamer presentation
- Matlab demo
- Blackboard (when is needed)

Previous övning

- Fixed point and fixed point iteration method
- Roots of a function / Rot (eller lösning) till ekvation
- Newton method
- Sensitivity analysis / Tillförlitlighetsbedömning

This övning

- Ordinär Differentialekvationer (Ordinary differential equation) [ODE]
- Explicit Euler method
- Trapezoid method (Trapetsmetoden)
- Implicit Euler method

ODE

Let $f(t, y)$ be a function, let t_0 and y_0 be numbers (initial time and initial condition), then we are looking for a function $y(t)$ such that

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

The function $y(t)$ is called solution of the differential equation.

Example:

- Let $f(t, y) = 3t^2$, $t_0 = 0$ and $y_0 = 0$, then

$$\begin{cases} y'(t) = 3t^2 \\ y(0) = 0 \end{cases}$$

has solution

$$y(t) = t^3$$

Ordinär Differentialekvationer

ODE

Let $f(t, y)$ be a function, let t_0 and y_0 be numbers (initial time and initial condition), then we are looking for a function $y(t)$ such that

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

The function $y(t)$ is called solution of the differential equation.

Example:

- Let $f(t, y) = -y$, $t_0 = 0$ and $y_0 = 1$, then

$$\begin{cases} y'(t) = -y(t) \\ y(0) = 1 \end{cases}$$

has solution $y(t) = e^{-t}$

Ordinär Differentialekvationer

ODE

Let $f(t, y)$ be a function, let t_0 and y_0 be numbers (initial time and initial condition), then we are looking for a function $y(t)$ such that

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

The function $y(t)$ is called solution of the differential equation.

Example:

- Let $f(t, y) = \lambda y$, $t_0 = 0$ and $y_0 = 1$, then

$$\begin{cases} y'(t) = \lambda y(t) \\ y(0) = 1 \end{cases}$$

has solution $y(t) = e^{\lambda t}$

ODE

Let $f(t, y)$ be a function, let t_0 and y_0 be numbers (initial time and initial condition), then we are looking for a function $y(t)$ such that

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

The function $y(t)$ is called solution of the differential equation.

Example:

- Let $f(t, y) = y^2$, $t_0 = 0$ and $y_0 = 1$, then

$$\begin{cases} y'(t) = y(t)^2 \\ y(0) = 1 \end{cases}$$

has solution

$$y(t) = \frac{1}{1 - t}$$

Ordinär Differentialekvationer

How to solve Ordinary differential equation:

- separation of variables
- direct integration
- ...
- numerical methods (numerical approximation of the solution)

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For $t_0 \leq t \leq T$ (T=final time)

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

Then let (time discretization)

$$t_0 < t_1 < t_2 < \cdots < t_N = T$$

We want to approximate $y(t_i) \approx y_i$

Explicit Euler method

Let the time discretization

$$t_0 < t_1 < t_2 < \cdots < t_N = T$$

such that $t_{i+1} = t_i + h$ (uniform time discretization). Then

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$$t_0 < t_1 < t_2 < \cdots < t_N = T$$

such that $t_{i+1} = t_i + h$ (uniform time discretization). Then

$$y'(t_i) = f(t_i, y(t_i)) \qquad\qquad i = 1, \ldots, N$$

$$\lim_{s \to 0} \frac{y(t_i + s) - y(t_i)}{s} = f(t_i, y(t_i)) \qquad\qquad i = 1, \ldots, N$$

Explicit Euler method

Let the time discretization

$$t_0 < t_1 < t_2 < \cdots < t_N = T$$

such that $t_{i+1} = t_i + h$ (uniform time discretization). Then

$$y'(t_i) = f(t_i, y(t_i)) \qquad\qquad i = 1, \ldots, N$$

$$\lim_{s \to 0} \frac{y(t_i + s) - y(t_i)}{s} = f(t_i, y(t_i)) \qquad\qquad i = 1, \ldots, N$$

If h is small enough

$$\frac{y(t_i + h) - y(t_i)}{h} \approx f(t_i, y(t_i)) \qquad\qquad i = 1, \ldots, N$$

then

$$y(t_i + h) \approx y(t_i) + hf(t_i, y(t_i)) \qquad\qquad i = 1, \ldots, N$$

Explicit Euler method

$$y(t_i + h) \approx y(t_i) + hf(t_i, y(t_i)) \qquad i = 1, \ldots, N$$

Then we have the explicit Euler method

$$\begin{aligned} y_0 &= \text{given initial condition} \\ y_{i+1} &= y_i + hf(t_i, y_i) \qquad i = 1, \ldots, N \end{aligned}$$

MATLAB DEMO

Exercise from the book

7.3 En fallskärmshoppare påverkas av en uppåtriktad kraft som är proportionell mot v^α där v är hastigheten (m/s) och α är en parameter ≥ 1. Hastigheten som funktion av tiden t lyder differentialekvationen $dv/dt = g\,(1 - (\frac{v}{v_\infty})^\alpha)$, där $g = 9.81$ och $v_\infty = 5$ är den konstanta sluthastigheten som uppnås. Då fallskärmen vecklas ut (vid $t = 0$) har hopparen hastigheten 50 m/s.
Låt $\alpha = 1.1$. Skriv ett MATLAB-program som med Eulers metod och tidssteget 0.05 beräknar och ritar upp hastighetskurvan för $0 \leq t \leq 1$. Vad har hastigheten sjunkit till vid $t = 1$?

Gör om beräkningarna två gånger med halverat tidssteg. Bedöm tillförlitligheten i det erhållna hastighetsvärdet vid $t = 1$.

Beräkna och rita hastighetskurvan även för α-värdena 1.3, 1.5 och 1.7. Notera i samtliga fall hastighetsvärdet vid $t = 1$.

Exercise from the book

$$\begin{cases} v'(t) = 9.81\left(1 - \frac{v(t)}{5}\right)^{\alpha} \\ v(0) = 50 \end{cases} \qquad 0 \leq t \leq 1$$

Parameters:

- $\alpha = 1.1$, $\alpha = 1.3$, $\alpha = 1.5$ and $\alpha = 1.7$
- $h = 0.05$, $h = 0.025$ and $h = 0.0125$
- $f(v) = 9.81\left(1 - \dfrac{v}{5}\right)^{\alpha}$

MATLAB DEMO

Exercise from the book

7.13 Enligt Newtons gravitationslag påverkar solen en planet med en kraft som är riktad mot solen och omvänt proportionell mot kvadraten på avståndet. När man delar upp kraften längs koordinataxlarna i ett fixt x-y system med solen i origo får man därför (om man valt lämpliga enheter)

$$d^2x/dt^2 = -\cos\phi/r^2, \quad d^2y/dt^2 = -\sin\phi/r^2$$

där ϕ är vinkeln mellan positiva x-axeln och ortsvektorn och r är avståndet från origo till planeten.

a) Skriv om differentialekvationerna för de beroende variablerna x och y till ett system av första ordningens differentialekvationer. Det gäller alltså bland annat att skriva högerleden som funktioner av x och y.

Derive the differential equation

It holds

$$r = (x^2 + y^2)^{1/2}$$
$$x = r \cos \phi$$
$$y = r \sin \phi$$

Derive the differential equation

It holds

$$r = (x^2 + y^2)^{1/2}$$
$$x = r\cos\phi$$
$$y = r\sin\phi$$

Then

$$x'' = -\frac{\cos\phi}{r^2} = -\frac{r\cos\phi}{r^3} = -\frac{x}{(x^2+y^2)^{3/2}}$$
$$y'' = -\frac{\sin\phi}{r^2} = -\frac{r\sin\phi}{r^3} = -\frac{y}{(x^2+y^2)^{3/2}}$$

Write as first order differential equation

$$x'' = -\frac{x}{(x^2 + y^2)^{3/2}}, \qquad y'' = -\frac{y}{(x^2 + y^2)^{3/2}}$$

$x(0), x'(0), y(0), y'(0)$ are the initial values

Write as first order differential equation

$$x'' = -\frac{x}{(x^2 + y^2)^{3/2}}, \qquad y'' = -\frac{y}{(x^2 + y^2)^{3/2}}$$

$x(0), x'(0), y(0), y'(0)$ are the initial values

$$u_1 := x, \qquad u_2 := x', \qquad u_3 = y, \qquad u_4 = y'$$

Write as first order differential equation

$$x'' = -\frac{x}{(x^2+y^2)^{3/2}}, \qquad y'' = -\frac{y}{(x^2+y^2)^{3/2}}$$

$x(0), x'(0), y(0), y'(0)$ are the initial values

$$u_1 := x, \qquad u_2 := x', \qquad u_3 = y, \qquad u_4 = y'$$

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{u_1}{(u_1^2+u_3^2)^{3/2}} \\ u_3' &= u_4 \\ u_4' &= -\frac{u_3}{(u_1^2+u_3^2)^{3/2}} \end{aligned}$$

Write as first order differential equation

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ -\frac{u_1}{u_1^2+u_3^2} \\ u_4 \\ -\frac{u_3}{u_1^2+u_3^2} \end{pmatrix} := f\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

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$$u := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Write as first order differential equation

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$$u := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$\begin{cases} \frac{d}{dt}u = f(u) \\ u(0) = u_0 \end{cases}$$

MATLAB DEMO

Trapetsmetoden

Let the ODE

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} \qquad [t_0, T]$$

Let the time discretization

$$t_0 < t_1 < t_2 < \cdots < t_N = T$$

such that $t_{i+1} = t_i + h$ (uniform time discretization). Then, consider the ODE in the subintervals

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_i) = y_i \end{cases} \qquad [t_i, t_{i+1}]$$

integrate the equation

$$\begin{cases} \int_{t_i}^{t_{i+1}} y'(t) = \int_{t_i}^{t_{i+1}} f(t, y(t)) \\ y(t_i) = y_i \end{cases} \qquad [t_i, t_{i+1}]$$

Trapetsmetoden

$$\begin{cases} \int_{t_i}^{t_{i+1}} y'(t) = \int_{t_i}^{t_{i+1}} f(t, y(t)) \\ y(t_i) = y_i \end{cases} \qquad [t_i, t_{i+1}]$$

Trapetsmetoden

$$\begin{cases} \int_{t_i}^{t_{i+1}} y'(t) = \int_{t_i}^{t_{i+1}} f(t, y(t)) \\ y(t_i) = y_i \end{cases} \qquad [t_i, t_{i+1}]$$

$$y(t_{i+1}) - y(t_i) = h\frac{f(t, y(t_i)) + f(t_{i+1}, y(t_{i+1}))}{2} + O(h^3)$$

Trapetsmetoden

$$\begin{cases} \int_{t_i}^{t_{i+1}} y'(t) = \int_{t_i}^{t_{i+1}} f(t, y(t)) \\ y(t_i) = y_i \end{cases} \qquad [t_i, t_{i+1}]$$

$$y(t_{i+1}) - y(t_i) = h\frac{f(t, y(t_i)) + f(t_{i+1}, y(t_{i+1}))}{2} + O(h^3)$$

Now we use: $t_{i+1} = t_i + h$ (uniform grid) and $y(t_{i+1})) = y(t_i) + hf(t_i, y_i)$ (explicit Euler).

$$y(t_{i+1}) - y(t_i) = h\frac{f(t, y(t_i)) + f(t_i + h, y(t_i) + hf(t_i, y(t_i)))}{2} + O(h^3)$$

Trapetsmetoden

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$$y(t_{i+1}) = y(t_i) + h\frac{f(t, y(t_i)) + f(t_i + h, y(t_i) + hf(t_i, y(t_i)))}{2} + O(h^3)$$

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$$y_{i+1} = y_i + h\frac{f(t, y_i) + f(t_i + h, y_i + hf(t_i, y_i))}{2}$$

Fel av Trapetsmetoden

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t, y_i) + f(t_i + h, y_i + hf(t_i, y_i))\right]$$

- Local error (error in each step) $O(h^3)$
- Global error (accumulated error) $O(h^2)$ [kvadratiskt fel]
  Why:
  - Numer of steps times error in each step $N \cdot O(h^3)$
  - $N = (T - t_0)/h$
  - Global error: $O(h^2)$

MATLAB DEMO

Implicit Euler method

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

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The implicit Euler method is given by

$$y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})$$

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- In each iteration solve a nonlinear equation (use Newton or fixed point iteration)
- Solve stiff problems (f' big in norm)

Implicit Euler method

$$\begin{cases} y'(t) = -20y(t) + sin(y(t)) \\ y(t_0) = y_0 \end{cases}$$

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The implicit Euler method is given by

$$y_{i+1} = y_i + h[-20y_{i+1} + sin(y_{i+1})]$$

At each step:

$$y = y_i + h[-20y + sin(y)]$$

Implicit Euler method

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The implicit Euler method is given by

$$y_{i+1} = y_i + h[-20y_{i+1} + sin(y_{i+1})]$$

At each step:

$$y = y_i + h[-20y + sin(y)]$$

- (Fixed point approach) Compute the fixed point of

$$g(y) = y_i + h[-20y + sin(y)]$$

- (Newton approach) Compute the zero of

$$f(y) = y - y_i - h[-20y + sin(y)]$$

MATLAB DEMO