# SF1544

# Övning 4

# This övning

- Newton method with several variables
- Linear systems and condition number (linjärt ekvationssystem och konditionstal)
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$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \text{initial guess} \qquad \begin{pmatrix} x_{j+1} \\ y_{j+1} \end{pmatrix} = \begin{pmatrix} x_{j+1} \\ y_{j+1} \end{pmatrix} - J \begin{pmatrix} x_j \\ y_j \end{pmatrix}^{-1} F \begin{pmatrix} x_j \\ y_j \end{pmatrix}$$

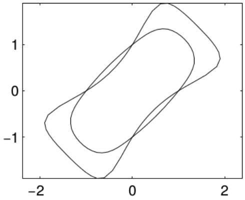
where

$$J(x, y) = \begin{pmatrix} \partial_x F_1(x, y) & \partial_y F_1(x, y) \\ \partial_x F_2(x, y) & \partial_y F_2(x, y) \end{pmatrix}$$

# A complete example

Man vill för olika värden på parametern $a$ studera den slutna kurvan som definieras av ekvationen $x^2 + y^2 = 1 + a \sin xy$.

Då $a = 0$ utgörs kurvan av enhetscirkeln. Figuren visar fallen $a = 1.6$ (ovalen) och $a = 3.2$ (blöjkurvan). Beräkna maxpunktens koordinater för de fyra fallen $a = 0.8,\ 1.6,\ 2.4,\ 3.2$.
Beskriv också lämpligt tillvägagångssätt för att räkna fram och rita upp de slutna kurvorna i de fyra fallen.



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Special case: $\theta = 0 \Rightarrow r = 1$

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The Newton method becomes

$$r_0 = \text{initial guess}$$
$$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_i)}$$

# Matlab implementation

```
close all
clear all
clc

n=100; theta=linspace(0,2*pi,n);

a=3.2;  R(1)=1;
for j=2:n
    h=1;   r=R(j-1);
    while abs(h)>1e-10
        f=r^2-1-a*sin(0.5*r^2*sin(2*theta(j)));
        fp=2*r-a*r*sin(2*theta(j))*cos(0.5*r^2*sin(2*theta(j)));
        h=-f/fp;    r=r+h;
    end
    R(j)=r;
end
x=R.*cos(theta); y=R.*sin(theta); plot(x,y)
axis equal
```

# MATLAB DEMO

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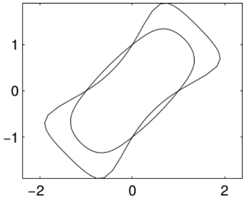

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Maximize with respect $x \rightarrow$ derive an set equal to zero

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In conclusion we have

$$x^2 + y^2 - 1 + a\sin(xy) = 0 \qquad \text{the point is on the curve}$$

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\begin{aligned}
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\end{aligned}
$$

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So we have the system of nonlinear equations

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\begin{cases}
x^2 + y^2 - 1 + a\sin(xy) & = 0 \\
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\end{cases}
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$$x^2 + y^2 - 1 + a\sin(xy) = 0 \qquad \text{the point is on the curve}$$
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So we have the system of nonlinear equations

$$\begin{cases} x^2 + y^2 - 1 + a\sin(xy) & = 0 \\ 2x - ay\cos(xy) & = 0 \end{cases}$$

The solutions are the roots of the function

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 1 + a\sin(xy) \\ 2x - ay\cos(xy) \end{pmatrix}$$

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The solutions are the roots of the function

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 1 + a\sin(xy) \\ 2x - ay\cos(xy) \end{pmatrix}$$

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + ay\cos(xy) & 2y + ax\cos(xy) \\ 2 + ay^2\sin(xy) & a[xy\sin(xy) - \cos(xy)] \end{pmatrix}$$

# Matlab implementation

```
plot_curve
grid on

% x=z(1) and y=z(2)

F=@(z) [z(1)^2+z(2)^2-1-a*sin(z(1)*z(2))
        2*z(1)-a*z(2)*cos(z(1)*z(2))];

J=@(z) [2*z(1)-a*z(2)*cos(z(1)*z(2)) 2*z(2)-a*z(1)*cos(z(1)*z(2))
        2+a*z(2)^2*sin(z(1)*z(2))    a*(z(1)*z(2)*sin(z(1)*z(2))-cos(z(1)*z(2)))]

z=[0.5; 1.5];            h=1;
while norm(h)>1e-15
    h=J(z)\F(z);    z=z-h;
end
hold on;    plot(z(1),z(2),'o');    axis([-3 3 -3 3]);
```

# MATLAB DEMO

# Linear system

Let $A$ a matrix and $b$ a vector, we look for $\bar{x}$ such that

$$A\bar{x} = b$$

Let $x$ an approximation of $\bar{x}$

- Relative forward error

$$\frac{\|x - \bar{x}\|}{\|\bar{x}\|}$$

- Relative backward error

$$\frac{\|Ax - b\|}{\|b\|}$$

- Condition number $\kappa(A) = \|A\|\|A^{-1}\|$
- Relation between relative forward error, relative backward error and condition number

$$\frac{\|x - \bar{x}\|}{\|\bar{x}\|} \leq \kappa(A)\frac{\|Ax - b\|}{\|b\|}$$

# Problem

**3.4** Ekvationssystemet $\mathbf{Ax} = \mathbf{b}$ ska lösas där $\mathbf{A}$ är en tridiagonal matris:

$$\mathbf{A} = \begin{pmatrix} -4 & 3 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 9.6 \\ -3.7 \\ 1.1 \\ -1.2 \\ 2.0 \\ -1.9 \end{pmatrix}$$

Lös systemet för hand eller med hjälp av MATLAB. Vi vill avgöra hur tillförlitlig den erhållna lösningsvektorn $\mathbf{x}$ är, då vi vet att högerledets komponenter är korrekt avrundade till en decimal. Undersökningen görs experimentellt med några olika störningar i $\mathbf{b}$-vektorn:

$$\mathbf{b}_1 = \mathbf{b} + \begin{pmatrix} -0.05 \\ 0.05 \\ -0.05 \\ 0.05 \\ -0.05 \\ 0.05 \end{pmatrix}, \quad \mathbf{b}_2 = \mathbf{b} + \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \\ -0.05 \\ -0.05 \\ -0.05 \end{pmatrix}, \quad \mathbf{b}_3 = \mathbf{b} + \begin{pmatrix} 0.05 \\ 0.05 \\ -0.05 \\ -0.05 \\ 0.05 \\ 0.05 \end{pmatrix}$$

Vilket av störningsexperimenten ger största förändring i lösningsvektorn? Beräkna det experimentellt erhållna konditionstalet för det värsta fallet. Beräkna även det teoretiska konditionstalet.

# Matlab implementation

```
% solve the linear system
xx=A\bb;

% condition number
cond(A,'inf')

% compute an approximation of the condition number
cond_est=1;
for j=1:10000

    b=bb+0.05*(-1 + 2*rand(6,1));
    x=A\b;

    FW=norm(x-xx,'inf')/norm(xx,'inf');
    BW=norm(A*x-bb,'inf')/norm(bb,'inf');
    t=FW/BW;
    cond_est=max(cond_est,t);
end
cond_est
```

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