# SF1544

# Övning 5

# This övning

- BVP (bounday value problem)
- Linear BVP
- Nonlinear BVP
- Shooting method

# Linear BVP

## Problem

$$\begin{cases} y''(t) + 2ty'(t) + y(t) = 2 + 5t^2 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

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Numerical approximations

# Numerical differentiation

Recall

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# Numerical differentiation

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$$y''(t) = \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} + O(h^2)$$

$$y'(t) = \frac{y(t+h) - y(t-h)}{2h} + O(h^2)$$

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Let

$$0 = t_1 < t_2 < \cdots < t_N = 1$$

# Linear BVP

## Problem

For $i = 1, \ldots, n$

$$\begin{cases} y''(t_i) + 2ty'(t_i) + y(t) = 2 + 5t_i^2 \\ y(t_1) = 0 \\ y(t_N) = 1 \end{cases}$$

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# Linear BVP

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For $i = 1, \ldots, n$

$$\begin{cases} \dfrac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} + 2t_i \dfrac{y(t_{i+1}) + y(t_{i-1})}{2h} + y(t_i) = 2 + 5t_i^2 \\ y(t_1) = 0 \\ y(t_N) = 1 \end{cases}$$

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LINEAR SYSTEM OF EQUATION!

# Linear BVP

$$D_2 := \frac{1}{h^2} \begin{pmatrix} 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \end{pmatrix}$$

$$G := \frac{1}{2h} \begin{pmatrix} 0 & 2t_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 2t_{N-1} & 0 \end{pmatrix}$$

$$D_1 := \frac{1}{2h} \begin{pmatrix} -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \end{pmatrix} \qquad y := \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

# Linear BVP

$$\begin{pmatrix} 1 & & \\ & D_2 + GD_1 + I & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} 0 \\ 2 + 5t_2^2 \\ \vdots \\ 2 + 5t_{N-1}^2 \\ 1 \end{pmatrix}$$

# Matlab implementation

```
n=20;                t=linspace(0,1,n);  h=t(2)-t(1);
f=@(t) 2+5*t.^2;     g=@(t) 2*t;

e  = ones(n,1);
DD = spdiags([e -2*e e], -1:1, n, n)/(h^2);
G  = diag(g(t));
D  = spdiags([-e 0*e e], -1:1, n, n)/(2*h);
I  = speye(n);

M = DD+G*D+I;
M(1,:)=0;         M(1,1)=1;
M(n,:)=0;         M(n,n)=1;
b=f(t');          b(1)=0; b(n)=1;
u=M\b;

plot(t,u);hold on; plot(t,t.^2,'--r')
```

MATLAB DEMO

# Nonlinear BVP

**Problem**

$$\begin{cases} y''(t) - y(t) + y(t)^2 = 0 \\ y(0) = 1 \\ y(1) = 4 \end{cases}$$

# Nonlinear BVP

## Problem

For $i = 1, \ldots, n$

$$\begin{cases} \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - y_i + y_i^2 = 0 & i = 2, \ldots, N-1 \\ y_1 = 1 \\ y_N = 4 \end{cases}$$

# Nonlinear BVP

## Problem

For $i = 1, \ldots, n$

$$\begin{cases} y_{i+1} - 2y_i + y_{i-1} - h^2 y_i + h^2 y_i^2 = 0 & i = 2, \ldots, N-1 \\ y_1 = 1 \\ y_N = 4 \end{cases}$$

# Nonlinear BVP

Compute the roots of

$$F\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} y_1 - 1 \\ y_3 - 2y_2 + y_1 - h^2 y_2 + h^2 y_2^2 \\ \vdots \\ y_{i+1} - 2y_i + y_{i-1} - h^2 y_i + h^2 y_i^2 \\ \vdots \\ y_N - 2y_{N-1} + y_{N-2} - h^2 y_{N-1} + h^2 y_{N-1}^2 \\ y_N - 4 \end{pmatrix}$$

# Newton method in several variables

We want to solve

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$$x_{k+1} = x_k - J(x_k)^{-1} F(x_k)$$

$$J(x) = \left( \frac{\partial F_i}{\partial x_j} \right)_{i,j=1}^{n}$$

# Nonlinear BVP

Compute the roots of

$$F\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} y_1 - 1 \\ y_3 - 2y_2 + y_1 - h^2 y_2 + h^2 y_2^2 \\ \vdots \\ y_{i+1} - 2y_i + y_{i-1} - h^2 y_i + h^2 y_i^2 \\ \vdots \\ y_N - 2y_{N-1} + y_{N-2} - h^2 y_{N-1} + h^2 y_{N-1}^2 \\ y_N - 4 \end{pmatrix}$$

# Nonlinear BVP

$$J(y) = \begin{pmatrix} 1 & & & & & & \\ 1 & 2h^2 y_2 - (2+h^2) & & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & 2h^2 y_i - (2+h^2) & 1 & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & 2h^2 y_{N-1} - (2+h^2) & 1 \\ & & & & & & 1 \end{pmatrix}$$

$$y_{k+1} = y_k - J(y_k)^{-1} F(y_k)$$

# Matlab implementation

```
close all
clear all
clc

n=100;
t=linspace(0,1,n);
h=t(2)-t(1);

y=zeros(n,1);
for jj=1:10
    y=y-Jac(y,h)\F(y,h);
    pause
    plot(t,y)
    norm(F(y,h))
end
```

# Matlab implementation

```
function [ J ] = Jac( y, h )

n=length(y);
e  = ones(n,1);
d  = 2*(h^2)*y-(2+h^2)*e;

J  = spdiags([e d e], -1:1, n, n);
J(1,:)=0;   J(n,:)=0;
J(1,1)=1;   J(n,n)=1;

end
```

MATLAB DEMO

# Shooting for nonlinear BVP

## Problem

$$\begin{cases} y''(t) - y(t) + y(t)^2 = 0 \\ y(0) = 1 \\ y(1) = 4 \end{cases}$$

Let

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ y - y^2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 - u_1^2 \end{pmatrix}$$

# Shooting for nonlinear BVP

## Problem

$$\begin{cases} y''(t) - y(t) + y(t)^2 = 0 \\ y(0) = 1 \\ y(1) = 4 \end{cases}$$

Let

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# Shooting for nonlinear BVP

## Problem

$$\begin{cases} y''(t) - y(t) + y(t)^2 = 0 \\ y(0) = 1 \\ y(1) = 4 \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u_1(0) = 1, u_2(0) = x$$

Choose $x$ such that $u_1(1) = 4$.

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 - u_1^2 \end{pmatrix}$$

# Matlab implementation

```
function [ y ] = euler_bvp( yp0 )

t=linspace(0,1,n);  h=t(2)-t(1);
u(1,1)=1;    u(2,1)=yp0;

F=@(u) [u(2); u(1)-u(1)^2];

for j=1:length(t)-1
    u(:,j+1)=u(:,j)+h*F(u(:,j));
end

y=u(1,:);
end
```

# Matlab implementation

```
function [ val ] = G( x )

[ y ] = euler_bvp( x );
val=y(end)-4;

end
```

# Matlab implementation

```
close all
clear all
clc

a=5;    b=6;    c=5;
while abs(G(c))>1e-6
    c=(a+b)/2;
    if G(c)<0
        a=c;
    elseif G(c)>0
        b=c;
    else
        break
    end
end

[ y ] = euler_bvp( c );
t=linspace(0,1,100);
plot(t,y)
```

MATLAB DEMO