Non-linear wave equations

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CHAPTER 1

Introduction

This course is devoted to the study of non-linear wave equations, so how are they defined? A natural problem arising in physics is to determine a function u of (t, x), where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, satisfying

$$\begin{cases} \Box u = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x), \end{cases}$$

where f and g are given functions and we have used the notation

$$\Box u = -u_{tt} + \Delta u, \quad u_t = \partial_t u = \frac{\partial u}{\partial t}, \quad u_{tt} = \partial_t^2 u = \frac{\partial^2 u}{\partial t^2}, \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial (x^i)^2}.$$

The equation $\Box u = 0$ is the **wave equation**. It can be used to model for instance the vibrations of a string, in which case one should impose boundary conditions. Formulating the Maxwell vacuum equations for an electric and magnetic potential also yields wave equations. In order to define non-linear wave equations, it is convenient to write the wave operator $\Box$ in a different form. Let us define the $(n+1) \times (n+1)$-matrix $\eta = \mathrm{diag}(-1, 1, ..., 1)$. This is the $n+1$-dimensional Minkowski metric. Note that the inverse of η equals η. We shall use the notation $\eta_{\mu\nu}$ for the components of η and $\eta^{\mu\nu}$ for the components of the inverse of η. Note that

$$\Box u = \eta^{\mu\nu}\partial_\mu \partial_\nu u \left(= \sum_{\mu,\nu=0}^{n} \eta^{\mu\nu}\partial_\mu\partial_\nu u \right),$$

where we use the notation $\partial_0 = \partial_t$, $\partial_i = \partial_{x^i}$ for $i = 1, ..., n$ and the Einstein summation convention, i.e. that we sum over repeated upstairs and downstairs indices. By a **non-linear wave equation**, we mean an equation of the following form:

$$g^{\mu\nu}(u, \partial u)\partial_\mu\partial_\nu u = F(u, \partial u),$$

where g is a symmetric matrix with one negative eigenvalue and n positive eigenvalues, depending on u and its first partial derivatives, and F is a function depending on u and its first partial derivatives (here ∂u is the vector $\partial u = (\partial_t u, \partial_1 u, ..., \partial_n u)$ of all first partial derivatives and $g^{\mu\nu}$ are the matrix elements of the inverse of g). The function u is allowed to be vector valued. Equations of this form arise in the study of Einstein's equations. The dependence of g on u and ∂u leads to additional technical complications that make the essential ideas less transparent. For this reason, we shall in this course focus on equations of the form

$$\Box u = F(u, \partial u).$$

The step from linear to non-linear partial differential equations (PDE:s) is quite big. The linear theory is based on the fact that by adding two solutions, one obtains a new solution. In the non-linear case, this is no longer true. One is consequently forced to develop new ideas and methods.

The **questions** of interest are first of all *local existence*: consider

$$\begin{cases} \Box u = F(u, \partial u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x). \end{cases}$$

Is there a solution to the problem for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$? A second question is that of *global existence*. Is it possible to find a solution for all t? In the case of the Einstein equations, this question is related to the existence of singularities (big bang and black hole type singularities). A related question is that of *global existence for small data*. Given that the initial data are small in some suitable sense, is there a global solution? In the case of general relativity, this question is related to that of stability of special solutions. If one perturbs the initial data corresponding to Minkowski space, does one get a solution that is roughly similar? Also, given an expanding cosmological spacetime, does one get something similar after perturbing the corresponding initial data?

Concerning the question of local existence, it is possible to develop a general theory, but the questions concerning global existence depend in a rather subtle way on the structure of the non-linearity, i.e. on the function F. Consider for instance the equations

$$\Box u = -u_t^2, \quad \Box u = \sum_{i=1}^{3} (\partial_i u)^2 - u_t^2$$

where $x \in \mathbb{R}^3$. The first equation does not admit any non-trivial global solutions for initial data that vanish outside a compact set, whereas the second admits global solutions if the initial data are small. Another problem which illustrates how ignorant we are is

$$\Box u = u^k$$

for $n = 3$ and k odd. If $k = 1, 3, 5$ one obtains global existence for arbitrary initial data. For $k \geq 7$, nothing is known. Even for simple equations, there is in other words very limited understanding.

The outline of the course is as follows. We begin by studying ordinary differential equations (ODE:s), starting by proving local existence and uniqueness. The reason for this is that the ideas behind the proofs are similar to those used when proving local existence and uniqueness for non-linear wave equations. There are however less technical complications. We also point out what the obstructions to global existence are. After that we illustrate how the existence of monotone quantities can be used to analyze the asymptotic behaviour of solutions to ODE:s. Monotone quantities, if they exist, are powerful tools, a statement which is also true for non-linear wave equations more generally. After this brief study of ODE:s, we turn to 1+1 non-linear wave equations (in the notation n+1-dimensional wave equation, the n refers to the number of space dimensions and the 1 refers to the time dimension). We start by making some comments on which types of functions are appropriate as initial data, and we define some natural metric spaces associated with these classes. Then we show how to solve the linear wave equation. This is necessary in order to be able the analyze the non-linear case. Then we discuss special types of estimates that one obtains in the $1+1$-dimensional case. These estimates can be used to prove local existence and uniqueness of solutions. We then discuss global existence, giving some examples in which one can prove it. Finally, we turn to $n+1$ dimensional wave equations, starting by mentioning the basic facts concerning the linear wave equation needed in the non-linear theory. The idea for finding a local solution is similar to the earlier problems; one carries out an iteration. The space of functions in which one obtains convergence is however different. One needs some basic knowledge of measure and integration theory in order to define these spaces, but we shall see to it that the material is readable also for those who are not familiar with this theory. To end the course, we give some examples of when one can prove global existence and some examples for which it is possible to prove global existence for small data.

In this course, we shall assume that the reader is familiar with the material covered in the book *Principles of Mathematical Analysis* by Walter Rudin, at least until and including Chapter 7 on Sequences and Series of Functions. However, for the convenience of the reader, we shall repeat some of the definitions and basic results proved there.

To end, a few words concerning what will not be contained in this course. We shall give several examples from General Relativity, but we shall not derive the equations starting with the Einstein equations. This would require significant amounts of differential geometry, and we do not have the time to cover the necessary material here. We shall simply state the equations in their end form and start working with them.

CHAPTER 2

Local existence and uniqueness for ODE:s

A natural starting point in the study of non-linear wave equations is local existence. Let us start by considering this question for ordinary differential equations. It is of interest in its own right, but we mainly consider it here since it illustrates some of the main ideas of the proof of local existence in general. Let f be a function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, which is at least continuous (later we shall impose stronger conditions on f). Consider the problem

$$\frac{dx}{dt}(t) = f[t, x(t)] \tag{1}$$

$$x(0) = x_0, \tag{2}$$

for some $x_0 \in \mathbb{R}^n$. We wish to prove the existence of a continuously differentiable x defined on the interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ satisfying (1) and (2). How? The idea is to define a sequence of approximations

$$x_n = x_0 + \int_0^t f[s, x_{n-1}(s)]ds$$

for $n \geq 1$ and then to prove that this sequence converges. We thus have a sequence, and we wish to prove that there is a function x to which the sequence converges. This makes it necessary to discuss the concept of metric spaces and completeness.

1. Background material

In order to make sense of the concept of convergence, it is natural to have a concept of distance (even though it is strictly speaking not absolutely necessary). Let X be a set. Intuitively, a distance function d on X should associate with each pair of points $x, y \in X$ a non-negative number $d(x, y)$:

$$d(x, y) \geq 0 \tag{3}$$

for all $x, y \in X$. Furthermore, we expect the distance between two points to be zero if and only if the points coincide:

$$d(x, y) = 0 \Leftrightarrow x = y. \tag{4}$$

Another natural condition is that the distance from x to y equals the distance from y to x:

$$d(x, y) = d(y, x) \tag{5}$$

for all $x, y \in X$. Finally, we require that the triangle inequality holds:

$$d(x, y) \leq d(x, z) + d(z, y) \tag{6}$$

for all $x, y, z \in X$. Let us make the following formal definition.

DEFINITION 1. Let X be a set. A function $d : X \times X \to \mathbb{R}$ satisfying (3)-(6) is called a *metric* on X. If d is a metric on X, then we shall refer to (X, d) as a *metric space*.

A situation that arises frequently in analysis is that we wish to find a solution to an equation and in order to do so, we define a sequence of approximations. The hope being that the sequence converges to a solution. The question is then how to characterize "convergence" of a sequence when one does not have the element to which the sequence is supposed to converge. We are naturally lead to the concept of a Cauchy sequence.

DEFINITION 2. Let (X, d) be a metric space. A sequence $x_n \in X$, $n \geq 1$, is called a *Cauchy sequence* if for every $\epsilon > 0$ there is an N such that for $n, m \geq N$,

$$d(x_n, x_m) \leq \epsilon.$$

As we mentioned above, it is quite common to try to solve an equation by constructing a sequence of approximations, and if one has set up the problem in a good way, one might be able to prove that the sequence is a Cauchy sequence. It is then crucial to prove that, given a Cauchy sequence, there is an element to which the sequence converges. This is however a property of the metric space.

DEFINITION 3. Let (X, d) be a metric space. If, for every Cauchy sequence $\{x_n\}$ $n \geq 1$, there is an $x \in X$ such that $x_n \to x$, then we say that the metric space is *complete*.

Remark. The notation $x_n \to x$ means that for every $\epsilon > 0$ there is an N such that $d(x_n, x) \leq \epsilon$ for all $n \geq N$.

For the reasons mentioned above, complete metric spaces are extremely important in analysis. The real numbers form a complete metric space with respect to the metric defined by $d(x, y) = |x - y|$. The rational numbers are however not complete, and this is the main distinction between them. Thanks to the completeness, it is possible to take the nth square root of a positive real number and to define the integral of a continuous function by a limit.

Often there is more structure available than simply the metric structure. Let us define the concept of a Banach space.

DEFINITION 4. A *normed linear space* is a vector space X (over $\mathbb{R}$ or $\mathbb{C}$) on which there is a function $\|\cdot\|$ defined, called a *norm*, with the following properties:

$$\begin{aligned} \|x\| &\geq 0, \quad \|x\| = 0 \Leftrightarrow x = 0, \\ \|\lambda x\| &= |\lambda|\|x\|, \\ \|x + y\| &\leq \|x\| + \|y\|. \end{aligned}$$

We leave it to the reader to check that if X is a normed linear space, with norm $\|\cdot\|$, then $d(x, y) = \|x - y\|$ is a metric on X.

For the sake of completeness, we remind the reader of the definition of a vector space. In the definition below, let F denote $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 5. A *vector space* over F consists of a set X, a mapping $(x, y) \to x + y$ of $X \times X$ into X, and a mapping $(\lambda, x) \to \lambda x$ of $F \times X$ into X, such that the following holds:

- X is an abelian group with the group operation $+$.
- $\lambda(\mu x) = (\lambda\mu)x$ for all $\lambda, \mu \in F$ and $x \in X$ (*associativity*).
- $(\lambda + \mu)x = \lambda x + \mu x$ and $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda, \mu \in F$ and $x, y \in X$ (*distributivity*).
- $1x = x$ for all $x \in X$.

That X is an abelian group with group operation $+$ means that

- $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$.
- $x + y = y + x$ for all $x, y \in X$.
- There is an element $0 \in X$ such that $0 + x = x$ for all $x \in X$.
- For every $x \in X$, there is an element $-x \in X$ such that $x + (-x) = 0$.

Definition 6. Let X be a normed linear space with norm $\|\cdot\|$ and let $d(x, y) = \|x - y\|$. Then we say that X is a *Banach space* if (X, d) is complete.

Note that $\mathbb{R}^n$ is a Banach space with respect to the usual norm

$$|x| = \left(\sum_{i=1}^{n} (x^i)^2\right)^{1/2}, \tag{7}$$

where $x = (x^1, ..., x^n)$. Let us prove this. It is clear that $\mathbb{R}^n$ is a vectorspace over $\mathbb{R}$. Of the conditions for a normed linear space, the only one which is not clear is the last one. In order to prove it we need Schwartz inequality. We shall use the notation

$$x \cdot y = \sum_{i=1}^{n} x^i y^i,$$

for $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$. Note that $x \cdot x = |x|^2$.

Theorem 1 (Schwarz inequality). *Let $x, y \in \mathbb{R}^n$. Then*

$$|x \cdot y| \leq |x||y|.$$

Proof. If $y = 0$, the inequality is obvious since both sides are zero. Assume therefore $y \neq 0$. One can compute that

$$||y|^2 x - (x \cdot y)y|^2 = |y|^2[|x|^2|y|^2 - (x \cdot y)^2].$$

Since the left hand side is non-negative and $|y|^2 > 0$, we obtain

$$|x|^2|y|^2 - (x \cdot y)^2 \geq 0.$$

This yields the theorem. □

Let us prove the last condition for a normed linear space. By Schwarz inequality we have

$$|x+y|^2 = |x|^2 + 2x \cdot y + |y|^2 \leq |x|^2 + 2|x \cdot y| + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2.$$

This proves that $\mathbb{R}^n$ is a normed linear space. All that remains to be proved is completeness. Assume $\{x_n\}$ is a Cauchy sequence. Then $\{x_n^i\}$ is a Cauchy sequence in $\mathbb{R}$. Thus there is an x^i such that x_n^i converges to x^i by the completeness of the real numbers. This defines the limit and $x_n \to x$. We have proved that $\mathbb{R}^n$ is a Banach space.

As was mentioned in the introduction, we assume the reader is familiar with the basic material in Rudin's book, but for the sake of completeness, we repeat some definitions and elementary results here.

DEFINITION 7. Let (X, d) be a metric space. Let us introduce some terminology:

(1) If $x \in X$ and $r > 0$, we define a *neighbourhood* of x to be a set of the form
$$B_r(x) = \{y \in X : d(x, y) < r\},$$
for some $r > 0$, and we refer to r as the *radius* of the neighbourhood.
(2) A subset U of X is said to be *open* if for every $x \in X$ there is an $r > 0$ such that $B_r(x) \subseteq U$.
(3) A point x is a *limit point* of the set E if *every* neighbourhood of x contains a $y \neq x$ such that $y \in E$.
(4) A subset C of X is said to be *closed* if every limit point of E is a point of E.
(5) Let K be a subset of X. An *open covering* of K is a collection $\{U_\alpha\}$ of open sets such that K is contained in the union of the U_α.
(6) K is said to be *compact* if every open covering of K has a finite subcovering, i.e. if there is a finite collection $U_{\alpha_1}, ..., U_{\alpha_n}$ such that K is contained in the union of the U_{α_i}, $i = 1, ..., n$.
(7) E is said to be *bounded* if there is a real number M and a $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
(8) If E is a subset of X, we denote by $\bar{E}$ the union of E and all its limit points. We call $\bar{E}$ the *closure* of E. Note that $\bar{E}$ is closed.

One important result that was proved in Rudin's book *Principles of Mathematical Analysis* is the Heine Borel theorem, which is the equivalence of the first two statements in the following theorem.

THEOREM 2. *If a set E in $\mathbb{R}^n$ has one of the following three properties, then it has the other two:*

(1) *E is closed and bounded.*
(2) *E is compact.*
(3) *Every infinite subset of E has a limit point in E.*

Let us note that one part of the theorem can be generalized. First we need a definition.

DEFINITION 8. Given a sequence $\{x_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < ...$. Then the sequence $\{x_{n_k}\}$ is called a *subsequence* of $\{x_n\}$.

A subset Y of a metric space X is called *sequentially compact* if every sequence $\{y_n\}$ in Y has a subsequence $\{y_{n_k}\}$ that converges to a point of Y, i.e. $d(y_{n_k}, y) \to 0$ for some $y \in Y$.

THEOREM 3. *A subset of a metric space is compact if and only if it is sequentially compact.*

We shall not prove this theorem here, nor shall we use it.

2. The space of continuous functions

Let us return to the problem at hand, i.e. that of proving local existence of solutions to ordinary differential equations. We have a sequence of approximations and we wish to prove that this sequence is a Cauchy sequence in a complete metric space. We need to find a suitable complete metric space. But first, let us define the concept of continuity.

DEFINITION 9. Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$. We say that f is *continuous at* x if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that $\rho[f(x), f(y)] < \epsilon$. If f is continuous at x for every $x \in X$, we say that f is *continuous.*

We shall sometimes use the fact that f is continous at x if and only if for every sequence $\{x_k\}$ such that $x_k \to x$, $f(x_k) \to f(x)$. To prove this is a good exercise, but the proof is also available in Rudin's book. Let f be a continuous function between metric spaces (X, d) and (Y, ρ). If $K \subseteq X$ is a compact subset then $f(K)$ is compact. For a proof, we refer again to Rudin's book. Note however that this is a rather immediate consequence of the definition and the fact that f is continuous if and only if $f^{-1}(U)$ is open for every open set U.

DEFINITION 10. Let (X, d) be a metric space and $(Y, \|\cdot\|)$ be a normed linear space. Define $C(X, Y)$ to be the set of continuous functions from X to Y and define the set of bounded continous functions from X to Y to be $C_b(X, Y)$. For $f \in C_b(X, Y)$, define

$$\|f\|_C = \sup_{x \in X} \|f(x)\|.$$

Note that this turns $C_b(X, Y)$ into a normed linear space (prove this).

Remark. That f is bounded means that there is a constant $C < \infty$ such that $\|f(x)\| \leq C$ for all $x \in X$.

Note that if X is compact, then a continuous function f from X to Y is automatically bounded. The reason for this is the following. Let $g(x) = \|f(x)\|$. Then g is a continuous function from X to the real numbers (prove this). By the observation made prior to the definition, we conclude that $g(X)$ is a compact subset of the real numbers. By Theorem 2, we conclude that $g(X)$ is closed and bounded. In particular, it is bounded. Thus if X is compact $C(X, Y) = C_b(X, Y)$.

THEOREM 4. *Let* (X, d) *be a metric space and let* $(Y, \|\cdot\|)$ *be a Banach space. Then* $[C_b(X, Y), \|\cdot\|_C]$ *is a Banach space.*

Proof. We need to prove completeness. Let the sequence $\{f_n\}$ of elements of $C_b(X, Y)$ be a Cauchy sequence. Let $x \in X$. Note that $f_n(x)$ is a Cauchy sequence in Y, so that $f_n(x)$ converges to, say, $f(x)$, due to the fact that Y is a Banach space. This defines the function f. We need to prove that f is continuous, that it is bounded and that f_n converges to f with respect to $\|\cdot\|_C$. Let us start with boundedness. There is an N such that for $n, m \geq N$,

$$\|f_n - f_m\|_C \leq 1.$$

In particular

$$\|f_n\|_C \leq 1 + \|f_N\|_C$$

for all $n \geq N$. Furthermore, since $\{1, ..., N\}$ is only a finite set, there is a constant $C < \infty$ such that

$$\|f_n\|_C \leq C$$

for all $n \leq N$. We conclude that

$$\|f_n\|_C \leq C + 1$$

for all n. Since

$$\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \leq C + 1,$$

we conclude that f is bounded. Let us turn to continuity. Let $x \in X$ and $\epsilon > 0$. Let N be such that $\|f_n - f_m\|_C < \epsilon/3$ for all $n, m \geq N$. Finally, let $\delta > 0$ be small enough that $d(x, y) < \delta$ implies $\|f_N(x) - f_N(y)\| < \epsilon/3$. Then, for $d(x, y) < \delta$,

$$\begin{aligned} \|f(x) - f(y)\| &\leq \|f(x) - f_N(x)\| + \|f_N(x) - f_N(y)\| + \|f_N(y) - f(y)\| \\ &= \lim_{n\to\infty} \|f_n(x) - f_N(x)\| + \|f_N(x) - f_N(y)\| \\ &\quad + \lim_{n\to\infty} \|f_N(y) - f_n(y)\| < \epsilon. \end{aligned}$$

Consequently, f is continuous. Finally, let us prove that f_n converges to f. Let $\epsilon > 0$. There is an N such that $n, m \geq N$ implies that $\|f_n - f_m\|_C < \epsilon$. Let $n \geq N$ and $x \in X$. Then

$$\|f(x) - f_n(x)\| = \lim_{m\to\infty} \|f_m(x) - f_n(x)\| < \epsilon.$$

Since the right hand side does not depend on x, we obtain $\|f - f_n\|_C < \epsilon$. This proves convergence. □

As a special example of this construction, let $X = [a, b]$ be a compact subinterval of $\mathbb{R}$ with the metric $d(x, y) = |x - y|$. Furthermore, let $Y = \mathbb{R}^n$ with the norm $|\cdot|$ defined in (7). We denote the resulting Banach space by $C([a, b], \mathbb{R}^n)$.

3. Local existence for ordinary differential equations

We are now in a position to prove local existence and uniqueness. In order to do so, it will however be necessary to demand more of f than that it simply be continuous.

THEOREM 5. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be continuous and assume also that it is continuously differentiable with respect to x^i, $i = 1, ..., n$. Consider the problem*

$$\frac{dx}{dt}(t) = f[t, x(t)] \tag{8}$$

$$x(t_0) = x_0, \tag{9}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Then there is an $\epsilon > 0$ and a unique, continuously differentiable $x : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n$ satisfying (8) and (9). Furthermore, there is a function $\epsilon_0(C_0) > 0$ which decreases when C_0 increases such that ϵ can be chosen to satisfy $\epsilon \geq \epsilon_0(C_0)$ where

$$C_0 = \sup_{|t-t_0|\leq 1,\ |x-x_0|\leq 1} \left(|f(t, x)|^2 + \sum_{i=1}^n |\partial_{x^i} f(t, x)|^2 \right)^{1/2}. \tag{10}$$

Remark. The first part of the theorem gives local existence, but what is the purpose of the second part? The second part is there to help us when we wish to go from local existence to global existence. Quite generally, local existence results come with two parts. First, there is the existence and uniqueness of a local solution. Second, there is statement giving the obstruction to extending this local solution to a global one. In the present case, the second statement will later be used to prove that either the solution becomes unbounded in finite time, or there is global existence.

Proof. As described earlier, we set up the iteration

$$x_n(t) = x_0 + \int_{t_0}^{t} f[s, x_{n-1}(s)]ds \tag{11}$$

for $n \geq 1$. The zeroth iterate x_0 is the constant vector given in the statement of the theorem. Let us start by obtaining some rough control of the iterates x_n.

Inductive assumption: $|x_n(t)-x_0| \leq 1$ for $|t-t_0| \leq \epsilon$. If $n = 0$, this is true. Assume it is true for n. By (11) and the definition of C_0 we get

$$|x_{n+1}(t) - x_0| \leq C_0|t - t_0|$$

for $|t - t_0| \leq \epsilon$. If we demand that $\epsilon \leq 1/(C_0 + 1)$, we conclude that the inductive assumption holds with n replaced by $n + 1$.

Rough control: If $\epsilon \leq 1/(C_0 + 1)$, then $|t - t_0| \leq \epsilon$ implies $|t - t_0| \leq 1$ and $|x_n(t) - x_0| \leq 1$ for all n. Note that due to the definition of C_0, we thus control f and its derivatives in $[t, x_n(t)]$.

Let us turn to convergence. Consider the difference of two successive iterates:

$$x_{n+1}(t) - x_n(t) = \int_{t_0}^{t} \{f[s, x_n(s)] - f[s, x_{n-1}(s)]\}ds. \tag{12}$$

Note that for $x, y \in \bar{B}_1(x_0)$ and s such that $|s - t_0| \leq 1$ we have

$$\begin{aligned} |f(s,x) - f(s,y)| &= \left| \int_0^1 \partial_\tau f[s, \tau x + (1-\tau)y]d\tau \right| \\ &= \left| \int_0^1 \sum_{i=1}^{n} (\partial_{x^i} f)[s, \tau x + (1-\tau)y](x^i - y^i)d\tau \right| \\ &\leq C_0|x - y|, \end{aligned} \tag{13}$$

where we have used the Schwartz inequality,

$$|z \cdot w| = \left| \sum_{i=1}^{n} z^i w^i \right| \leq |z||w|,$$

for $z, w \in \mathbb{R}^n$ and the definition of C_0. Inserting this information into (12), we obtain

$$|x_{n+1}(t) - x_n(t)| \leq \int_{t_0}^{t} C_0|x_n(s) - x_{n-1}(s)|ds. \tag{14}$$

Let us define ϵ by $\epsilon = 1/[2(C_0 + 1)]$ and let

$$\|x\| = \sup_{|t-t_0|\leq\epsilon} |x(t)|$$

for continuous $x : [t_0 - \epsilon, t_0 + \epsilon] \to \mathbb{R}^n$. Note that all the iterates x_n are continuous. Due to (14), we get the conclusion that

$$\|x_{n+1} - x_n\| \leq C_0 \|x_n - x_{n-1}\| \epsilon \leq \frac{1}{2} \|x_n - x_{n-1}\|.$$

Iterating this estimate, we obtain

$$\|x_{n+1} - x_n\| \leq \frac{1}{2^n} \|x_1 - x_0\|.$$

Assume that $m > n$. Then

$$\begin{aligned}\|x_m - x_n\| &\leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{m-1} 2^{-k} \|x_1 - x_0\| \\ &\leq 2^{-n} \|x_1 - x_0\| \sum_{j=0}^{\infty} 2^{-j} \leq 2^{-n+1} \|x_1 - x_0\|.\end{aligned}$$

Clearly, the x_n constitute a Cauchy sequence. Thus there is a continuous function $x : [t_0 - \epsilon, t_0 + \epsilon] \to \mathbb{R}^n$ such that x_n converges to x with respect to $\| \cdot \|$.

Since x_n converges to x, we conclude that $|x(t) - x_0| \leq 1$ for all t such that $|t - t_0| \leq \epsilon$. By (13), we conclude that

$$|f[t, x(t)] - f[t, x_n(t)]| \leq C_0 \|x - x_n\|.$$

Thus $f(\cdot, x_n)$ converges uniformly to $f(\cdot, x)$. Due to (11), we conclude that

$$x(t) = x_0 + \int_{t_0}^{t} f[s, x(s)] ds.$$

This proves that x is continuously differentiable and that it satisfies (8) and (9). For a proof of uniqueness, see Theorem 6. Note that we can take $\epsilon_0(C) = 1/[2(C+1)]$, which has the desired properties. □

In order to prove uniqueness, it will be convenient to have Grönwall's lemma. It will also be useful in the analysis of non-linear wave equations more generally.

LEMMA 1 (Grönwall's lemma). *Let $f, g : [t_0, t_0 + T] \to \mathbb{R}$ be continuous non-negative functions, where T is some positive number. Assume there is a constant $C \geq 0$ such that*

$$f(t) \leq C + \int_{t_0}^{t} g(s) f(s) ds \tag{15}$$

for all $t \in [t_0, t_0 + T]$. Then

$$f(t) \leq C \exp\left[\int_{t_0}^{t} g(s) ds\right]$$

for all $t \in [t_0, t_0 + T]$.

Proof. Let $\epsilon > 0$ and define a function h by

$$h(t) = C + \epsilon + \int_{t_0}^{t} g(s) f(s) ds.$$

The reason we include an ϵ is technical; we wish to be sure that h is positive. Since f and g are continuous functions, h is continuously differentiable. Estimate, using (15) and the definition of h,

$$\frac{dh}{dt} = gf \leq gh.$$

Since h is positive, we are allowed to divide by it. Dividing by h and integrating, we obtain

$$\ln \frac{h(t)}{h(t_0)} = \int_{t_0}^{t} \frac{1}{h(s)} \frac{dh}{ds}(s)ds \leq \int_{t_0}^{t} g(s)ds.$$

Taking the exponential of this inequality, we obtain

$$f(t) \leq h(t) \leq h(t_0) \exp\left[\int_{t_0}^{t} g(s)ds\right] = (C+\epsilon) \exp\left[\int_{t_0}^{t} g(s)ds\right],$$

where we have used (15) in the first step. Since this inequality holds for any $\epsilon > 0$, we can take the limit $\epsilon \to 0$ in order to obtain the desired conclusion. □

Exercise. Prove the following analogue of Grönwall's lemma for going backwards in time: Let $f, g : [t_0 - T, t_0] \to \mathbb{R}$ be continuous non-negative functions, where T is some positive number. Assume there is a constant $C \geq 0$ such that

$$f(t) \leq C + \int_{t}^{t_0} g(s)f(s)ds$$

for all $t \in [t_0 - T, t_0]$. Then

$$f(t) \leq C \exp\left[\int_{t}^{t_0} g(s)ds\right]$$

for all $t \in [t_0 - T, t_0]$.

Grönwall's lemma is one example of how to obtain conclusions from inequalities. In the study of non-linear problems, this is very important, and we shall see more examples of how to obtain conclusions from integral and differential inequalities as the course progresses. Let us turn to uniqueness for ordinary differential equations.

THEOREM 6. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be continuous and assume also that it is continuously differentiable with respect to x^i, $i = 1, ..., n$. Consider the problem*

$$\frac{dx}{dt}(t) = f[t, x(t)] \tag{16}$$

$$x(t_0) = x_0, \tag{17}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Assume $x, y : (t_-, t_+) \to \mathbb{R}^n$ are two continuously differentiable solutions to (16) and (17), where we assume $t_0 \in (t_-, t_+)$. Then $x = y$ on (t_-, t_+).

Proof. We have

$$x(t) = x_0 + \int_{t_0}^{t} f[s, x(s)]ds, \quad y(t) = x_0 + \int_{t_0}^{t} f[s, y(s)]ds. \tag{18}$$

Let $t_1 \in [t_0, t_+)$. Since x and y are continuous, there is a constant $C < \infty$ such that $|x(t) - x_0| \leq C$ and $|y(t) - x_0| \leq C$ for $t \in [t_0, t_1]$. Let

$$C_0 = \sup_{t_0 \leq t \leq t_1,\ |x - x_0| \leq C} \left(\sum_{i=1}^{n} |\partial_{x^i} f(t, x)|^2 \right)^{1/2}.$$

Subtracting the two equations (18) from each other and using an estimate of the form (13), we obtain

$$|x(t) - y(t)| \leq \int_{t_0}^{t} |f[s, x(s)] - f[s, y(s)]| ds \leq C_0 \int_{t_0}^{t} |x(s) - y(s)| ds$$

for $t \in [t_0, t_1]$. Applying Grönwall's lemma with $f(t) = |x(t) - y(t)|$, $g(t) = C_0$ and $C = 0$, we obtain the conclusion that $f(t) = 0$ for $t \in [t_0, t_1]$. Since $t_1 \in [t_0, t_+)$ was arbitrary, we obtain $x = y$ in $[t_0, t_+)$. The argument to prove that the functions coincide in $(t_-, t_0]$ is similar. □

Thanks to this result, it makes sense to speak of a *maximal existence interval* for a solution to (16) and (17). Let S be the set of solutions to (16) and (17). With each solution there is an associated existence interval (t_-, t_+). Let A be the set of right endpoints of existence intervals of solutions in S. There are two cases. Either A is unbounded, in which case we define $t_{\max} = \infty$ or it is bounded, in which case we let $t_{\max} = \sup A$. We define $t_{\min}$ similarly as the infimum of the left endpoints of existence intervals. Let us prove that we get a solution to (16) and (17) on $(t_{\min}, t_{\max})$. Let $t \in [t_0, t_{\max})$. Since $t < t_{\max}$, there is a solution y in S with $t_+ > t$. Define $x(t) = y(t)$. Due to Theorem 6, it does not matter which y one takes. Furthermore x is continuously differentiable in a neighbourhood of t. This defines a solution for $t \in [t_0, t_{\max})$. The definition of x in $(t_{\min}, t_0]$ is similar.

It is of interest to find out when one can take $t_{\max} = \infty$ and when one can take $t_{\min} = -\infty$. That there are sometimes obstructions to this is clear from the equation

$$\frac{dx}{dt} = x^2,$$

one solution of which is $x(t) = 1/(1-t)$. In fact, say that we have a continuously differentiable x such that $x(0) > 0$ and

$$\frac{dx}{dt} \geq x^2$$

for $t \geq 0$. Then x has to blow up in finite time. To see this, note first that $x(t) > 0$ for $t > 0$. Thus we are allowed to divide the inequality by x^2 and integrate in order to obtain

$$\frac{1}{x(0)} - \frac{1}{x(t)} \geq t.$$

This can be rearranged to yield

$$x(t) \geq \frac{x(0)}{1 - x(0)t},$$

which clearly blows up in finite time. In fact, the only thing that can go wrong is that the solution becomes unbounded.

THEOREM 7. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be continuous and assume also that it is continuously differentiable with respect to x^i, $i = 1, ..., n$. Consider the problem*

$$\frac{dx}{dt}(t) = f[t, x(t)] \tag{19}$$

$$x(t_0) = x_0, \tag{20}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Let $(t_{\min}, t_{\max})$ be the maximal existence interval. If $t_{\max} < \infty$, then $|x(t)|$ is unbounded on $[t_0, t_{\max})$. Similarly, if $t_{\min} > -\infty$, then $|x(t)|$ is unbounded on $(t_{\min}, t_0]$.

Proof. Let us assume that $t_{\max} < \infty$ but that $|x(t)| \le C < \infty$ for some constant C and all $t \in [t_0, t_{\max})$. Define

$$C_1 = \sup_{t_0-1\le t\le t_{\max}+1,\ |x|\le C+1} \left(|f(t,x)|^2 + \sum_{i=1}^{n} |\partial_{x^i} f(t,x)|^2 \right)^{1/2}.$$

Let $\epsilon = \min\{\epsilon_0(C_1), t_{\max} - t_0\}$, where ϵ_0 is the function given in Theorem 5. Consider the problem

$$\frac{dy}{dt}(t) = f[t, y(t)] \tag{21}$$

$$y(t_{\max} - \epsilon/2) = x(t_{\max} - \epsilon/2). \tag{22}$$

Let C_0 be defined as in (10) with t_0 replaced by $t_0' = t_{\max} - \epsilon/2$ and with $x_0 = x(t_{\max} - \epsilon/2)$. By Theorem 5 we obtain a solution to (21) and (22) for $|t - t_0'| < \epsilon_0(C_0)$. Since $|x_0| \le C$ and $t_0 \le t_0' < t_{\max}$, we conclude that $C_0 \le C_1$ (C_0 and C_1 are defined by taking the supremum of the same object, but the set over which we take the supremum in C_1 contains the set over which we take the supremum in C_0). Thus $\epsilon_0(C_0) \ge \epsilon_0(C_1)$. In other words, the existence interval of y includes $(t_{\max} - 3\epsilon/2, t_{\max} + \epsilon/2)$. Since x also satisfies (21) and (22), we conclude that x and y coincide on their common domain of definition, due to Theorem 6. In other words, y can be used to extend x beyond its maximal existence interval. This is a contradiction. The argument in the other time direction is similar. □

The above result can be interpreted as a *continuation criterion*. Say that we have a solution x to (19) and (20) on an interval (t_-, t_+). What the theorem says is that if $x(t)$ is bounded for $t \in [t_0, t_+)$, then we can extend the solution beyond t_+. The continuation criterion is thus that x remains bounded. Note that the criterion is an immediate consequence of the local existence theorem in the form we have proved it; i.e. with an estimate of the existence time in terms of C_0 defined in (10). It is possible to improve the statement of the theorem.

Exercise. Prove the following statements. Say that we have a solution on an interval (t_-, t_+), and assume there is a sequence $t_k \in (t_-, t_+)$ with $t_k \to t_+$ and a constant $C < \infty$ such that $|x(t_k)| \le C$. Then we can extend the solution beyond t_+. This has the following consequence: say that the maximal existence interval for x is $(t_{\min}, t_{\max})$ and assume that $t_{\max} < \infty$. Then $|x(t)| \to \infty$ as $t \to t_+$, i.e. for every $M > 0$ there is a $t_M \in [t_0, t_+)$ such that $|x(t)| \ge M$ for all $t \in [t_M, t_+)$.

Example 1. Assume f satisfies an estimate

$$f(t,x) \cdot x \le g(t)(|x|^2 + 1) \tag{23}$$

for $t \ge t_0$, where $g : [t_0, \infty) \to \mathbb{R}$ is a continuous, non-negative function. Let x solve (19) and (20) and let $(t_{\min}, t_{\max})$ be the maximal existence interval. Then $t_{\max} = \infty$. In order to prove this statement, define

$$h(t) = |x(t)|^2 + 1.$$

Then h is a strictly positive, continuously differentiable function. Compute

$$\frac{dh}{dt} = 2\frac{dx}{dt} \cdot x = 2f(t,x) \cdot x \le 2gh,$$

where we have used (23). Dividing by h, which is allowed since h is strictly positive, and integrating yields

$$\ln \frac{h(t)}{h(t_0)} = \int_{t_0}^{t} \frac{1}{h(s)} \frac{dh}{ds}(s)ds \leq 2 \int_{t_0}^{t} g(s)ds$$

for all $t \in [t_0, t_{\max})$. Exponentiating this inequality, we obtain

$$h(t) \leq h(t_0) \exp \left[2 \int_{t_0}^{t} g(s)ds \right].$$

If $t_{\max} < \infty$, the right hand side is bounded, so that h, and therefore $|x|$, is bounded on $[t_0, t_{\max})$. This contradicts Theorem 7.

Exercise. Let $n = 3$. Find the maximal existence interval if the function f is given by

$$a)\ f(t,x) = -|x|^2 x, \quad b)\ f(t,x) = |x|^2 x, \quad c)\ f(t,x) = |x|^2 (v \times x),$$

where $v \in \mathbb{R}^3$ is a fixed vector. Note that the answer depends on the initial data.

Example 2, Hamiltonian dynamics. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function, which we shall refer to as the *potential energy*. Define the *Hamiltonian* associated with V by

$$H(p,q) = \frac{1}{2}|p|^2 + V(q),$$

where $p \in \mathbb{R}^n$. Then $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a twice continuously differentiable function. The *Hamiltonian equations* are then given by

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}(p,q) \tag{24}$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(p,q) \tag{25}$$

$$p(t_0) = p_0 \tag{26}$$

$$q(t_0) = q_0. \tag{27}$$

An important point to note is that if p, q are solutions to (24)-(27), then $H[p(t), q(t)]$ is constant.

Exercise. Prove that if there is a positive constant $C < \infty$ such that V satisfies

$$V(q) \geq -C(|q|^2 + 1), \tag{28}$$

then solutions to (24)-(27) have a maximal existence interval $(-\infty, \infty)$ for arbitrary initial data p_0, q_0. *Hint*: Use the fact that the Hamiltonian is conserved along a trajectory and start by trying to control $q(t)$. Note that for arbitrary $z, w \in \mathbb{R}^n$,

$$|z \cdot w| \leq \frac{1}{2}[|z|^2 + |w|^2].$$

Prove this inequality.

It is interesting to note that (28) cannot be improved to

$$V(q) \geq -C(|q|^2 + 1)^{1+\epsilon}$$

for $\epsilon > 0$. In fact, let $n = 1$ and

$$V(q) = -\left(q^2 + \frac{1}{2}\right)^{1+\epsilon}.$$

Consider (24)-(27) with initial data $q_0 = 1/\sqrt{2}$ and $p_0 = \sqrt{2}$. Then $H(p_0, q_0) = 0$ so that

$$\frac{1}{2}p^2 - \left(q^2 + \frac{1}{2}\right)^{1+\epsilon} = 0 \tag{29}$$

for all $t \in (t_{\min}, t_{\max})$. Using the Hamiltonian equations, one can prove that the time derivative of the product pq is non-negative. Since pq starts out positive it consequently has to remain positive. We conclude that both p and q have to remain positive to the future. By (29), we conclude that

$$p = \sqrt{2}\left(q^2 + \frac{1}{2}\right)^{1/2+\epsilon/2}.$$

Combining this with (25), we obtain

$$\frac{dq}{dt} = \sqrt{2}\left(q^2 + \frac{1}{2}\right)^{1/2+\epsilon/2}. \tag{30}$$

Exercise. Prove that (30), together with the condition $q(t_0) > 0$ implies that q blows up in finite time.

Example 3, homogeneous cosmologies. In cosmology, one often makes the assumption that the universe is spatially homogeneous and isotropic. Under these assumptions, the only freedom left is one function of one variable. The function describes the overall scale of the universe and Einstein's equations is an ODE for it. If one is interested in studying something more complicated, one can drop the isotropy condition but keep the spatial homogeneity. This leads to a system of ordinary differential equations. There is a formulation of Einstein's equations for a large class of spatially homogeneous spacetimes, the so called Bianchi class A spacetimes. Unfortunately it does not cover all of them. The equations are given by

$$\begin{aligned} N_1' &= (q - 4\Sigma_+)N_1 \\ N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \\ N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3 \\ \Sigma_+' &= -(2-q)\Sigma_+ - 3S_+ \\ \Sigma_-' &= -(2-q)\Sigma_- - 3S_- \end{aligned} \tag{31}$$

where the prime denotes derivative with respect to τ and

$$\begin{aligned} q &= 2(\Sigma_+^2 + \Sigma_-^2) \\ S_+ &= \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)] \\ S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3). \end{aligned} \tag{32}$$

The initial data should furthermore be such that

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_1N_3)] = 1 \tag{33}$$

is fulfilled. The above equations are a reformulation of Einstein's vacuum equations. In other words, no matter terms appear. Unfortunately, the variables have a rather intricate geometric interpretation which is not easy to explain without a significant amount of differential geometry. A derivation of the above formulation starting with the Einstein equations can be found in

Wainwright J and Hsu L 1989 A dynamical systems approach to Bianchi cosmologies: orthogonal models of class A *Class. Quantum Grav.* **6** 1409–31.

Exercise. Prove that if (33) is satisfied for the initial data, then it is satisfied for all τ. *Hint.* Define g to be the left hand side minus 1. It is enough to prove that g' can be expressed as a multiple of itself (cf. Grönwall's lemma). The computation is long.

Exercise. Prove that if N_1 is zero initially, then it is always zero and similarly for N_2 and N_3. *Hint*: use Grönwall's lemma.

As a consequence of the last exercise, if N_1 is positive initially, it is always positive, and if it is negative initially, it is always negative. The statements for N_2 and N_3 are similar.

Exercise. Assume that $N_1 \leq 0$ and $N_2, N_3 \geq 0$. Prove that (33) implies that Σ_+ and Σ_- remain bounded for all time. Prove that the boundedness of Σ_+ and Σ_- imply that N_1, N_2, N_3 cannot become unbounded in finite time.

The above exercise proves that if $N_1 \leq 0$ and $N_2, N_3 \geq 0$, then one obtains global existence. In fact, it is possible to prove that one always gets global existence for solutions to (31)-(33) unless all the N_i are strictly positive or all are strictly negative. In that case, one has past global existence (which is not so easy to prove) but blow up to the future (which is much more difficult to prove). Let us comment briefly on the geometric meaning of this. *Please note that the following discussion is not an important part of this course, it simply serves the purpose of adding some flavour.* If all the N_i are non-zero and have the same sign, the interpretation of the blow up to the future is simply that the spacetime reaches a point of maximal expansion after which it recollapses. The future global existence in the other cases corresponds to infinite expansion of the corresponding cosmological models. The global existence to the past may seem strange since one expects there to be a big bang to the past. In fact, $\tau \to -\infty$ does correspond to a singularity, and a freely falling observer going into the past will reach the singularity in a finite proper time. The time coordinate in the formulation of the above equations has however been chosen to be such that the singularity is infinitely far to the past with respect to the τ-time. There are examples of initial data to (31)-(33) for which the gravitational field remains bounded as $\tau \to -\infty$. This is considered to be a pathological feature of cosmological spacetimes. Consequently, one is interested in analyzing the asymptotic behaviour of solutions in order to determine for what initial data the gravitational fields become arbitrarily strong as one approaches the singularity.

4. Generalizations

The hypothesis of some of the results mentioned above can be weakened. In all the results, the condition that f be continuously differentiable with respect to x can be replaced by a Lipschitz condition: f is said to be *Lipschitz* with respect to x if, for all non-negative K, there is a constant $C < \infty$, depending on K, such that

$$|f(t,x) - f(t,y)| \leq C|x - y|$$

for all t, x, y such that $|t|, |x|, |y| \leq K$. This condition can replace the condition of continuous differentiability with respect to x in Theorems 5, 6 and 7. In order to obtain existence, it is actually not necessary to demand that f be Lipschitz.

THEOREM 8 (Peano's theorem). *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a continuous function. Consider the problem*

$$\begin{aligned} \frac{dx}{dt}(t) &= f[t, x(t)] && (34)\\ x(t_0) &= x_0, && (35) \end{aligned}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Then there is an $\epsilon > 0$ and a continuously differentiable function $x : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n$ satisfying (34) and (35).

We shall not prove this theorem here. If one drops the Lipschitz condition, one does however get a problem with uniqueness. To illustrate this, let

$$x_1(t) = \begin{cases} (2t/3)^{3/2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and let $x_2(t) = 0$. Then both x_1 and x_2 are continuously differentiable functions, and both satisfy

$$\begin{aligned} \frac{dx}{dt} &= x^{1/3} \\ x(0) &= 0. \end{aligned}$$

Note that $x^{1/3}$ is a continuous function from $\mathbb{R}$ to itself.

5. Regularity

In Theorem 5, we only proved the existence of a continuously differentiable solution. If we demand that f be more regular, do we obtain greater regularity of the solution? Say that f is continuously differentiable with respect to t and x. Then it is clear that if x is continuously differentiable solution, then the right hand side of (1) is continuously differentiable. In other words, x is twice continuously differentiable. Similarly, by an induction argument, we can prove that if f is k times continuously differentiable with respect to t and x, then the solution is $k+1$ times continuously differentiable.

CHAPTER 3

Monotone quantities

1. Vector fields and flows

In this chapter we shall carry on the study of ordinary differential equations. We shall restrict our attention to equations of the form

$$\frac{dx}{dt}(t) = f[x(t)] \tag{36}$$

$$x(0) = x_0, \tag{37}$$

i.e. we consider only autonomous systems. The starting point in the analysis of equations of this form is the study of fixed points, i.e. points x_0 for which $f(x_0) = 0$. One considers the derivative of f at the fixed point, computes the eigenvalues and so on and so forth. This study is very important in the analysis of non-linear ODE:s, but carrying out such an analysis here would take us too far away from the purpose of the course. We shall however comment on how the existence of monotone quantities can be used in the analysis. When considering non-linear ODE:s or non-linear wave equations, it is difficult to make progress unless one can find quantities that are conserved or have monotonicity properties. However, in many special situations it is possible to find such quantities and to use them to good effect. We shall see examples of this for ODE:s but also for non-linear wave equations.

Note that there is a geometric interpretation of (36). The function f can be interpreted as a vector field, or as the velocity field of particles; at every point x the velocity of a particle at that point is $f(x)$. A solution of (36) and (37) is then the trajectory of a particle moving in the velocity field. Given a vector field, we can define its flow.

Definition 11. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth (infinitely differentiable). We say that f is a *complete vectorfield* if all solutions to (36) and (37) have maximal existence interval $(-\infty, \infty)$. If f is a complete vectorfield, we define the *flow* of f to be the function $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined as follows: for a given (t, x_0), $\Phi(t, x_0) = x(t)$, where x is the solution to (36) and (37).

We leave it as an exercise for the reader to prove that the flow has the following property:

$$\Phi[t_1, \Phi(t_2, x)] = \Phi(t_1 + t_2, x) \tag{38}$$

for all $t_1, t_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$. The statement is a direct consequence of uniqueness, Theorem 6. Below, we shall define the flow for vectorfields that are not complete. The only difference is that, in general, one has to be careful about the domain of definition of Φ. One can prove that Φ is a smooth function of both its variables. We shall not do so here, but be satisfied with proving that it is a continuous function.

THEOREM 9. *Let f be a complete vectorfield and let $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be its flow. Then Φ is continuous.*

Proof. We start by proving that for a fixed x_0, Φ is continuous for $|t| \leq \epsilon$ and $x \in \bar{B}_1(x_0)$ for some $\epsilon > 0$. We shall refer to this property as *local continuity.* Define

$$C_1 = \sup_{|x-x_0|\leq 2} \left(|f(x)|^2 + \sum_{i=1}^n |\partial_{x^i} f(x)|^2 \right)^{1/2}.$$

Rough control: The first step is to get some rough control of $\Phi(t,x)$. By the proof of Theorem 5, we know that if $x_1 \in \bar{B}_1(x_0)$, then

$$|\Phi(t,x_1) - x_1| \leq 1$$

for $|t| \leq \epsilon_0(C_1) = 1/[2(C_1+1)]$. In other words,

$$|\Phi(t,x_1) - x_0| \leq 2 \tag{39}$$

for $(t,x) \in S$, where

$$S = \{(t,x) : |x_1 - x_0| \leq 1, \quad |t| \leq \epsilon_0(C_1)\}.$$

Local continuity. Let $(t_k, y_k) \in S$ converge to (t,y). We wish to prove that $\Phi(t_k,y_k) \to \Phi(t,y)$. Note that

$$\Phi(s,x) = x + \int_0^s f[\Phi(u,x)]du.$$

For $(s,y_1), (s,y_2) \in S$, $s \geq 0$, we thus get

$$|\Phi(s,y_1) - \Phi(s,y_2)| \leq |y_1 - y_2| + C_1 \int_0^s |\Phi(u,y_1) - \Phi(u,y_2)|du,$$

where we have used (13) and (39). Applying Grönwall's lemma with $C = |y_1 - y_2|$,

$$f(s) = |\Phi(s,y_1) - \Phi(s,y_2)|$$

and $g(s) = C_1$, we conclude that

$$|\Phi(s,y_1) - \Phi(s,y_2)| \leq |y_1 - y_2| \exp[C_1 \epsilon_0(C_1)].$$

Note that this proves that Φ is Lipschitz with respect to x. Above, we have assumed $s \geq 0$, but if $s < 0$, we can proceed similarly, cf. the exercise following Grönwall's lemma. We conclude that

$$\lim_{k\to\infty} |\Phi(t_k,y_k) - \Phi(t_k,y)| = 0.$$

By the continuity of $\Phi(s,y)$ with respect to s for a fixed y,

$$\lim_{k\to\infty} |\Phi(t_k,y) - \Phi(t,y)| = 0.$$

We conclude that Φ is continuous at $(t,y) \in S$. Thus Φ is continuous in S.

Continuity. Let us prove that Φ is continuous at an arbitrary point (t,y). Assume therefore that $(t_k,y_k) \to (t,y)$. Let K be such that

$$\sup_{|s|\leq|t|+1} |\Phi(s,y)| \leq K.$$

Note that the left hand side is bounded, since $\Phi(s, y)$ is continuous in s for a fixed y. Let

$$C_K = \sup_{|x| \leq K+2} \left(|f(x)|^2 + \sum_{i=1}^n |\partial_{x^i} f(x)|^2 \right)^{1/2}$$

and define $\epsilon = \epsilon_0(C_K)$. Due to the local continuity, if $|x_1| \leq K$, then Φ is continuous for $x \in \bar{B}_1(x_1)$ and $|t| \leq \epsilon$. For the sake of convenience, let us assume that $t > 0$. Then there is an integer $l \geq 0$ such that

$$0 \leq t - l\epsilon < \epsilon.$$

Let us make the inductive assumption that

$$\lim_{k \to \infty} \Phi(m\epsilon, y_k) = \Phi(m\epsilon, y) \tag{40}$$

for $m = 0, ..., j \leq l - 1$. For $j = 0$ it is true. Assume it is true for $j \leq l - 1$. Due to (38),

$$\Phi[(j+1)\epsilon, y_k] = \Phi[\epsilon, \Phi(j\epsilon, y_k)].$$

Since we have (40) with $m = j$, we conclude in particular that $|\Phi(j\epsilon, y_k) - \Phi(j\epsilon, y)| \leq 1$ for k large enough. Due to the continuity of Φ for $|t| \leq \epsilon$ and $|x - \Phi(j\epsilon, y)| \leq 1$, we conclude that

$$\lim_{k \to \infty} \Phi[(j+1)\epsilon, y_k] = \lim_{k \to \infty} \Phi[\epsilon, \Phi(j\epsilon, y_k)] = \Phi[\epsilon, \Phi(j\epsilon, y)] = \Phi[(j+1)\epsilon, y],$$

where we have used (38) again in the last step. By induction, we obtain (40) for $m = l$. Since

$$\Phi(t_k, y_k) = \Phi[t_k - l\epsilon, \Phi(l\epsilon, y_k)]$$

and $|t_k - l\epsilon| \leq \epsilon$ for k large enough, we obtain the desired convergence by invoking the local continuity once more. □

The condition of completeness is not necessary. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. Let D_Φ be the set of (t_0, x_0) such that if x is the solution of (36)-(37), then t_0 belongs to the maximal existence interval. If $(t_0, x_0) \in D_\Phi$, we define $\Phi(t_0, x_0) = x(t_0)$. As an immediate consequence of the definition, if $(t_0, x_0) \in D_\Phi$, $t_0 > 0$, then $(t, x_0) \in D_\Phi$ for all $t \in [0, t_0]$ and similarly for $t_0 < 0$. Furthermore, $(0, x_0) \in D_\Phi$ for all $x_0 \in \mathbb{R}^n$. Note that we still have (38) whenever the right hand side is defined.

THEOREM 10. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Then D_Φ is open and $\Phi : D_\Phi \to \mathbb{R}^n$ is continuous.*

Proof. Let $(t_0, x_0) \in D_\Phi$. We need to prove that there is a neighbourhood of (t_0, x_0) contained in D_Φ and that if $(t_k, x_k) \to (t_0, x_0)$, $k \geq 1$, then $\Phi(t_k, x_k) \to \Phi(t_0, x_0)$. Let x be the solution corresponding to the initial data x_0. For convenience, let us assume $t_0 \geq 0$. Let $C < \infty$ be a constant such that $|x(t)| \leq C$ for all $t \in [0, t_0]$. The idea of the proof is to reduce it to the previous theorem. In order to do so, we modify the vectorfield f for $|x| \geq C + 1$, in other words, away from the trajectory of x for $t \in [0, t_0]$. The modified vectorfield f_1 is complete and coincides with f for $|x| \leq C + 1$. Consequently, the corresponding flow Φ_1 is continuous by the previous result. By showing that Φ coincides with Φ_1 in a neighbourhood of (t_0, x_0), we can draw the desired conclusion. Let us write down the technical details. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\phi(x) = 1$ for $|x| \leq C + 1$ and $\phi(x) = 0$

for $|x| \geq C+2$. We shall later in the course prove that there are such functions. Define

$$f_1 = \phi f.$$

Then f_1 is bounded and consequently it is complete (prove this). Let Φ_1 be the associated flow. Note that since $f_1(x) = f(x)$ for $|x| \leq C+1$, the solution x satisfies

$$\frac{dx}{dt}(t) = f_1[x(t)], \quad x(0) = x_0,$$

for $t \in [0, t_0]$. In other words $\Phi(t, x_0) = \Phi_1(t, x_0)$ for $t \in [0, t_0]$. Let $U = \Phi_1^{-1}[B_{C+1}(0)]$. Since Φ_1 is continuous, U is an open subset of $\mathbb{R}^{n+1}$. Note that $S = [0, t_0] \times \{x_0\}$ is contained in U. Since U is open and S is compact, there is an $\epsilon > 0$ such that $S_\epsilon = (-\epsilon, t_0 + \epsilon) \times B_\epsilon(x_0)$ is contained in U (we leave it to the reader to prove this; one needs to use the compactness of S). Note that for $(t, x_1) \in S_\epsilon$,

$$\frac{d\Phi_1}{dt}(t, x_1) = f_1[\Phi_1(t, x_1)] = f[\Phi_1(t, x_1)],$$

since $|\Phi_1(t, x_1)| \leq C+1$ for $(t, x_1) \in S_\epsilon$. Consequently $S_\epsilon \subseteq D_\Phi$ and $\Phi_1 = \Phi$ on S_ϵ. In particular, D_Φ is open. Finally, if $(t_k, x_k) \to (t_0, x_0)$, where $k \geq 1$ and $(t_k, x_k) \in D_\Phi$, then for k large enough, $(t_k, x_k) \in S_\epsilon$ so that for k large enough

$$\Phi(t_k, x_k) = \Phi_1(t_k, x_k) \to \Phi_1(t_0, x_0) = \Phi(t_0, x_0),$$

where we used the continuity of Φ_1. The conclusion of the theorem follows. $\square$

2. Limit sets

In complicated situations, it is impossible to solve (36) and (37) explicitly. However, one is often only interested in the behaviour as $t \to t_{\max}$ or $t \to t_{\min}$, i.e. the asymptotic behaviour, which is an easier problem. In the analysis of the asymptotics, it is convenient to have the concepts of α-limit set and ω-limit set at our disposal.

Definition 12. Consider a solution x to (36) and (37) with maximal existence interval $(t_{\min}, t_{\max})$. We say that x_1 is an *α-limit point* of x if there is a sequence $t_k \in (t_{\min}, t_{\max})$ such that $t_k \to t_{\min}$ and $x(t_k) \to x_1$. Similarly, we say that x_1 is an *ω-limit point* of x if there is a sequence $t_k \in (t_{\min}, t_{\max})$ such that $t_k \to t_{\max}$ and $x(t_k) \to x_1$. The *α-limit set* of x is defined to be the set of α-limit points and the *ω-limit set* of x is defined to be the set of ω-limit points.

Note that if $t_{\max} < \infty$, then the ω-limit set is empty, due to the exercise following Theorem 7. Similarly, if $t_{\min} > -\infty$, the α-limit set is empty. That $t_{\max} = \infty$ is a necessary, but not sufficient condition for the non-emptiness of the ω-limit set is illustrated by the following example.

Example. Let x solve

$$\frac{dx}{dt} = x, \quad x(0) = 1.$$

The solution is of course $x(t) = e^t$, so that the α-limit set consists of the point 0 and the ω limit set is empty. The solution to

$$\frac{dx}{dt} = -x, \quad x(0) = 1$$

has an empty α-limit set and an ω-limit set consisting of the point 0.

Exercise. Consider the equation

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -y \\ x \end{pmatrix} \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} &= \begin{pmatrix} r_0 \\ 0 \end{pmatrix} \end{aligned}$$

where x and y take values in $\mathbb{R}$ and $r_0 \geq 0$. Compute the α-limit set and the ω-limit set depending on r_0.

Note that if x is a solution to (36) and (37) such that there is a sequence $t_k \to \infty$ with the property that $x(t_k)$ is bounded, then the ω-limit set is non-empty. The statement for the α-limit set is similar. The proof is an application of Theorem 2. Note that there are two cases to consider.

Let us prove some general properties of the α-limit set and the ω-limit set.

THEOREM 11. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth, let Φ be the associated flow and let x be a solution to (36)-(37). Then the α-limit sets and the ω-limit sets are closed and invariant under the flow. The last statement means that if x_0 is an α-limit point (ω-limit point) and $(t, x_0) \in D_\Phi$, then $\Phi(t, x_0)$ is an α-limit point (ω-limit point).*

Proof. We only prove the statements for the ω-limit set, the proofs for the α-limit set are similar. Let x be a solution with maximal existence interval (t_-, t_+). If $t_+ < \infty$ the ω-limit set is empty, which means that it is closed. Furthermore, the empty set is invariant under the flow and nothing remains to be proved. Assume therefore that $t_+ = \infty$.

Let y_k be a sequence of ω-limit points converging to a point y. In order to prove that the ω-limit set is closed, all we need to prove is that y is an ω-limit point. Let $l \geq 1$ be a given integer. There is a y_{k_l} such that $|y - y_{k_l}| \leq 1/(2l)$. Furthermore, there is a t_l with $t_l \geq l$ and $|x(t_l) - y_{k_l}| \leq 1/(2l)$ (since y_{k_l} is an ω-limit point). By construction $t_l \to \infty$ and $|y - x(t_l)| \leq 1/l$. We conclude that y is an ω-limit point.

Let us prove that the ω-limit set is invariant under the flow. Let x_0 be an ω-limit point and assume that $(t, x_0) \in D_\Phi$. By definition, there is a sequence $t_k \to \infty$ such that $x(t_k) \to x_0$. Define $s_k = t_k + t$ and note that $s_k \to \infty$. Since $t_+ = \infty$, $[t, x(t_k)] \in D_\Phi$. By the continuity of Φ,

$$\Phi(t, x_0) = \lim_{k\to\infty} \Phi[t, x(t_k)] = \lim_{k\to\infty} x(s_k).$$

Consequently, $\Phi(t, x_0)$ is an ω-limit point. □

Under some circumstances, one can also prove that the ω-limit set is connected. Let us remind the reader of the concept of connectedness.

DEFINITION 13. Two subsets A and B of a metric space X are said to be *separated* if both $\bar{A} \cap B$ and $A \cap \bar{B}$ are empty. A set $E \subseteq X$ is said to be *connected* if E is *not* a union of two non-empty separated sets.

As illustrating examples, $(0, 1)$ and $(1, 2)$ are separated sets, whereas $(0, 1)$ and $[1, 2)$ are not.

THEOREM 12. *Consider a solution x to (36) and (37), where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Assume there is a T and a $C < \infty$ such that $|x(t)| \leq C$ for all $t \geq T$.*

Then the ω-limit set is connected. Similarly, if $|x(t)| \leq C$ for all $t \leq T$, the α-limit set is connected.

Proof. We prove only the statement for the ω-limit set. Denote this set by Ω. By Theorem 11, Ω is a closed set and by the hypothesis of the theorem, Ω is bounded. By the Heine-Borel theorem, we conclude that Ω is compact. Assume that Ω is not connected. Then there are non-empty subsets A and B such that $\Omega = A \cup B$ and $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. Since $A \subset \Omega$ and Ω is closed, $\bar{A} \subset \Omega = A \cup B$. Since no limit point of A can belong to B, we conclude that $A = \bar{A}$. In other words, A is closed. Similarly B is closed. Since Ω is compact, we conclude that A and B are compact. Define

$$d_0 = \inf_{x \in A, y \in B} |x - y|.$$

Note that $g(x, y) = |x - y|$ is a continuous function from $\mathbb{R}^{2n}$ to $\mathbb{R}$. Since $A \times B$ is a compact set, g attains its minimum on $A \times B$. In other words, there are $x_1 \in A$ and $y_1 \in B$ such that $d_0 = |x_1 - y_1|$. Since A and B are disjoint, we conclude that $d_0 > 0$. Define

$$A_0 = \{z \in X : \inf_{x \in A} |z - x| < d_0/4\}, \quad B_0 = \{z \in X : \inf_{x \in B} |z - x| < d_0/4\}.$$

Note that A_0 and B_0 are both open sets. Note also that $\bar{A}_0 \cap \bar{B}_0$ is empty. Since A is non-empty, we can, for every integer $k \geq 0$ find a $t_k \geq k, T$ such that $x(t_k) \in A_0$. Since B is non-empty, x must leave A_0. Let

$$t'_k = \sup_{t \in [t_k, \infty)} \{s : x(s) \in A_0\}.$$

By the comments made, the set over which we take supremum is bounded from above, consequently $T, k \leq t'_k < \infty$. By definition $x(t'_k) \notin A_0$ and $x(t'_k) \in \bar{A}_0$. Since $\bar{B}_0 \cap \bar{A}_0$ is empty, we conclude that $x(t'_k) \in B_0^c \cap A_0^c$. Since $t'_k \to \infty$ and $x(t'_k)$ is a bounded sequence, there is a convergent subsequence, converging to, say, x_0. Since $A_0^c \cap B_0^c$ is closed, $x_0 \in A_0^c \cap B_0^c$. In other words, we have constructed an ω-limit point, x_0, which does not belong to $\Omega \subseteq A_0 \cup B_0$. We have a contradiction to the assumption that the ω-limit set is disconnected. ☐

3. The monotonicity principle

So far, we have established some basic facts concerning the flow and the α- and ω-limit sets of solutions to (36) and (37). In order to be able to say something more, it is very useful to have monotone quantities at one's disposal. One reason for this is the following result, which goes under the name of the monotonicity principle.

THEOREM 13 (Monotonicity principle). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth and consider (36) and (37). Let U be an open subset of $\mathbb{R}^n$ and M a closed subset which is invariant under the flow of the vectorfield f. Assume there is a continuous function $F : U \to \mathbb{R}$ such that $F[x(t)]$ is strictly monotone for any solution x to (36) and (37), as long as $x(t) \in U \cap M$. Then no solution whose image is contained in $U \cap M$ has an α- or ω-limit point in U.*

Remark. That M is *invariant under the flow* simply means that if the initial data belongs to M, then the entire solution is contained in M. A function $g : (a, b) \to \mathbb{R}$ is said to be *strictly monotonically increasing* if $t_1 < t_2$ implies $g(t_1) < g(t_2)$. The concept *strictly monotonically decreasing* is defined similarly and a function is said to be *strictly monotone* if it is either strictly monotonically increasing or decreasing. The reason we include the closed invariant set M in the statement is that we wish to apply it to the cosmology example given in the previous chapter, for which there is a constraint filling the role of M.

Proof. Let x be a solution which is contained in $U \cap M$. Let us prove the statement for the ω-limit set. Assume that there is an ω-limit point $p \in U$. We wish to prove that this leads to a contradiction. By definition there is a sequence $t_k \to \infty$ such that $x(t_k) \to p$. Due to this fact and the fact that F is continuous in U, we conclude that $F[x(t_k)] \to F(p)$. Note also that since M is closed, $p \in M$. By hypothesis, $F[x(t)]$ is strictly monotone. Consequently $F[x(t)] \to F(p)$. In other words, if q is an ω-limit point, $F(q) = F(p)$. Due to Theorem 11, the solution $\bar{x}$ solving (36) with initial value p is contained in the ω-limit set. By the above argument $F[\bar{x}(t)] = F(p)$. On the other hand, there is an $\epsilon > 0$ such that $\bar{x}(t) \in U \cap M$ for $|t| < \epsilon$ so that $F[\bar{x}(t)]$ is strictly monotonically increasing or decreasing for $|t| < \epsilon$. We have a contradiction. $\square$

4. Applications

The rest of this chapter is devoted to applications of the monotonicity principle. We shall apply it to the equations (31)-(33). Let us make some comments concerning the structure of these equations. As we know, due to one of the exercises, the sets $N_1 < 0$, $N_1 = 0$ and $N_1 > 0$ are invariant under the flow and similarly for N_2 and N_3. When analyzing the equations it thus makes sense to start by looking at solutions with $N_i = 0$, $i = 1, 2, 3$, then look at solutions with one N_i non-zero and so on and so forth. Let us look at a solution with all $N_i = 0$. We shall refer to such solutions as type I solutions. By (32) and the constraint (33), $q = 2$ and $S_+ = S_- = 0$. Considering (31), we thus see that all solutions of type I are fixed points; any constant (σ_+, σ_-) with $\sigma_+^2 + \sigma_-^2 = 1$ yields a solution $[\Sigma_+(\tau), \Sigma_-(\tau)] = (\sigma_+, \sigma_-)$. Solutions with one N_i non-zero are called type II solutions. Let us apply the monotonicity principle in order to analyze the asymptotic behaviour of a type II solution.

PROPOSITION 1. *Consider a solution of (31)-(33) with $N_1 > 0$ and $N_2 = N_3 = 0$. Then*

$$\lim_{\tau \to \infty} N_1(\tau) = 0 \tag{41}$$

and there is a $(\sigma_+, \sigma_-) \in \mathbb{R}^2$ with $\sigma_+^2 + \sigma_-^2 = 1$ and $\sigma_+ > 1/2$ such that

$$\lim_{\tau \to \infty} (\Sigma_+, \Sigma_-)(\tau) = (\sigma_+, \sigma_-). \tag{42}$$

Proof. Using the constraint (33) we deduce that

$$\Sigma_+' = \frac{3}{2} N_1^2 (2 - \Sigma_+),$$

(prove this). We wish to apply the monotonicity principle. We view the system as being in $\mathbb{R}^3$. Let U be defined by $N_1 > 0$, M be defined as the set of N_1, Σ_+ and Σ_- satisfying (33) and $F(\Sigma_+, \Sigma_-, N_1) = \Sigma_+$. Note that as long as the solution is in $U \cap M$, $N_1 > 0$ and $|\Sigma_+| \leq 1$ so that $\Sigma_+' > 0$, with the consequence that Σ_+ is strictly monotonically increasing. We can thus apply the monotonicity principle.

Equation (41) follows, since if N_1 does not converge to zero, we can construct an ω-limit point in U due to the fact that the variables are contained in a compact set (prove this). Combining this with the constraint, we deduce

$$\lim_{\tau\to\infty} q = 2.$$

Since Σ_+ is monotone and bounded, it converges. Let us call the limit σ_+. Since $q \to 2$, we conclude that $|\Sigma_-|^2$ converges to $1-\sigma_+^2$. If $|\sigma_+| \neq 1$, the ω-limit set could thus in principle consist of two different points. However, since the ω-limit set is connected by Theorem 12, it consists of only one point. Consequently, Σ_- has to converge. We call the limit σ_- and obtain (42), but we do not yet know anything about the limit (σ_+, σ_-). Compute

$$\left(\frac{\Sigma_-}{2-\Sigma_+}\right)' = 0. \tag{43}$$

Note that this equality can be used to give another proof of the fact that Σ_- has to converge. We get

$$\frac{\Sigma_-}{2-\Sigma_+} = \frac{\sigma_-}{2-\sigma_+} \tag{44}$$

for arbitrary (Σ_+, Σ_-) belonging to the solution. Since $N_1' = (q-4\Sigma_+)N_1$ and $N_1 \to 0$ we have to have $\sigma_+ \geq 1/2$ (why?). If $\sigma_+ = 1/2$ then $\sigma_- = \pm\sqrt{3}/2$. The two corresponding lines in the $\Sigma_+\Sigma_-$-plane obtained by substituting (σ_+, σ_-) into (44) do not intersect any points interior to the Kasner circle. Therefore $\sigma_+ = 1/2$ is not an allowed limit point and the proposition follows. □

Exercise. Add the missing details to the proof above, i.e. fill in the details where we have written "prove this" and "why?". Analyze the asymptotics for solutions to (31)-(33) with $N_1 > 0$ and $N_2 = N_3 = 0$ as $\tau \to -\infty$. Draw a picture of the orbits in the $\Sigma_+\Sigma_-$-plane.

The answer to the above exercise yields conclusions concerning the character of the Big Bang singularities in the corresponding cosmological spacetimes. If $\Sigma_- = 0$ initially, then by (43) it remains zero for the entire solution. For such a solution, one can prove that the gravitational field remains bounded as one approaches the singularity. For all the other solutions, the gravitational field becomes arbitrarily strong.

Consider a solution to (31)-(33) with $N_1 = 0$, $N_2 > 0$ and $N_3 < 0$. Such solutions are said to be of type VI_0.

Exercise. Analyze the asymptotic behaviour of a VI_0 solution as $\tau \to \infty$. *Hint*: Proceed in the following steps. Prove that the solution is contained in a compact set. Prove that Σ_+ can be used as a strictly monotone quantity in the monotonicity principle. Prove that these observations lead to $N_2, N_3 \to 0$. Prove that (Σ_+, Σ_-) has to converge. Prove that there is only one value of the limit which is consistent with $N_2, N_3 \to 0$.

CHAPTER 4

Local existence for $1+1$-dimensional wave equations

1. Introduction

The main object of study in this course is the equation

$$\begin{cases} \Box u = F(u, \partial u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x). \end{cases} \tag{45}$$

In general, we wish to consider this equation in $n+1$ dimensions, i.e. in n spatial and 1 time dimensions. As we mentioned in the introduction, the natural starting point is to prove local existence of solutions to these equations. One question then arises. What type of functions f and g are reasonable to choose as initial data? Let us for the moment restrict our attention to smooth (infinitely differentiable) initial data. It is however not clear that this is enough. Say for instance that f and g become larger and larger as $|x| \to \infty$. Is it then not possible that the equation does not admit local solutions? In fact we shall give an example for which there is no local existence. The example is however such that g grows as $|x| \to \infty$. One convenient class of functions to consider are such that are smooth and vanish outside of a compact set. By considering such a class, one can avoid the problem of the behaviour for large x. However, we need to prove that there are such functions. Furthermore, in order to prove local existence, we set up an iteration, similarly to the ODE case. Consequently, we need to find suitable metric spaces in which the sequence converges. These problems will be the subject of the first two sections.

2. Smooth functions with compact support

Let us start by proving that there are smooth functions with compact support. We shall use the following notation.

Definition 14. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The set of smooth (infinitely differentiable) $f : \Omega \to \mathbb{R}^k$ will be denoted $C^\infty(\Omega, \mathbb{R}^k)$. The set of $f \in C^\infty(\Omega, \mathbb{R}^k)$ such that there is a compact set $K \subset \Omega$ with $f = 0$ for $f \notin K$ is denoted $C_0^\infty(\Omega, \mathbb{R}^k)$. We shall also use the notation $C^\infty(\Omega, \mathbb{R}) = C^\infty(\Omega)$ and $C_0^\infty(\Omega, \mathbb{R}) = C_0^\infty(\Omega)$. An element $f \in C_0^\infty(\Omega, \mathbb{R}^k)$ is called a *smooth function with compact support*. Similarly, if $m \geq 0$, we denote the set of m times differentiable functions by $C^m(\Omega, \mathbb{R}^k)$. The notation $C_0^m(\Omega, \mathbb{R}^k)$, $C_0^m(\Omega)$ and $C^m(\Omega)$ is analogous to the above. Finally, we shall use the notation $C^0(\Omega) = C(\Omega)$, $C_0^0(\Omega) = C_0(\Omega)$ etc.

Let us prove that, given an open set $\Omega \subseteq \mathbb{R}^n$ and a compact set $K \subset \Omega$, there is a function $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ on K. We shall do so in several steps.

LEMMA 2. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases} \tag{46}$$

Then f *is smooth.*

Proof. Note that f is smooth for $t > 0$ and for $t < 0$. The only problem is $t = 0$. Note that for $t > 0$ and any integer $m \geq 0$,

$$e^{-1/t} = \frac{1}{e^{1/t}} = \left(\sum_{k=0}^{\infty} \frac{t^{-k}}{k!}\right)^{-1} \leq \left(\frac{t^{-m}}{m!}\right)^{-1} = m!t^m.$$

Since $f(t) = 0$ for $t \leq 0$, we obtain

$$|f(t)| \leq m!|t|^m \tag{47}$$

for all integers $m \geq 0$ and $t \in \mathbb{R}$. As a consequence, it is clear that f is continuous at 0. Let us turn to the differentiability of f at 0. Estimate, for $h \neq 0$,

$$\left|\frac{f(h) - f(0)}{h}\right| = \left|\frac{f(h)}{h}\right| \leq m!|h|^{m-1}, \tag{48}$$

where we have used (47). We conclude that f is differentiable at zero and the derivative is zero.

Exercise. Prove that f is continuously differentiable. By an induction argument, prove that f is smooth. □

Note that all the derivatives of f are zero at the origin. In other words, if we do a series expansion of f around the origin, all the coefficients in the expansion are zero. However, if $t > 0$, then $f(t) > 0$. In other words, the class of smooth functions is very different from the class of analytic functions we are familiar with from complex analysis.

LEMMA 3. *Let* $\epsilon > 0$. *There is a* $g \in C^\infty(\mathbb{R})$ *such that* $0 \leq g(t) \leq 1$ *for all* $t \in \mathbb{R}$, $g(t) = 0$ *for* $t \leq 0$ *and* $g(t) = 1$ *for* $t \geq \epsilon$.

Proof. Define

$$\phi(t) = f(t)f(\epsilon - t),$$

where f is as in (46). Note that $\phi \in C^\infty(\mathbb{R})$, $\phi \geq 0$, $\phi(\epsilon/2) > 0$, and that $\phi(t) = 0$ for $t \leq 0$ and for $t \geq \epsilon$. Define

$$g(t) = \int_0^t \phi(s)ds \left[\int_0^\epsilon \phi(s)ds\right]^{-1}.$$

Note that the integral in the denominator is non-zero. Then g has the desired properties. □

LEMMA 4. *Let* $x_0 \in \mathbb{R}^n$ *and* $0 < r_1 < r_2$. *Then there is a* $\phi \in C^\infty(\mathbb{R}^n)$ *such that* $\phi(x) = 1$ *for* $x \in \bar{B}_{r_1}(x_0)$, $\phi(x) = 0$ *for* $x \notin B_{r_2}(x_0)$ *and* $0 \leq \phi \leq 1$.

Remark. Due to this lemma, there are functions of the form needed in the proof of Theorem 10.

Proof. Let f be as in (46) and define

$$\phi_1(x) = f[r_2^2 - |x - x_0|^2].$$

Then $\phi_1 \in C^\infty(\mathbb{R}^n)$. Furthermore, $\phi_1(x) = 0$ if $x \notin B_{r_2}(x_0)$ and $\phi_1(x) \geq f(r_2^2 - r_1^2)$ for $x \in \bar{B}_{r_1}(x_0)$, since f is monotonically increasing. Let $\epsilon = f(r_2^2 - r_1^2) > 0$ and let g be a function as in Lemma 3. Define

$$\phi(x) = g[\phi_1(x)].$$

Then ϕ has the desired properties. □

PROPOSITION 2. *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $K \subset \Omega$ be compact. Then there is a $\phi \in C_0^\infty(\Omega)$ such that $\phi(x) = 1$ for $x \in K$ and $0 \leq \phi \leq 1$.*

Proof. For every $x \in K$, let $r_x > 0$ be such that $\bar{B}_{2r_x}(x) \subset \Omega$. There is such an $r_x > 0$ since Ω is open. The collection of sets $B_{r_x}(x)$ for $x \in K$ is an open covering of K. By compactness, there is a finite subcovering $B_{r_i}(x_i)$ $i = 1, ..., m$, where $r_i = r_{x_i}$. Let K_1 be the union of the $\bar{B}_{2r_i}(x_i)$. Since K_1 is a union of a finite number of compact sets, it is compact. Furthermore, it is contained in Ω by construction. Let $\phi_i \in C_0^\infty(\Omega)$ satisfy $\phi_i(x) = 1$ for $x \in \bar{B}_{r_i}(x_i)$ and $\phi_i = 0$ for $x \notin B_{2r_i}(x_i)$. That there are such functions follows from Lemma 4. Define

$$\psi = \sum_{i=1}^m \phi_i.$$

Note that $\psi \in C^\infty(\Omega)$, $\psi(x) = 0$ for $x \notin K_1$ and $\psi(x) \geq 1$ for $x \in K$ (since for each $x \in K$, there is at least one i such that $x \in B_{r_i}(x_i)$). Let $\epsilon = 1$ and let g be the function one obtains as a result of Lemma 3. Define

$$\phi(x) = g[\psi(x)].$$

Then ϕ has the desired properties. □

3. Metrics and function spaces

Before we define the needed function spaces, we need to introduce some notation.

DEFINITION 15. If $\alpha = (\alpha_1, ..., \alpha_n)$, where the $\alpha_i \geq 0$ are integers, we shall say that α is a *multi index*. Define

$$|\alpha| = \alpha_1 + ... + \alpha_n, \quad \alpha! = \alpha_1! \cdot ... \cdot \alpha_n!.$$

If $f \in C^{|\alpha|}(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$, we define

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial(x^1)^{\alpha_1} \cdots \partial(x^n)^{\alpha_n}}.$$

Let us try to find a suitable function space for our initial data. Consider the space $C^\infty(\Omega)$ for an open set Ω. Note that functions in this space need not necessarily be bounded. For instance, $f(x) = 1/x$ defines a perfectly good element of $C^\infty[(0,1)]$ and e^x certainly defines a member of $C^\infty(\mathbb{R})$. As was mentioned in the introduction, initial data that are unbounded are not always desirable, since they can constitute and obstruction even to local existence. One candidate for a function space would then be $C_0^\infty(\Omega)$. However, we shall see that this class does not lead to a complete metric space, at least not if one uses the most primitive choice of metric.

3.1. Finite degree of differentiability. We shall mostly be interested in working with a finite degree of differentiability. By comments made above, in order to obtain local existence, we need to have at least bounded functions. This leads to the following definition.

DEFINITION 16. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $m, k \geq 0$ be integers. Define $C^m_b(\Omega, \mathbb{R}^k)$ to be the set of $f \in C^m(\Omega, \mathbb{R}^k)$ such that for each multi index α with $|\alpha| \leq m$, there is a $C_\alpha < \infty$ such that

$$|\partial^\alpha f(x)| \leq C_\alpha$$

for all $x \in \Omega$. For $f \in C^m_b(\Omega, \mathbb{R}^k)$, define

$$\|f\|_{C^m_b(\Omega,\mathbb{R}^k)} = \sum_{|\alpha|\leq m} \sup_{x\in\Omega} |\partial^\alpha f(x)|.$$

We shall also use the notation $C^0_b(\Omega) = C_b(\Omega)$ etc.

PROPOSITION 3. *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $m, k \geq 0$ be integers. Then the space $[C^m_b(\Omega, \mathbb{R}^k), \|\cdot\|_{C^m_b(\Omega,\mathbb{R}^k)}]$ is a Banach space.*

Proof. We leave it as an exercise to the reader to prove that the space is a normed linear space. If $m = 0$, the result is a consequence of Theorem 4. Let us prove the theorem by induction. Assume the statement is true for $m \geq 0$ and let $\{f_l\}$ be a Cauchy sequence in $C^{m+1}_b(\Omega, \mathbb{R}^k)$. Then $\{f_l\}$ and $\{\partial_i f_l\}$, $i = 1, ..., n$, are Cauchy sequences in $C^m_b(\Omega, \mathbb{R}^k)$. By the induction hypothesis, there are functions $f, g_i \in C^m_b(\Omega, \mathbb{R}^k)$ for $i = 1, ..., n$ such that $f_l \to f$ and $\partial_i f_l \to g_i$ with respect to $\|\cdot\|_{C^m_b(\Omega,\mathbb{R}^k)}$. We wish to prove that $f \in C^{m+1}_b(\Omega, \mathbb{R}^k)$. Let $x \in \Omega$ and let $h \in \mathbb{R}^n$ be such that $\bar{B}_{|h|}(x) \subset \Omega$. Compute

$$\begin{aligned} f(x+h) - f(x) &= \lim_{l\to\infty} [f_l(x+h) - f_l(x)] \\ &= \lim_{l\to\infty} \int_0^1 \partial_t [f_l(x+th)] dt \\ &= \lim_{l\to\infty} \int_0^1 (\partial_i f_l)(x+th) h^i dt = \int_0^1 g_i(x+th) h^i dt. \end{aligned}$$

This proves that f is continuously differentiable in Ω and that $\partial_i f = g_i$. Since $g_i \in C^m_b(\Omega, \mathbb{R}^k)$, we conclude that $f \in C^{m+1}_b(\Omega, \mathbb{R}^k)$. That $f_l \to f$ follows from the above (prove this). □

Let $Y = C^m_b(\mathbb{R}^n, \mathbb{R}^k)$ and let $\|\cdot\| = \|\cdot\|_{C^m_b(\mathbb{R}^n,\mathbb{R}^k)}$. Then the space $C([a, b], Y) = C_b([a, b], Y)$ is a Banach space due to Theorem 4. If $F : \mathbb{R}^{n+1} \to \mathbb{R}^k$ is m times continuously differentiable with respect to the last n variables, we can can view F as a map from $\mathbb{R}$ into $C^m(\mathbb{R}^n, \mathbb{R}^k)$; we have a map $t \mapsto F(t, \cdot)$, where we view $F(t, \cdot)$ as a function of n variables and t as a fixed parameter. If this map is in $C\{[-T, T], Y\}$ for some $T > 0$, we shall write $F \in C\{[-T, T], Y\}$. We shall also use a similar abuse of notation for similar function spaces. The space $C([a, b], Y)$ is a possible candidate for a metric space in which we could obtain convergence of an iteration in the case of a non-linear wave equation. This construction may seem unnatural at first sight, but it is in some sense forced upon us by the equation, cf. the section on estimates. For this reason, the above considerations will be very important in what follows. In the case of $n + 1$-dimensions, we shall encounter similar function spaces, the only difference being that we shall have to replace Y with another function space.

It is important to keep in mind that the above spaces have some perhaps unexpected properties.

PROPOSITION 4. *Let Y and $\|\cdot\|$ be as above. There is a $\phi \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^k)$ such that*

- *For each fixed t, $\phi(t,\cdot) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^k)$.*
- *There is a constant $C < \infty$ such that for all $t \in \mathbb{R}$, $\|\phi(t,\cdot)\| \leq C$.*
- *Let f be defined by $f(t) = \phi(t,\cdot)$. By the above, $f : \mathbb{R} \to Y$. However, $f \notin C([-1,1], Y)$.*

Remark. The purpose of the proposition is to point out that one has to be very careful when dealing with spaces of the form $C([a,b], Y)$. One's intuition might very well be misleading.

Proof. Let $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ satisfy $|\phi_1| \leq 1$, $\phi_1(0) = 1$ and $\phi(x) = 0$ for $|x| \geq 1$. Let

$$\phi_2(t,x) = \begin{cases} \phi_1(x^1 - 1/t, x^2, ..., x^n) & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Note that for $t > 0$ and $t < 0$, ϕ_2 is clearly smooth. For $t = 0$ it is not so clear. However, if $x \in \mathbb{R}^n$, then ϕ_2 is zero in a neighbourhood of $(0,x)$, since $|x^1 - 1/t| \geq 1$ implies $\phi_2(t,x) = 0$. In other words ϕ_2 is smooth in a neighbourhood of $(0,x)$. We conclude that ϕ_2 is smooth. By construction, $\phi_2(t,\cdot)$ has compact support in x for every t. Define

$$\phi = (\phi_2, 0, ..., 0).$$

Then $\phi \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^k)$ and ϕ has the first property of the proposition. The second property follows from the fact that

$$\sup_{x\in\mathbb{R}^n} |(\partial^\alpha \phi_2)(t,x)| = \sup_{x\in\mathbb{R}^n} |(\partial^\alpha \phi_1)(x)| \leq C < \infty$$

for $t > 0$. If $t \leq 0$, the supremum is of course 0. Assume that the third statement is not true. Then $g(t) = \|\phi(t,\cdot)\|$ would be a continuous function from $[-1,1]$ to the real numbers. However, $g(t) = c_0 > 0$ for $t > 0$ and $g(t) = 0$ for $t \leq 0$. □

There is another problem with these spaces which is more serious for our purposes.

PROPOSITION 5. *There is a function $f \in C_b(\mathbb{R})$ such that the function $\psi(t,x) = f(t+x)$ satisfies $\psi \notin C\{[-1,1], C_b(\mathbb{R})\}$.*

Remark. Note that if $f \in C^2(\mathbb{R})$, then ψ is a solution of the homogeneous wave equation $\psi_{tt} - \psi_{xx} = 0$. Furthermore, we can view ψ as a map from $\mathbb{R}$ to $C_b(\mathbb{R})$.

Proof. Let $f(x) = \sin x^2$ and define ψ to be as in the statement of the proposition. Let $k \geq 1$ and define

$$x_k = \left(2k\pi + \frac{\pi}{2}\right)^{1/2}, \quad t_k = [(2k+1)\pi]^{1/2} - x_k.$$

Then $\psi(t_k, x_k) = 0$ and $\psi(0, x_k) = 1$. Consequently

$$\|\psi(t_k,\cdot) - \psi(0,\cdot)\|_{C_b(\mathbb{R})} \geq 1.$$

However, $t_k \to 0$. The proposition follows. □

We shall see that as a consequence of this proposition, it is not meaningful to use $C\{[-\epsilon,\epsilon],C_b^k(\mathbb{R})\}$ as the space in which our iteration converges. In fact, the proposition has the consequence that we cannot even prove that the 0:th iterate is in the right space. We therefore define the following spaces.

DEFINITION 17. Define $C_d^m(\mathbb{R}^n,\mathbb{R}^k)$ to be the set of $f\in C_b^m(\mathbb{R}^n,\mathbb{R}^k)$ such that for every $\epsilon>0$ there is an M such that $|x|\geq M$ implies

$$\sum_{|\alpha|\leq m}\sup_{|x|\geq M}|\partial^\alpha f(x)|\leq\epsilon.$$

We shall also use the notation $C_d^0(\mathbb{R}^n,\mathbb{R}^k)=C_d(\mathbb{R}^n,\mathbb{R}^k)$ etc.

The space $C_d^m(\mathbb{R}^n,\mathbb{R}^k)$ is in other words the subspace of $C_b^m(\mathbb{R}^n,\mathbb{R}^k)$ consisting of functions that tend to zero as x tends to infinity. The letter b signifies that that the functions in the corresponding space are bounded and the d is for decay. Let us note the following fact.

PROPOSITION 6. *The space $C_d^m(\mathbb{R}^n,\mathbb{R}^k)$ with the norm $\|\cdot\|_{C_b^m(\mathbb{R}^n,\mathbb{R}^k)}$ is a Banach space.*

Proof. The only thing we need to prove is completeness. Let f_n be a Cauchy sequence. Due to the completeness of $C_b^m(\mathbb{R}^n,\mathbb{R}^k)$, we know that there is a function $f\in C_b^m(\mathbb{R}^n,\mathbb{R}^k)$ such that $f_n\to f$. The only problem is to prove that f is in the right space. Let $\epsilon>0$. There is an N such that $\|f_N-f\|_{C_b^m(\mathbb{R}^n,\mathbb{R}^k)}<\epsilon/2$. Since f_N is in the right space, there is an M_N such that

$$\sum_{|\alpha|\leq m}\sup_{|x|\geq M_N}|\partial^\alpha f_N(x)|<\frac{\epsilon}{2}.$$

Adding up these facts, we obtain the desired conclusion. □

For our purposes, the following result will be of interest.

PROPOSITION 7. *Assume $f\in C_d(\mathbb{R})$. Let $\psi_\pm(t,x)=f(x\pm t)$. Then $\psi_\pm\in C[\mathbb{R},C_d(\mathbb{R})]$.*

Proof. The idea of the proof is to divide $\mathbb{R}$ into two parts. On one part (for $|x|$ large), $|f(x)|$ is small. On the remaining compact part, f is uniformly continuous. Note that the problem with the counterexample in Proposition 5 is that f is not uniformly continuous.

Since the two cases are similar, so let us consider only ψ_+. Let $t\in\mathbb{R}$ and assume $t_k\to t$. **Let** $\epsilon>0$. There is an M such that $|x|\geq M$ implies $|f(x)|<\epsilon/2$. Let $I=[-M-2,M+2]$. Since I is a compact interval, f is uniformly continuous on I, so that there is a $1>\delta>0$ such that if $x,y\in I$ and $|x-y|<\delta$, then $|f(x)-f(y)|<\epsilon$. Let N be such that for $k\geq N$, $|t_k-t|<\delta$. **Let** $k\geq N$. There are two cases to consider. Assume $|x+t|\leq M+1$. Then $x+t_k\in I$ so that

$$|\psi_+(t_k,x)-\psi_+(t,x)|=|f(t_k+x)-f(t+x)|<\epsilon.$$

If $|x+t|\geq M+1$, then $|x+t_k|\geq M$ so that

$$|\psi_+(t_k,x)-\psi(t,x)|<\epsilon.$$

We conclude that

$$\|\psi_+(t,\cdot)-\psi_+(t_k,\cdot)\|_{C_b(\mathbb{R})}\leq\epsilon.$$

The proposition follows. ☐

We immediately obtain the following consequence.

COROLLARY 1. *Assume* $f \in C_d^m(\mathbb{R})$. *Let* $\psi_\pm(t,x) = f(x \pm t)$. *Then* $\psi_\pm \in C[\mathbb{R}, C_d^m(\mathbb{R})]$.

Later, it will be convenient to have the following result.

LEMMA 5. *Assume* $F \in C[I, C_d^m(\mathbb{R})]$ *for some compact interval* $I = [a,b]$. *Then for every* $\epsilon > 0$ *there is an* M *such that*

$$\sum_{|\alpha| \le m} \sup_{|x| \ge M} |\partial^\alpha F(t,x)| \le \epsilon \tag{49}$$

for all $t \in I$.

Proof. Let $\epsilon > 0$. Since I is compact, F is uniformly continuous. In other words, there is a $\delta > 0$ such that $t, s \in I$ and $|s - t| < \delta$ implies

$$\|F(t,\cdot) - F(s,\cdot)\|_{C_b^m(\mathbb{R})} \le \epsilon/2. \tag{50}$$

Let $l \ge 0$ be such that $l\delta \le b - a < (l+1)\delta$. Define $t_i = a + i\delta$ for $i = 0, ..., l$. Let M_i be such that

$$\sum_{|\alpha| \le m} \sup_{|x| \ge M_i} |\partial^\alpha F(t_i,x)| \le \epsilon/2. \tag{51}$$

Let $M = \max\{M_1, ..., M_l\}$. Then (49) follows from (50) and (51). ☐

3.2. Infinite degree of differentiability. The spaces $C_b^m(\Omega, \mathbb{R}^k)$ are only for a finite degree of differentiability. What about smooth functions? It is possible to something in this case as well, but one does not obtain a Banach space. Let us note that this subsection is not of central importance in the course, but it is sometimes useful to know that one can endow the smooth functions with a complete metric. Let us use the notation

$$C_b^\infty(\Omega, \mathbb{R}^k) = \bigcap_{m \ge 0} C_b^m(\Omega, \mathbb{R}^k).$$

We shall define a complete metric on this space, but we need some preliminaries.

Exercise. Let (X,d) be a metric space. Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that (X, ρ) is a metric space and that $\{x_n\}$ is a Cauchy sequence with respect to d if and only if $\{x_n\}$ is a Cauchy sequence with respect to ρ.

Note that the metric ρ has the property that $\rho(x,y) \le 1$ for all $x, y \in X$. This is the purpose of the construction. For an integer $m \ge 0$ and $f, g \in C_b^m(\Omega, \mathbb{R}^k)$, define

$$d_m(f,g) = \frac{\|f - g\|_{C_b^m(\Omega,\mathbb{R}^k)}}{1 + \|f - g\|_{C_b^m(\Omega,\mathbb{R}^k)}}.$$

For $f, g \in C_b^\infty(\Omega, \mathbb{R}^k)$, we define

$$d_\infty(f,g) = \sum_{m=0}^{\infty} 2^{-m} d_m(f,g).$$

Note that since $d_m(f,g) \leq 1$, the series is convergent.

Exercise. Prove that d_∞ is a metric.

PROPOSITION 8. *Let $X = C_b^\infty(\Omega, \mathbb{R}^k)$. Then (X, d_∞) is a complete metric space.*

Remark. As opposed to the case with a finite degree of differentiability, X is *not* a Banach space.

Proof. We shall prove that d_∞ is complete. Let $\{f_l\}$ be a Cauchy sequence. As a consequence, $\{f_l\}$ is a Cauchy sequence with respect to d_m for every $m \geq 0$ and thus with respect to $\|\cdot\|_{C_b^m(\Omega,\mathbb{R}^k)}$. Due to Proposition 3, we conclude that there is a $g_m \in C_b^m(\Omega, \mathbb{R}^k)$ such that $f_l \to g_m$ with respect to $\|\cdot\|_{C_b^m(\Omega,\mathbb{R}^k)}$. Since $f_l \to g_m$ with respect to $\|\cdot\|_{C_b(\Omega,\mathbb{R}^k)}$ for all $m \geq 0$, we conclude that all the g_m coincide. Let $f = g_0$. Since all the g_m coincide, $f \in X$ and for any $m \geq 0$, $f_l \to f$ with respect to $\|\cdot\|_{C_b^m(\Omega,\mathbb{R}^k)}$. For a fixed m and a fixed $\epsilon > 0$, there is thus an N such that for $l \geq N$,

$$d_m(f_l, f) < \epsilon.$$

Let us prove that f_l converges to f with respect to d_∞. Let $\epsilon > 0$. Then there is an M such that $2^{-M} < \epsilon/2$. Let N be such that $d_m(f_l, f) < \epsilon/4$ for all $m = 0, ..., M$ and all $l \geq N$. Let $l \geq N$ and estimate

$$d_\infty(f_l, f) = \sum_{m=0}^{\infty} 2^{-m} d_m(f, f_l) \leq \sum_{m=0}^{M} 2^{-m}\frac{\epsilon}{4} + \sum_{m=M+1}^{\infty} 2^{-m} \leq \frac{\epsilon}{2} + 2^{-M} < \epsilon,$$

where we have used the fact that $d_m(f, f_l) \leq 1$. $\square$

In a similar fashion, one can construct a metric that turns $C^\infty(\Omega, \mathbb{R}^k)$ into a complete metric space. Note that $Y = C_0^\infty(\mathbb{R}^n, \mathbb{R}^k)$ is a subspace of $X = C_b^\infty(\mathbb{R}^n, \mathbb{R}^k)$. However, Y is not a closed subspace. In other words, if $f_l \in Y$ converges to $f \in X$ with respect to d_∞, then f need not be in Y. To prove this, let $f(x) = \exp(-|x|^2)$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ equal 1 for $|x| \leq 1$ and 0 for $|x| \geq 2$. That there is such a function follows from Lemma 4. Define, for $R > 0$,

$$\phi_R(x) = \phi(x/R).$$

Then ϕ_R equals 1 for $|x| \leq R$ and 0 for $|x| \geq 2R$. Define, for $l = 1, 2, ...$,

$$f_l = \phi_l f.$$

Exercise. Prove that f_l converges to f with respect to d_∞. *Hint*: By using the ideas of the proof of Proposition 8, it is enough to prove that f_l converges to f with respect to $\|\cdot\|_{C_b^m(\mathbb{R}^n,\mathbb{R}^k)}$ for every fixed $m \geq 0$.

As a consequence of the above exercise, we see that (Y, d_∞) is not a complete metric space.

4. The wave equation in $1 + 1$ dimensions

The linear wave equation in $1 + 1$-dimensions is simply the equation $u_{tt} - u_{xx} = 0$. In order to be able to solve non-linear equations, we however need to consider the more general problem of an inhomogeneous wave equation:

$$\begin{cases} u_{tt} - u_{xx} = F \\ u(t_0, x) = f(x) \\ u_t(t_0, x) = g(x). \end{cases} \tag{52}$$

Here F is a given function of t and x and f and g are given functions of x. We shall be more specific concerning the regularity conditions on these functions below, but in the meantime, let us assume that f, g and F are smooth functions. Is it possible to solve (52)? The answer is yes, and we shall now write down the solution.

PROPOSITION 9. *Let $f \in C^{k+1}(\mathbb{R})$, $g \in C^k(\mathbb{R})$ and $F \in C^k[(T_-, T_+) \times \mathbb{R}]$ for some $k \geq 1$ and $T_-, T_+ \in \mathbb{R}$. Assuming $t_0 \in (T_-, T_+)$, there is a unique $u \in C^{k+1}[(T_-, T_+) \times \mathbb{R}]$ solving (52). It is given by*

$$
\begin{aligned}
u(t,x) &= \frac{1}{2}[f(x+t-t_0) + f(x-t+t_0)] + \frac{1}{2}\int_{x-t+t_0}^{x+t-t_0} g(s)ds \\
&\quad + \frac{1}{2}\int_{t_0}^{t}\left[\int_{x+s-t}^{x+t-s} F(s,v)dv\right] ds.
\end{aligned} \tag{53}
$$

Proof. Let us start by assuming we have a solution to (52). Define

$$h_-(s) = (u_t - u_x)(s, x_0 + s), \tag{54}$$

where we consider x_0 to be a parameter. The curve $(s, x_0 + s)$ is an example of a *characteristic* and in physical terms it describes a line along which light travels, assuming we are in 2-dimensional Minkowski space. The reason we consider the above expression is that

$$\frac{dh_-}{ds} = (u_{tt} + u_{tx} - u_{xt} - u_{xx})(s, x_0 + s) = (u_{tt} - u_{xx})(s, x_0 + s) = F(s, x_0 + s).$$

Note that since u is at least C^2, we are allowed to differentiate h_-. In other words, dh_-/ds can be expressed purely in terms of known quantities. Integrating this equality, we obtain

$$
\begin{aligned}
(u_t - u_x)(t, x_0 + t) &= h_-(t) = (u_t - u_x)(t_0, x_0 + t_0) + \int_{t_0}^{t} F(s, x_0 + s)ds \\
&= (g - \partial_x f)(x_0 + t_0) + \int_{t_0}^{t} F(s, x_0 + s)ds,
\end{aligned}
$$

where the last equality is due to the initial conditions. Inserting $x_0 = x - t$ in this equality, we obtain

$$(u_t - u_x)(t,x) = (g - \partial_x f)(x - t + t_0) + \int_{t_0}^{t} F(s, x + s - t)ds. \tag{55}$$

In analogy with the above, let us define

$$h_+(s) = (u_t + u_x)(s, x_0 - s). \tag{56}$$

The curve $(s, x_0 - s)$ is another characteristic in 2-dimensional Minkowski space. By a similar computation, we obtain

$$(u_t + u_x)(t,x) = (g + \partial_x f)(x + t - t_0) + \int_{t_0}^{t} F(s, x + t - s)ds. \tag{57}$$

Note that in this case, we put $x_0 = x + t$. Adding up the equations (57) and (55), we obtain

$$
\begin{aligned}
u_t(t,x) &= \frac{1}{2}[(g - \partial_x f)(x - t + t_0) + (g + \partial_x f)(x + t - t_0)] \\
&\quad + \frac{1}{2}\left[\int_{t_0}^{t} [F(s, x + t - s)ds + F(s, x + s - t)]ds\right].
\end{aligned} \tag{58}
$$

Integrating this equality, we obtain

$$\begin{aligned} u(t,x) &= f(x) + \frac{1}{2}\int_{t_0}^{t} [(g - \partial_x f)(x - s + t_0) + (g + \partial_x f)(x + s - t_0)]ds \\ &\quad + \frac{1}{2}\int_{t_0}^{t} \left[\int_{t_0}^{v} [F(s, x + v - s) + F(s, x + s - v)]ds \right] dv. \end{aligned}$$

Note that

$$\int_{t_0}^{t} [\partial_x f(x + s - t_0) - \partial_x f(x - s + t_0)]ds = f(x + t - t_0) + f(x - t + t_0) - 2f(x).$$

Furthermore, by suitable changes of variable,

$$\int_{t_0}^{t} [g(x - s + t_0) + g(x + s - t_0)]ds = \int_{x-t+t_0}^{x+t-t_0} g(s)ds.$$

Finally, by changing the order of integration and then changing variables, one obtains

$$\int_{t_0}^{t} \left[\int_{t_0}^{v} [F(s, x + v - s) + F(s, x + s - v)]ds \right] dv = \int_{t_0}^{t} \left[\int_{x+s-t}^{x+t-s} F(s, v)dv \right] ds.$$

We conclude that (53) holds. This equality proves uniqueness, since the right hand side only depends on the given functions. To prove existence, define u by (53).

Exercise. Prove that u defined by (53) is a $C^2[(T_-, T_+) \times \mathbb{R}]$-solution to (52).

We need to prove that the solution is in $C^{k+1}[(T_-, T_+) \times \mathbb{R}]$. Since we already know that the solution is in C^2, we can carry out the derivation leading to (55) and (57). From these equations and the assumptions, it is clear that u is in the desired space. $\square$

The essential idea of the above proof is to consider $(u_t - u_x)$ along the characteristic $(s, x_0 + s)$ and $(u_t + u_x)$ along the characteristic $(s, x_0 - s)$. The reason we consider these objects is of course that dh_-/ds and dh_+/ds yield $u_{tt} - u_{xx}$, which in the case of the inhomogeneous wave equation is a known function. This is a special technique which is only available in $1 + 1$ dimensions, and thanks to this technique much more can be said in $1 + 1$ dimensions than in the general case. It will play an essential role in the results of this chapter.

The formula (53) is very interesting due to the fact that it says something about the propagation of information. Let $(t, x) \in \mathbb{R}^2$ with $t > t_0$. Then $u(t, x)$ only depends on f and g in the interval $[x - t + t_0, x + t - t_0]$ and on F in the triangle with base $\{0\} \times [x - t + t_0, x + t - t_0]$ and vertex (t, x). If we start at $t = t_0$, no information outside of this region can affect the value of u at (t, x). In other words, information cannot propagate at a speed higher than 1. This is consistent with special relativity where we have set the speed of light equal to 1. Let us write down some consequences. Assume we have two solutions u_i, $i = 1, 2$ to (52) corresponding to initial data f_i, g_i, $i = 1, 2$ at $t_0 = 0$ respectively. Assume that $f_1 = f_2$ and $g_1 = g_2$ in an interval $[x - |t|, x + |t|]$. Then $u_1(t, x) = u_2(t, x)$. In particular, if u is a solution to (52), where $t_0 = 0$ and $F = 0$, and $f(x) = g(x) = 0$ for $|x| \geq C$, then $u(t, x) = 0$ for $|x| \geq C + |t|$.

Let us contrast the above properties of the wave equation with the heat equation. Consider the equation

$$\begin{cases} u_t - \Delta u = 0 \\ u(0,x) = f(x), \end{cases}$$

where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Let $f \in C_0^\infty(\mathbb{R}^n)$ and assume that $f \geq 0$, $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| \geq 2$. The solution is given by

$$u(t,x) = (4\pi t)^{-n/2} \int \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)dy.$$

We shall not prove this statement here. We refer instead to Fritz John's book on Partial Differential Equations. Note that if $t = 0$, then $u(t,x) = 0$ for $|x| \geq 2$. However, if $t > 0$, then $u(t,x) > 0$ for all x. The information that $u(0,x) \neq 0$ for $|x| \leq 1$ has thus propagated arbitrarily far in an arbitrarily short time. In the case of the heat equation information is in other words allowed to travel with infinite speed.

5. Domain of dependence

Due to the propagation of information, there are certain sets that appear again and again. For this reason it is convenient to introduce some terminology for them.

Definition 18. Consider an interval $I = [a,b]$. Assuming $|t - t_0| \leq (b-a)/2$, define

$$I_{t_0,t} = [a + |t-t_0|, b - |t-t_0|].$$

If $t_0 \leq t_1$ and $|t_1 - t_0| \leq (b-a)/2$, we define

$$\mathcal{D}_{I,t_0,t_1} = \{(s,x) : s \in [t_0,t_1], x \in I_{t_0,s}\}.$$

Similarly, if $t_0 \geq t_1$ and $|t_1 - t_0| \leq (b-a)/2$, we define

$$\mathcal{D}_{I,t_0,t_1} = \{(s,x) : s \in [t_1,t_0], x \in I_{t_0,s}\}.$$

We shall take it to be understood that if we speak of $I_{t_0,t}$, then $|t-t_0| \leq (b-a)/2$. The observations concerning the propagation of information made in the previous section can be summarized by saying that information concerning the initial data in an interval I for $t = t_0$ and concerning the function F in $\mathcal{D}_{I,t_0,t_1}$ determines the solution in $\{t_1\} \times I_{t_0,t_1}$

It will be convenient to say that we have a solution on a set of the form $\mathcal{D}_{I,t_0,t_1}$. Since this set is closed, we do however need to be careful. We say that $u \in C^k(\mathcal{D}_{I,t_0,t_1})$ if it is C^k in the interior of $\mathcal{D}_{I,t_0,t_1}$ and all derivatives up to order k can be extended to be continuous functions on all of $\mathcal{D}_{I,t_0,t_1}$.

6. Estimates

In order to prove local existence for non-linear wave equations we carry out an iteration exactly as in the ODE case. In order to prove that the iteration converges, we need to develop some tools. Let us define the following quantity:

$$\mathcal{E}[u](t) = \|(u_t - u_x)(t,\cdot)\|_{C_b(\mathbb{R})} + \|(u_t + u_x)(t,\cdot)\|_{C_b(\mathbb{R})}. \tag{59}$$

Here we assume that $u_t(t,\cdot), u_x(t,\cdot) \in C_b(\mathbb{R})$ so that the right hand side is finite. We shall also use the notation $\mathcal{E}(t)$ when the function is understood from the context.

What is the reason for considering this object? In the proof of Proposition 9 we saw the use of the quantities $u_t - u_x$ and $u_t + u_x$. Let us give another illustration of their importance. Assume u is a solution to (52) with $F = 0$. Assume furthermore that f_x and g are bounded. By the arguments presented in the proof of Proposition 9, h_+ and h_- are constant. In particular,

$$(u_t - u_x)(t, x_0 + t) = (g - f_x)(x_0 + t_0), \quad (u_t + u_x)(t, x_0 - t) = (g + f_x)(x_0 - t_0).$$

As a consequence,

$$\|(u_t - u_x)(t, \cdot)\|_{C_b(\mathbb{R})} = \|g - f_x\|_{C_b(\mathbb{R})}, \quad \|(u_t + u_x)(t, \cdot)\|_{C_b(\mathbb{R})} = \|g + f_x\|_{C_b(\mathbb{R})}$$

In other words, we see that $\mathcal{E}$ defined by (59) is a conserved quantity for a solution to the wave equation. Note also that it bounds the sup norm of $|u_t| + |u_x|$. In general, we shall need to control more derivatives, and therefore, we need to define

$$\mathcal{E}_j[u](t) = \|(\partial_x^j \partial_t u - \partial_x^{j+1} u)(t, \cdot)\|_{C_b(\mathbb{R})} + \|(\partial_x^j \partial_t u + \partial_x^{j+1} u)(t, \cdot)\|_{C_b(\mathbb{R})}. \tag{60}$$

When it is clear what function is intended, we shall also write $\mathcal{E}_j$. The $\mathcal{E}_j$ defined above are naturally associated with the equation. In particular, for a solution to the wave equation, $\mathcal{E}_j[u]$ is conserved. They do however have one drawback: they do not give immediate control over the function u itself. We therefore define the following quantity:

$$E_k[u](t) = \sum_{j=0}^{k} \mathcal{E}_j[u](t) + \|u(t, \cdot)\|_{C_b(\mathbb{R})}. \tag{61}$$

When it is clear what function is intended, we shall also write E_k.

In order to prove the convergence of the iteration, we need to have estimates for E_k in situations where u is a solution to (52) with $F \neq 0$.

LEMMA 6. *Let $f \in C_d^{k+1}(\mathbb{R})$, $g \in C_d^k(\mathbb{R})$ and $F \in C^k[(T_-, T_+) \times \mathbb{R}]$ for some $k \geq 1$ and $T_-, T_+ \in \mathbb{R}$ such that $t_0 \in (T_-, T_+)$. Assume $F \in C[(T_-, T_+), C_d^k(\mathbb{R})]$. Then if u is the solution to (52),*

$$u \in C[(T_-, T_+), C_d^{k+1}(\mathbb{R})] \quad \textit{and} \quad \partial_t u \in C[(T_-, T_+), C_d^k(\mathbb{R})]. \tag{62}$$

Furthermore, for $j = 0, ..., k$,

$$\mathcal{E}_j(t) \leq \mathcal{E}_j(t_0) + 2 \left| \int_{t_0}^{t} \|\partial_x^j F(s, \cdot)\|_{C_b(\mathbb{R})} ds \right| \tag{63}$$

and

$$\begin{aligned} E_k(t) \quad &\leq \quad E_k(t_0) + 2 \sum_{j=0}^{k} \left| \int_{t_0}^{t} \|\partial_x^j F(s, \cdot)\|_{C_b(\mathbb{R})} ds \right| \\ &\quad + \frac{1}{2} \mathcal{E}(t_0) |t - t_0| + \int_{t_0}^{t} \left[\int_{t_0}^{s} \|F(u, \cdot)\|_{C_b(\mathbb{R})} du \right] ds. \end{aligned} \tag{64}$$

Remark. The last row in the estimate (64) arises due to the fact that we want to estimate $\|u(t, \cdot)\|_{C_b(\mathbb{R})}$.

Proof. Let us prove (62), starting with u_t. Consider the expression (58) (note that this expression holds since u is a C^2 solution). That the first line on the right hand side defines a function in the right space follows from Corollary 1. What remains is the integral. The integrand is a sum of two terms, but since the terms are similar, we consider only one of them. Let $t_1 \in (T_-, T_+)$ and consider

$$\int_{t_0}^{t_1} F(s, x + t_1 - s)ds - \int_{t_0}^{t} F(s, x + t - s)ds.$$

We wish to prove that this object converges to zero uniformly in x as $t \to t_1$. Due to the continuity properties of F, there is a $\delta > 0$ and an M such that

$$\|F(t, \cdot)\|_{C_b(\mathbb{R})} \leq M$$

for all $t \in [t_1 - \delta, t_1 + \delta] \subset (T_-, T_+)$. For $t \in [t_1 - \delta, t_1 + \delta]$, we thus obtain

$$\left| \int_{t_1}^{t} F(s, x + t - s)ds \right| \leq M|t - t_1|,$$

which clearly converges to zero uniformly in x. Consider

$$\left| \int_{t_0}^{t_1} [F(s, x + t - s) - F(s, x + t_1 - s)]ds \right|. \tag{65}$$

Due to Lemma 5, for every $\epsilon > 0$, there is an M such that $|F(s, x)| \leq \epsilon$ for all $|x| \geq M$ and all s in the interval defined by t_0 and t_1. For $|x| \leq M$ and s in the interval defined by t_0 and t_1, F is uniformly continuous. These two observations can be combined to prove that (65) converges to zero uniformly in x as $t \to t_1$. Note that we can differentiate (58) $j \leq k$ times with respect to x and use the same argument to prove continuity in t. We conclude that

$$\partial_t u \in C[(T_-, T_+), C_d^k(\mathbb{R})].$$

Integrating this, we obtain

$$u \in C[(T_-, T_+), C_d^k(\mathbb{R})].$$

In order to get the last derivative, observe that (55) and (57) can be combined to yield an expression for u_x similar to the expression for u_t. By an argument similar to the argument for u_t, we obtain

$$\partial_x u \in C[(T_-, T_+), C_d^k(\mathbb{R})].$$

This proves (62).

Consider (55). Differentiating this equality $j \leq k$ times and taking the supremum, we obtain

$$\|(\partial_x^j \partial_t u - \partial_x^{j+1} u)(t, \cdot)\|_{C_b(\mathbb{R})} \leq \|(\partial_x^j \partial_t u - \partial_x^{j+1} u)(t_0, \cdot)\|_{C_b(\mathbb{R})} + \left| \int_{t_0}^{t} \|\partial_x^j F(s, \cdot)\|_{C_b(\mathbb{R})} ds \right|$$

Starting with (57), we obtain a similar estimate. Adding the two estimates, we obtain (63). Finally, note that

$$u(t, x) = u(t_0, x) + \int_{t_0}^{t} u_t(s, x)ds,$$

so that

$$\|u(t, \cdot)\|_{C_b(\mathbb{R})} \leq \|u(t_0, \cdot)\|_{C_b(\mathbb{R})} + \frac{1}{2} \left| \int_{t_0}^{t} \mathcal{E}(s)ds \right|,$$

since $2|u_t(s, x)| \leq \mathcal{E}(s)$. We can use (63) for $j = 0$ in order to estimate the right hand side. Adding the resulting estimate to the sum of (63) for $j = 0, ..., k$, we obtain (64). $\square$

7. Localized versions of the estimates

Sometimes it is of interest to have local versions of the above estimates. Let $I = [a, b]$ be a compact subinterval of $\mathbb{R}$. Recall the terminology of Definition 18.

(66) $\mathcal{E}_{I,t_0,j}[u](t) = \|(\partial_x^j \partial_t u - \partial_x^{j+1} u)(t, \cdot)\|_{C_b(I_{t_0,t})} + \|(\partial_x^j \partial_t u + \partial_x^{j+1} u)(t, \cdot)\|_{C_b(I_{t_0,t})}.$

Similarly, we define

$$E_{I,t_0,k}[u](t) = \sum_{j=0}^{k} \mathcal{E}_{I,t_0,j}[u](t) + \|u(t, \cdot)\|_{C_b(I_{t_0,t})}. \tag{67}$$

We shall also use the notation $\mathcal{E}_{I,t_0}$ instead of $\mathcal{E}_{I,t_0,0}$ and the notation E_{I,t_0} instead of $E_{I,t_0,0}$.

Lemma 7. *Let $I = [a, b]$ be a compact subinterval of $\mathbb{R}$ and assume that $f \in C^{k+1}(I)$, $g \in C^k(I)$, $F \in C^k(\mathcal{D}_{I,t_0,t_1})$ and that $u \in C^{k+1}(\mathcal{D}_{I,t_0,t_1})$ is a solution of (52). Then*

$$\mathcal{E}_{I,t_0,j}(t) \le \mathcal{E}_{I,t_0,j}(t_0) + 2 \left| \int_{t_0}^{t} \|\partial_x^j F(s, \cdot)\|_{C_b(I_{t_0,s})} ds \right| \tag{68}$$

and

$$\begin{aligned} E_{I,t_0,k}(t) \quad &\le \quad E_{I,t_0,k}(t_0) + 2 \sum_{j=0}^{k} \left| \int_{t_0}^{t} \|\partial_x^j F(s, \cdot)\|_{C_b(I_{t_0,s})} ds \right| \\ &\quad + \frac{1}{2} \mathcal{E}_{I,t_0}(t_0)|t - t_0| + \int_{t_0}^{t} \left[\int_{t_0}^{s} \|F(u, \cdot)\|_{C_b(I_{t_0,u})} du \right] ds. \end{aligned} \tag{69}$$

Remark. In Lemma 6 it was necessary to have $F \in C[(T_-, T_+), C_d^{k+1}(\mathbb{R})]$. No analogous condition is necessary in the present lemma for the simple reason that we are taking supremum over a compact set.

Proof. The proof is for all practical purposes identical to the proof of Lemma 6; one only needs to be careful concerning the intervals over which one takes the supremum. □

8. Uniqueness

Let us illustrate how one can use Lemma 7 in order to prove uniqueness.

Theorem 14. *Let $I = [a, b]$ be a compact subinterval of $\mathbb{R}$. Assume that $F \in C^1(\mathbb{R}^3)$ and that $u_i \in C^2(\mathcal{D}_{I,t_0,t_1})$, $i = 1, 2$ are solutions to the equation*

$$u_{tt} - u_{xx} = F(u, \partial u).$$

Then, if $u_1(t_0, x) = u_1(t_0, x)$ and $\partial_t u_1(t_0, x) = \partial_t u_2(t_0, x)$ for $x \in I$, we have $u_1(t, x) = u_2(t, x)$ for all $(t, x) \in \mathcal{D}_{I,t_0,t_1}$.

Proof. Let $u = u_1 - u_2$. By (69), we have for $t \ge t_0$,

$$E_{I,t_0}[u](t) \le 2 \int_{t_0}^{t} \|\hat{F}(s, \cdot)\|_{C_b(I_{t_0,s})} ds + \int_{t_0}^{t} \left[\int_{t_0}^{s} \|\hat{F}(u, \cdot)\|_{C_b(I_{t_0,u})} du \right] ds, \tag{70}$$

where $\hat{F} = F(u_1, \partial u_1) - F(u_2, \partial u_2)$. Note that

$$|F(u_1, \partial u_1) - F(u_2, \partial u_2)| \le C[|u| + |\partial u|] \tag{71}$$

in $\mathcal{D}_{I,t_0,t_1}$ since u and ∂u are bounded on this set and F is C^1, cf. (85) below. Combining this observation with (70), we conclude that

$$E_{I,t_0}[u](t) \leq C\int_{t_0}^{t} E_{I,t_0}[u](s)ds + C\int_{t_0}^{t}\left[\int_{t_0}^{s} E_{I,t_0}[u](v)dv\right]ds. \tag{72}$$

We wish to apply Grönwall's lemma, but this is not immediately possible, since we have a double integral. There is however a simple remedy. Define, for $s \in [t_0, t_1]$,

$$h(s) = \sup_{u\in[t_0,s]} E_{I,t_0}(u).$$

Since the right hand side of (72) is monotonically increasing with t, we obtain

$$h(t) \leq C\int_{t_0}^{t} h(s)ds + C\int_{t_0}^{t} |t_1 - t_0| h(s)ds.$$

Now we can apply Grönwall's lemma in order to conclude that $h(s) = 0$ for all $s \in [t_0, t_1]$. The desired conclusion follows. If $t_1 \leq t_0$, the argument is similar. $\square$

Due to this result it makes sense to speak of a maximal existence interval for C^2-solutions to the equation. We could also define a maximal existence interval for C^{k+1}-solutions for $k \geq 1$. Note that these intervals could be different for different k. In principle, a solution could cease to be C^3 but remain C^2 with the consequence that the maximal existence interval for C^3-solutions would be shorter than the existence interval for C^2-solutions.

9. Local existence for $1+1$ non-linear wave equations

Let us consider an equation of the form

$$\begin{cases} u_{tt} - u_{xx} = F(u, \partial u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases} \tag{73}$$

where F is a smooth function with the property that $F(0,0) = 0$ and we shall specify the regularity properties of f and g later. We wish to prove that we have local existence. Note that we have taken $t_0 = 0$ above. We can do so without loss of generality since if u is a solution to (73), then $v(t,x) = u(t-t_0, x)$ is a solution to (73) with the initial time 0 replaced by t_0. We define a sequence of approximations in the following way. Let u_0 be defined by

$$\begin{cases} \partial_t^2 u_0 - \partial_x^2 u_0 = 0 \\ u_0(0,x) = f(x) \\ \partial_t u_0(0,x) = g(x). \end{cases} \tag{74}$$

We then define u_n for $n \geq 1$ by

$$\begin{cases} \partial_t^2 u_n - \partial_x^2 u_n = F(u_{n-1}, \partial u_{n-1}) \\ u_n(0,x) = f(x) \\ \partial_t u_n(0,x) = g(x). \end{cases} \tag{75}$$

We wish to prove that there is a $k \geq 1$ and an $\epsilon > 0$ such that u_n converges in $C\{[-\epsilon,\epsilon], C_d^{k+1}(\mathbb{R})\}$ and that $\partial_t u_n$ converges in $C\{[-\epsilon,\epsilon], C_d^k(\mathbb{R})\}$. In order to do so, we need to prove that the iterates are in the right spaces.

LEMMA 8. *Assume that $f \in C_d^{k+1}(\mathbb{R})$ and that $g \in C_d^k(\mathbb{R})$ for some $k \geq 1$. Then (74) and (75) defines a sequence $u_n \in C^{k+1}(\mathbb{R}^2)$, $n = 0, \ldots$. Furthermore,*

$$u_n \in C[\mathbb{R}, C_d^{k+1}(\mathbb{R})] \quad and \quad \partial_t u_n \in C[\mathbb{R}, C_d^k(\mathbb{R})]. \tag{76}$$

Proof. To start with, let us prove that the $u_n \in C^{k+1}(\mathbb{R}^2)$. For u_0 this follows from Proposition 9. Assume it is true for u_n. Then $F(u_n, \partial u_n)$ is in $C^k(\mathbb{R}^2)$, so that the statement follows for u_{n+1} by another application of Proposition 9. That (76) holds for $n = 0$ is a consequence of Lemma 6. Assume it is true for n. In order to prove the statement for $n + 1$, all we need to do is to prove that

$$F(u_n, \partial u_n) \in C[\mathbb{R}, C_d^k(\mathbb{R})], \tag{77}$$

due to Lemma 6. Let us start by proving that $[F(u_n, \partial u_n)](t, \cdot) \in C_d^k(\mathbb{R})$ for every fixed t. For $k = 0$, this follows since F is smooth, $F(0, 0) = 0$ and $u_n, \partial u_n$ tend to zero as $|x|$ tends to infinity. For $k \geq 1$, it follows from the fact that $\partial_x^j[F(u_n, \partial u_n)]$ can be written as a sum of terms of the form

$$(\partial_{z^1}^{j_1}\partial_{z^2}^{j_2}\partial_{z^3}^{j_3}F)(u_n, \partial u_n)\partial_x^{l_1}u_n \cdots \partial_x^{l_m}u_n\partial_x^{p_1}\partial_t u_n \cdots \partial_x^{p_o}\partial_t u_n, \tag{78}$$

where we have denoted the variables upon which F depends z^i, $i = 1, 2, 3$. Here $l_i, p_i \geq 1$. Note however that m or o could be zero, but if $k \geq 1$, one of them has to be non-zero. Since $l_i \leq k + 1$ and $p_i \leq k$ all the terms $\partial_x^{l_i}u_n$ and $\partial_x^{p_i}\partial_t u_n$ converge to zero as $|x|$ tends to infinity. Furthermore, since u_n and $\partial_t u_n$ are bounded for a fixed t, the expression involving derivatives of F is bounded. Consequently, the expression (78) is in $C_d(\mathbb{R})$ for every fixed t. We conclude that $[F(u_n, \partial u_n)](t, \cdot) \in C_d^k(\mathbb{R})$ for every fixed t. We need to prove continuity in t. Since F is smooth and $u_n, \partial u_n$ is uniformly bounded on sets of the form $[t_0 - \epsilon, t_0 + \epsilon] \times \mathbb{R}$, the continuity follows from the continuity of u_n and ∂u_n.

Exercise. Finish the proof of (77). □

THEOREM 15. *Let $F \in C^\infty(\mathbb{R}^3)$ have the property that $F(0, 0) = 0$. Let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for some $k \geq 1$. Then there is an $\epsilon_k > 0$, depending on $\|f\|_{C_b^{k+1}(\mathbb{R})}$ and $\|g\|_{C_b^k(\mathbb{R})}$ and the function F, such that the equation*

$$\begin{cases} u_{tt} - u_{xx} = F(u, \partial u) \\ u(0, x) = f(x) \\ u_t(0, x) = g(x) \end{cases} \tag{79}$$

has a unique solution in $C^{k+1}[(-\epsilon_k, \epsilon_k) \times \mathbb{R}]$. Furthermore,

$$u \in C\{(-\epsilon_k, \epsilon_k), C_d^{k+1}(\mathbb{R})\} \quad and \quad \partial_t u \in C\{(-\epsilon_k, \epsilon_k), C_d^k(\mathbb{R})\}. \tag{80}$$

Remark. Note that at first sight, this result is very unsatisfactory. The reason is that the existence time depends on the degree of differentiability. Given the above theorem, it is not immediately clear that if one has smooth initial data, then one gets a smooth local solution; ϵ_k could very well converge to zero as $k \to \infty$.

Proof. The uniqueness follows from Theorem 14. What remains to be proved is existence. Define the sequence of functions u_n, $n = 0, \ldots$ by (74) and (75). Due to Lemma 8, we know that this sequence is well defined and that (76) holds.

Rough control. Just as in the ODE case, we start by proving that we have rough control over the sequence of approximations. Consider $E_k[u_n]$, where E_k is defined in (61). Note that $E_k[u_n](0) = E_k[u_0](0)$ for all n, since $E_k[u_n](0)$ only depends on the initial data, which coincide for all n. Define $c_k = E_k[u_0](0)$. Assuming that $\epsilon \leq 2$, (64) yields

(81)

$$E_k[u_n](t) \leq 2c_k + 2\sum_{j=0}^{k} \left| \int_0^t \|\partial_x^j F_n(s,\cdot)\|_{C_b(\mathbb{R})} ds \right| + \int_0^t \left[\int_0^s \|F_n(v,\cdot)\|_{C_b(\mathbb{R})} dv \right] ds,$$

where $F_0 = 0$ and $F_n = F(u_{n-1}, \partial u_{n-1})$ for $n \geq 1$. Let us make the *inductive assumption* that

$$E_k[u_n](t) \leq 2c_k + 1 \tag{82}$$

for all $t \in [-\epsilon, \epsilon]$ for some $0 < \epsilon \leq 2$. First, we need to prove that it is true for $n = 0$. Consider (81). Since $F_0 = 0$ we obtain (82). Assume that (82) is true for n. We wish to prove that it is true for $n+1$. Note that $\partial_x^j F_{n+1}$ is a sum of terms of the form (78). All the factors in this expression are controlled by the inductive hypothesis. Thus there is an α_k, depending on c_k and F such that

$$2\sum_{j=0}^{k} \left| \int_0^t \|\partial_x^j F_{n+1}(s,\cdot)\|_{C_b(\mathbb{R})} ds \right| + \int_0^t \left[\int_0^s \|F_{n+1}(v,\cdot)\|_{C_b(\mathbb{R})} dv \right] ds \leq \alpha_k \epsilon$$

for all $t \in [-\epsilon, \epsilon]$. Choosing $\epsilon \leq \min\{1, 1/(\alpha_k + 1)\}$, we obtain (82) for $n+1$ due to (81).

Convergence. We are interested in the differences $\hat{u}_n = u_{n+1} - u_n$. It will be convenient to consider $E_k[\hat{u}_n]$. The reson for considering this object is that it dominates the C_b^{k+1}-norm of $\hat{u}_n$ and the C_b^k-norm of $\partial_t \hat{u}_n$. In other words, if we can prove that there is an $\epsilon > 0$ and a constant C_k (which is allowed to depend on k but not on n) such that

$$\sup_{t\in[-\epsilon,\epsilon]} E_k[\hat{u}_n] \leq \frac{1}{2^n} C_k, \tag{83}$$

then u_n is a Cauchy sequence in $C\{[-\epsilon, \epsilon], C_b^{k+1}(\mathbb{R})\}$ and $\partial_t u_n$ is a Cauchy sequence in $C\{[-\epsilon, \epsilon], C_b^k(\mathbb{R})\}$. The argument to prove this is identical to an argument presented in the proof of local existence for ODE:s. Since $E_k[\hat{u}_n](0) = 0$, (64) implies

$$E_k[\hat{u}_n](t) \leq 2\sum_{j=0}^{k} \left| \int_0^t \|\partial_x^j \hat{F}_n(s,\cdot)\|_{C_b(\mathbb{R})} ds \right| + \int_0^t \left[\int_0^s \|\hat{F}_n(v,\cdot)\|_{C_b(\mathbb{R})} dv \right] ds, \tag{84}$$

where

$$\hat{F}_n = F(u_n, \partial u_n) - F(u_{n-1}, \partial u_{n-1}).$$

Let us consider, for $n \geq 1$,

$$
\begin{aligned}
(85)\quad & F(u_n, \partial u_n) - F(u_{n-1}, \partial u_{n-1}) \\
= & \int_0^1 \partial_\tau \{F[\tau u_n + (1-\tau)u_{n-1}, \tau\partial u_n + (1-\tau)\partial u_{n-1}]\} d\tau \\
= & \int_0^1 \partial_{z^1} F[\tau u_n + (1-\tau)u_{n-1}, \tau\partial u_n + (1-\tau)\partial u_{n-1}] d\tau \cdot \hat{u}_{n-1} \\
& + \int_0^1 \partial_{z^2} F[\tau u_n + (1-\tau)u_{n-1}, \tau\partial u_n + (1-\tau)\partial u_{n-1}] d\tau \cdot \partial_x \hat{u}_{n-1} \\
& + \int_0^1 \partial_{z^3} F[\tau u_n + (1-\tau)u_{n-1}, \tau\partial u_n + (1-\tau)\partial u_{n-1}] d\tau \cdot \partial_t \hat{u}_{n-1}.
\end{aligned}
$$

Here $F(u, \partial u) = F(u, u_x, u_t)$ and we name the three coordinates z^i, $i = 1, 2, 3$. Since F is a smooth function and we have (82), we obtain

$$
\left| \partial_x^j \left(\int_0^1 \partial_{z^i} F[\tau u_n + (1-\tau)u_{n-1}, \tau\partial u_n + (1-\tau)\partial u_{n-1}] d\tau \right) \right| \leq \beta_{k,i}
$$

for some constants $\beta_{k,i}$ which only depend on c_k and F. The ∂_x^j derivative of the factors $\hat{u}_{n-1}$, $\partial_x \hat{u}_{n-1}$ and $\partial_t \hat{u}_{n-1}$ for $j \leq k$ on the other hand, are dominated by $E_k[\hat{u}_{n-1}]$. In other words, there is a constant β_k depending only on c_k and F such that

$$
\begin{aligned}
& 2\sum_{j=0}^{k} \left| \int_0^t \|\partial_x^j \hat{F}_n(s, \cdot)\|_{C_b(\mathbb{R})} ds \right| + \int_0^t \left[\int_0^s \|\hat{F}_n(v, \cdot)\|_{C_b(\mathbb{R})} dv \right] ds \\
\leq \ & \frac{\beta_k}{2} \left| \int_0^t E_k[\hat{u}_{n-1}](s) ds \right| + \frac{\beta_k}{2} \int_0^t \left[\int_0^s E_k[\hat{u}_{n-1}](v) dv \right] ds.
\end{aligned}
$$

Combining this inequality with (84), we obtain (for $|t| \leq \epsilon \leq 1$),

$$
E_k[\hat{u}_n](t) \leq \beta_k \epsilon \sup_{t \in [-\epsilon, \epsilon]} E_k[\hat{u}_{n-1}](t).
$$

Define $\epsilon_k = \min\{1, 1/(\alpha_k + 1), 1/(2\beta_k + 1)\}$. Then

$$
\sup_{t \in [-\epsilon_k, \epsilon_k]} E_k[\hat{u}_n](t) \leq \frac{1}{2} \sup_{t \in [-\epsilon_k, \epsilon_k]} E_k[\hat{u}_{n-1}](t)
$$

for $n \geq 1$. Just as in the proof of local existence for ODE:s, this leads to (83). As noted above, this in its turn leads to the consequence that u_n is a Cauchy sequence in $C\{[-\epsilon_k, \epsilon_k], C_b^{k+1}(\mathbb{R})\}$ and $\partial_t u_n$ is a Cauchy sequence in $C\{[-\epsilon_k, \epsilon_k], C_b^k(\mathbb{R})\}$.

The limit is in C^{k+1}. We wish to prove that the limit is in $C^{k+1}[(-\epsilon_k, \epsilon_k) \times \mathbb{R}]$. In order to do so, we prove that the u_n form a Cauchy sequence in $C_b^{k+1}[(-\epsilon_k, \epsilon_k) \times \mathbb{R}]$. What we know so far is that $\partial_x^j u_n$ and $\partial_x^l \partial_t u_n$ form Cauchy sequences in $C_b[(-\epsilon_k, \epsilon_k) \times \mathbb{R}]$ for $j \leq k+1$ and $l \leq k$. This is however not enough. We need to prove that $\partial_x^j \partial_t^m u_n$ is a Cauchy sequence for $j + m \leq k + 1$. By the equation, we obtain

$$
\partial_x^j \partial_t^2 (u_n - u_m) = \partial_x^{j+2}(u_n - u_m) + \partial_x^j [F(u_{n-1}, \partial u_{n-1}) - F(u_{m-1}, \partial u_{m-1})].
$$

Assume $j + 2 \leq k + 1$. Then, by arguments similar to the ones made in connection with the proof of convergence, we conclude that $\partial_x^j \partial_t^2 u_n$ is a Cauchy sequence for

$j+2 \leq k+1$. Differentiating the equation further with respect to t and repeating the argument, one inductively obtains the desired conclusion. $\square$

10. Continuation criteria

As was noted after the statement of Theorem 15, it does not imply local existence of smooth solutions given smooth initial data. This is clearly unsatisfactory. Secondly, we wish to have a a criterion which, when fulfilled, allows us to take the step from local to global existence. It turns out that one can acheive these two goals in one step.

THEOREM 16. *Let $F \in C^\infty(\mathbb{R}^3)$ have the property that $F(0,0) = 0$ and let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for some $k \geq 1$. Let u be a $C^{k+1}[(T_-, T_+) \times \mathbb{R}]$-solution to*

$$\begin{cases} u_{tt} - u_{xx} = F(u, \partial u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

for some $T_- < 0 < T_+$. Then if there is a real constant $c_0 > 0$ such that $E_0[u](t) \leq c_0$ for all $t \in [0, T_+)$, where $T_+ < \infty$, then there is a real constant C_k depending only on F, c_0, T_+ and $E_k[u](0)$ such that

$$E_k[u](t) \leq C_k \tag{86}$$

for all $t \in [0, T_+)$. The statement concerning T_- is similar.

Proof. Let us prove (86) by induction. By assumption, it is true if we replace k by 0. Assume it is true if we replace k with j for some $j \leq k-1$. We wish to prove that it is true for $j+1$. Due to (64), we have

$$\begin{aligned} E_{j+1}(t) \quad &\leq \quad (1 + T_+/2)E_{j+1}(0) + 2\sum_{l=0}^{j+1} \left| \int_0^t \|\partial_x^l F(s,\cdot)\|_{C_b(\mathbb{R})} ds \right| \\ &+ \int_0^t \left[\int_0^s \|F(v,\cdot)\|_{C_b(\mathbb{R})} dv \right] ds, \end{aligned}$$

where we have written F instead of $F(u, \partial u)$ and E_{j+1} instead of $E_{j+1}[u]$. Since u and ∂u are bounded by assumption, F is bounded, so that the last term is bounded. Letting α_j be the sum of the supremum of the first and last terms on the right hand side, we thus obtain

$$E_{j+1}(t) \leq \alpha_j + 2\sum_{l=0}^{j+1} \left| \int_0^t \|\partial_x^l F(s,\cdot)\|_{C_b(\mathbb{R})} ds \right| \tag{87}$$

for all $t \in [0, T_+)$. Note that α_j only depends on the constants mentioned in the statement of the theorem. Consider $\partial_x^l F$. It can be written as a sum of terms of the form

$$[\partial_{z^1}^{i_1} \partial_{z^2}^{i_2} \partial_{z^3}^{i_3} F](u, \partial u) \partial_x^{l_1} u \cdots \partial_x^{l_m} u \partial_x^{p_1} \partial_t u \cdots \partial_x^{p_o} \partial_t u.$$

Here $l_1 + ... + l_m + p_1 + ... + p_o = l$, $l_i \leq l+1$ and $p_i \leq l$. By induction, if all of the $l_i < j+2$ and all the $p_i < j+1$, then this term can be bounded by a constant (depending only on F, c_0, T_+ and $E_k[u](0)$). If one $l_i = j+2$ or one $p_i = j+1$, all the other l_i and p_i have to be zero. The term will thus consist of two factors, one of which is $\partial_{z^2} F$ or $\partial_{z^3} F$ and the other of which is $\partial_x^{j+2} u$ or $\partial_x^{j+1} \partial_t u$ respectively. The first term is bounded by the induction hypothesis and the second term is bounded by $E_{j+1}[u]$. Consequently, there are constants β_j and γ_j (depending only on F, c_0, T_+ and $E_k[u](0)$), such that

$$E_{j+1}(t) \leq \alpha_j + \left| \int_0^t \{\beta_j + \gamma_j E_{j+1}(s)\} ds \right| \leq \alpha_j + \beta_j T_+ + \left| \int_0^t \gamma_j E_{j+1}(s) ds \right|.$$

By Grönwall's lemma, the theorem follows. □

In the above theorem, we obtain control over an arbitrary number of derivatives starting with control of only one derivative. As a consequence, we obtain a continuation criterion.

COROLLARY 2. *Let $F \in C^\infty(\mathbb{R}^3)$ have the property that $F(0,0) = 0$ and let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for all k. Let $u \in C^2[(T_-, T_+) \times \mathbb{R}]$ be a solution to*

$$\begin{cases} u_{tt} - u_{xx} = F(u, \partial u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

where (T_-, T_+) is the maximal existence interval. Then $u \in C^\infty[(T_-, T_+) \times \mathbb{R}]$ and either $T_+ = \infty$ or $E_0[u](t)$ is unbounded on $[0, T_+)$. The statement concerning T_- is similar.

Proof. Fix $k \geq 1$. Assume the maximal existence interval for C^{k+1} solutions is $(T_{-,k}, T_{+,k})$. Clearly, this interval is contained in (T_-, T_+). We wish to prove that the two intervals coincide. Since there are two similar cases, let us only prove that $T_+ = T_{+,k}$. In order to obtain a contradiction, let us assume $T_{+,k} < T_+$. Then $E_0[u]$ is bounded on $[0, T_{+,k}]$. Due to Theorem 16, $E_k[u](t)$ satisfies a uniform bound on $[0, T_{+,k})$. As a consequence of Theorem 15, we conclude that there is an $\epsilon_k > 0$ such that for every $t \in [0, T_{+,k})$, we can find a solution to the equation with initial data $u(t, \cdot)$, $u_t(t, \cdot)$ with existence time at least ϵ_k. Similarly to the ODE case, we conclude that we can extend the solution beyond the maximal existence interval. Thus u is smooth. To prove that either $T_+ = \infty$ or $E_0[u]$ is unbounded, one proceeds in a similar fashion. □

The corollary represents a continuation criterion in the sense that if $E_0[u]$ remains bounded on $[0, T_+)$, then the solution can be continued beyond T_+.

If F only depends on u, one gets a better result.

THEOREM 17. *Let $F \in C^\infty(\mathbb{R})$ have the property that $F(0) = 0$ and let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for some $k \geq 1$. Let u be a $C^{k+1}[(T_-, T_+) \times \mathbb{R}]$-solution to*

$$\begin{cases} u_{tt} - u_{xx} = F(u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

for some $T_- < 0 < T_+$. Then if there is a real constant $c_0 > 0$ such that $\|u(t, \cdot)\|_{C_b(\mathbb{R})} \leq c_0$ for all $t \in [0, T_+)$, with $T_+ < \infty$, then there is a real constant C_k depending only on F, c_0, T_+ and $E_k[u](0)$ such that

$$E_k[u](t) \leq C_k$$

for all $t \in [0, T_+)$. The statement concerning T_- is similar.

The proof is similar to the proof of Theorem 16.

Exercise. Prove the above theorem.

COROLLARY 3. *Let $F \in C^\infty(\mathbb{R})$ have the property that $F(0) = 0$ and let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for all k. Let $u \in C^2[(T_-, T_+) \times \mathbb{R}]$ be a solution to*

$$\begin{cases} u_{tt} - u_{xx} = F(u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

where (T_-, T_+) is the maximal existence interval. Then $u \in C^\infty[(T_-, T_+) \times \mathbb{R}]$ and either $T_+ = \infty$ or $\|u(t,\cdot)\|_{C_b(\mathbb{R})}$ is unbounded on $[0, T_+)$. The statement concerning T_- is similar.

The proof is similar to the proof of Corollary 2.

Exercise. Prove the above corollary.

11. A counterexample to local existence

Let us illustrate how local existence can fail if we do not impose conditions on the behaviour of the initial data for large x.

PROPOSITION 10. *Consider the equation*

$$\begin{cases} u_{tt} - u_{xx} = u_t^2 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x). \end{cases} \tag{88}$$

There are initial data $f, g \in C^\infty(\mathbb{R})$ such that for any $\epsilon > 0$, there is no $u \in C^\infty[(-\epsilon, \epsilon) \times \mathbb{R}]$ solving (88).

Proof. Consider the equation $u_{tt} = u_t^2$. Say that the initial data for u_t is $k > 0$. Then

$$u_t = \frac{k}{1 - kt},$$

and

$$u = u(0) - \ln(1 - kt).$$

In other words, the solution blows up at $t = 1/k$. Consider a solution such that $g(x) = k$ and $f = 0$ for $x \in [a - 1/k, a + 1/k]$. Then $u = -\ln(1 - kt)$ in the triangle with base $\{0\} \times [a - 1/k, a + 1/k]$ and vertex $(1/k, a)$ due to Theorem 14. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Let $k \geq 1$ be an integer, and consider $g_k(x) = k\phi(x - 4k)$, $f(x) = 0$. Then $g_k(x) = k$ for $x \in [4k - 1/k, 4k + 1/k]$. Consequently the corresponding solution blows up in time $1/k$. Note also that if $k_1 \neq k_2$ for integers k_1, k_2, then $g_{k_1}(x) \neq 0$ implies $g_{k_2}(x) = 0$. Consequently, we can define

$$g(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then g is a smooth function and $g(x) = k$ for $x \in [4k - 1/k, 4k + 1/k]$. Define $f(x) = 0$. Then f and g are the desired initial data. Since any solution to (88) has to blow up in time $1/k$ for any integer $k \geq 1$, it is clear that there can be no local solution. $\square$

12. Generalizations

The results of this chapter can be generalized in a number of ways. The proof of local existence of solutions to (79) where $f \in C_d^{k+1}(\mathbb{R}, \mathbb{R}^m)$, $g \in C_d^k(\mathbb{R}, \mathbb{R}^m)$ and $F \in C^\infty(\mathbb{R}^{3m}, \mathbb{R}^m)$ is a function such that $F(0,0) = 0$ is practically identical to the proof in the case $m = 1$. One can also generalize and let F depend on x and t. This requires some modifications, but the argument is essentially the same.

We shall use the exercises below in an application to Einstein's equations.

Exercise. Prove the following local existence theorem.

THEOREM 18. *Let $F \in C^\infty[(T_-, T_+) \times \mathbb{R}^3, \mathbb{R}^m)$ have the property that $F(t, 0, 0) = 0$ for all $t \in (T_-, T_+)$. Let $f \in C_d^{k+1}(\mathbb{R}, \mathbb{R}^m)$ and $g \in C_d^k(\mathbb{R}, \mathbb{R}^m)$ for some $k \geq 1$. Then there is an $\epsilon_k > 0$, depending on $\|f\|_{C_b^{k+1}(\mathbb{R})}$, $\|g\|_{C_b^k(\mathbb{R})}$, t_0 and the function F, such that the equation*

$$\begin{cases} u_{tt} - u_{xx} = F(t, u, \partial u) \\ u(t_0, x) = f(x) \\ u_t(t_0, x) = g(x) \end{cases}$$

has a unique solution in $C^{k+1}[(t_0 - \epsilon_k, t_0 + \epsilon_k) \times \mathbb{R}]$. Furthermore,

$$u \in C\{(t_0 - \epsilon_k, t_0 + \epsilon_k), C_d^{k+1}(\mathbb{R}, \mathbb{R}^m)\} \quad \text{and} \quad \partial_t u \in C\{(t_0 - \epsilon_k, t_0 + \epsilon_k), C_d^k(\mathbb{R}, \mathbb{R}^m)\}.$$

Exercise. Formulate and prove uniqueness more generally for the type of functions F that appear in the statement of the above theorem.

Exercise. Prove the following theorem.

THEOREM 19. *Let $F \in C^\infty[(T_-, T_+) \times \mathbb{R}^{3m}, \mathbb{R}^m)$ have the property that $F(t, 0, 0) = 0$ for all $t \in (T_-, T_+)$ and let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for some $k \geq 1$. Let u be a $C^{k+1}[(t_-, t_+) \times \mathbb{R}, \mathbb{R}^m]$-solution to*

$$\begin{cases} u_{tt} - u_{xx} = F(t, u, \partial u) \\ u(t_0, x) = f(x) \\ u_t(t_0, x) = g(x) \end{cases}$$

where $t_0 \in (t_-, t_+)$. Then if there is a real constant $c_0 > 0$ such that $E_0[u](t) \leq c_0$ for all $t \in [t_0, t_+)$, where $t_+ < \infty$, then there is a real constant C_k depending only on F, c_0, t_0, t_+ and $E_k[u](t_0)$ such that

$$E_k[u](t) \leq C_k$$

for all $t \in [t_0, t_+)$. The statement concerning t_- is similar.

Exercise. Prove the corollary below.

COROLLARY 4. *Let $F \in C^\infty[(T_-, T_+) \times \mathbb{R}^{3m}, \mathbb{R}^m)$ have the property that $F(t, 0, 0) = 0$ for all $t \in (T_-, T_+)$ and let $f \in C_d^{k+1}(\mathbb{R})$ and $g \in C_d^k(\mathbb{R})$ for all k. Let $u \in C^2[(t_-, t_+) \times \mathbb{R}, \mathbb{R}^m]$ be a solution to*

$$\begin{cases} u_{tt} - u_{xx} = F(t, u, \partial u) \\ u(t_0, x) = f(x) \\ u_t(t_0, x) = g(x) \end{cases}$$

where (t_-, t_+) *is the maximal existence interval and* $t_0 \in (t_-, t_+)$*. Then* $u \in C^\infty[(t_-, t_+) \times \mathbb{R}, \mathbb{R}^m]$ *and either* $t_+ = T_+$ *or* $E_0[u](t)$ *is unbounded on* $[t_0, t_+)$*. The statement concerning* t_- *is similar.*

CHAPTER 5

Global existence for $1+1$-dimensional wave equations

In this chapter, we shall consider global existence of solutions to non-linear wave equations. There are essentially no general theorems guaranteeing global existence, so we shall simply consider some examples of interest.

1. Wave map equations

One family of equations that has been studied extensively is the class of wave map equations. We shall not define it in all generality, since this requires knowledge of Riemannian geometry, but only consider a special case. First we need to define the concept of a Riemannian metric on $\mathbb{R}^m$.

DEFINITION 19. Let $g : \mathbb{R}^m \to \mathbb{R}^{m^2}$ be a smooth map. We shall view g as a map that takes a point $x \in \mathbb{R}^m$ into an $m \times m$-matrix at that point, and we shall denote the matrix components of $g(x)$ by $g_{ij}(x)$. If $g(x)$ is a symmetric and positive definite matrix for each $x \in \mathbb{R}^m$, we shall call g a *Riemannian metric* on $\mathbb{R}^m$. Then $g(x)$ is invertible for each $x \in \mathbb{R}^m$ and we shall denote the matrix components of the inverse of $g(x)$ by $g^{ij}(x)$.

Remark. A matrix A is positive definite if and only if $v^t A v > 0$ for all $v \neq 0$. This implies in particular that the matrix is injective so that it is invertible.

DEFINITION 20. Let g be a Riemannian metric on $\mathbb{R}^m$. Given a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^m$, we define its *Lagrangian density* by

$$\mathcal{L}[f](t,x) = g_{ij}[f(t,x)][f_t^i f_t^j - f_x^i f_x^j](t,x). \tag{89}$$

We shall say that f satisfies the *Euler-Lagrange equations corresponding to* $\mathcal{L}$ if for every $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^m)$,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I_\epsilon = 0, \tag{90}$$

where

$$I_\epsilon = \int_K \mathcal{L}[f + \epsilon\phi](t,x)\,dtdx,$$

and K is any compact set such that $\phi = 0$ outside of K. A solution to the Euler Lagrange equations corresponding to the Lagrangian density (89) is called a $1+1$ dimensional *wave map* with *target* $(\mathbb{R}^m, g)$.

Remark. The definition of I_ϵ depends on K which is quite arbitrary. However, if we define $I_{i,\epsilon}$ for $i = 1,2$ by replacing K with K_i, $i = 1,2$, then $I_{1,\epsilon} - I_{2,\epsilon}$ is independent of ϵ by construction, so that the equation (90) remains the same. For those familiar with Riemannian geometry, let us note that $(\mathbb{R}^m, g)$ can be replaced by an arbitrary Riemannian manifold in this definition. Furthermore, the definition can be generalized to higher dimensions than 1+1. Finally, in the physics literature, the terminology σ-model is used instead of wave map.

Wave map equations are a special case of a wider class of variational problems. One reason they have received so much attention is probably due to the fact that (89) is the simplest Lagrangian density one can write down which leads to non-linear wave equations. However, equations of wave map type appear frequently in physics and in particular in General Relativity. Let us derive the Euler Lagrange equations.

PROPOSITION 11. *Let g be a Riemannian metric on $\mathbb{R}^m$. Then $f : \mathbb{R}^2 \to \mathbb{R}^m$ is a solution to the Euler Lagrange equations corresponding to the Lagrangian density (89) if and only if*

$$f^l_{tt} - f^l_{xx} = -\Gamma^l_{kj} \circ f[f^k_t f^j_t - f^k_x f^j_x], \tag{91}$$

where

$$\Gamma^l_{jk} = \frac{1}{2} g^{lm} \left[\frac{\partial g_{km}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m} \right]. \tag{92}$$

Remark. In the statement of the proposition, f^i are the components of f and we use the Einstein summation convention of summing repeated upstairs and downstairs indices. In particular, in the right hand side we sum over all indices except l. Furthermore we use the notation $\Gamma^l_{ij} \circ f$ to denote the function whose value at (t, x) is $\Gamma^l_{ij}[f(t, x)]$. The objects Γ^l_{kj} are called the *Christoffel symbols* of the metric g and they are very important in Riemannian geometry.

Proof. Fix $\phi \in C^\infty_0(\mathbb{R}^2, \mathbb{R}^m)$. Let us compute

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_K g_{ij}[(f + \epsilon\phi)(t, x)][(f + \epsilon\phi)^i_t (f + \epsilon\phi)^j_t - (f + \epsilon\phi)^i_x (f + \epsilon\phi)^j_x](t, x) dx dt,$$

where $\phi = 0$ outside of K. We obtain

$$\int_K \left\{ \frac{\partial g_{ij}}{\partial x^k}[f(t, x)]\phi^k(t, x)[f^i_t f^j_t - f^i_x f^j_x](t, x) + 2g_{ij}[f(t, x)][\phi^i_t f^j_t - \phi^i_x f^j_x](t, x) \right\} dx dt.$$

Here, we have used the fact that $g[f(t, x)]$ is a symmetric matrix. Below we take it to be understood that f and ϕ are to be evaluated at (t, x) and g and its derivatives are to be evaluated at $f(t, x)$. Let us integrate by parts in the part that contains derivatives of ϕ. Note that when doing so, there are no boundary terms since ϕ is zero on the boundary of K. We obtain

$$\int_K g_{ij}[\phi^i_t f^j_t - \phi^i_x f^j_x] dx dt = -\int_K \left\{ \frac{\partial g_{ij}}{\partial x^k}[f^k_t f^j_t - f^k_x f^j_x] + g_{ij}[f^j_{tt} - f^j_{xx}] \right\} \phi^i dx dt$$

By renaming indices, we can write

$$\int_K \frac{\partial g_{ij}}{\partial x^k}\phi^k [f^i_t f^j_t - f^i_x f^j_x] dt dx = \int_K \frac{\partial g_{kj}}{\partial x^i}\phi^i [f^k_t f^j_t - f^k_x f^j_x] dt dx.$$

Adding the pieces, we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I_\epsilon = \int_K \left\{ \frac{\partial g_{kj}}{\partial x^i}[f_t^k f_t^j - f_x^k f_x^j] - 2\frac{\partial g_{ij}}{\partial x^k}[f_t^k f_t^j - f_x^k f_x^j] \right. \\ \left. -2g_{ij}[f_{tt}^j - f_{xx}^j] \right\} \phi^i dxdt.$$

Since ϕ is arbitrary, Lemma 9 below implies that (90) is equivalent to

$$0 = \frac{\partial g_{kj}}{\partial x^i}[f_t^k f_t^j - f_x^k f_x^j] - 2\frac{\partial g_{ij}}{\partial x^k}[f_t^k f_t^j - f_x^k f_x^j] - 2g_{ij}[f_{tt}^j - f_{xx}^j].$$

Let us multiply this equation by $-g^{li}/2$ (and sum over i). Since $g^{il}g_{ij} = \delta_j^l$, we obtain

$$(f_{tt}^l - f_{xx}^l) = -g^{li}\frac{\partial g_{ij}}{\partial x^k}[f_t^k f_t^j - f_x^k f_x^j] + \frac{1}{2}g^{li}\frac{\partial g_{kj}}{\partial x^i}[f_t^k f_t^j - f_x^k f_x^j]. \tag{93}$$

Note that we can interchange the names of the indices j and k in order to obtain

$$g^{li}\frac{\partial g_{ij}}{\partial x^k}[f_t^k f_t^j - f_x^k f_x^j] = g^{li}\frac{\partial g_{ik}}{\partial x^j}[f_t^j f_t^k - f_x^j f_x^k].$$

Since we are summing over both j and k, the names are of course not important. As a consequence, we have

$$-g^{li}\frac{\partial g_{ij}}{\partial x^k}[f_t^k f_t^j - f_x^k f_x^j] = -\frac{1}{2}g^{li}\left[\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j}\right][f_t^k f_t^j - f_x^k f_x^j].$$

Combining this equality with (93), we obtain (91). □

LEMMA 9. *Let $f \in C^\infty(\mathbb{R}^2, \mathbb{R}^m)$ have the property that for every $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^m)$,*

$$\int f_i \phi^i dxdt = 0,$$

where f_i and ϕ^i are the components of f and ϕ respectively. Then $f = 0$.

Remark. The regularity condition on f in this lemma is absurdly strong. Much weaker conditions lead to the same conclusion, but we shall not need to know that.

Proof. Assume there is a $(t_0, x_0) \in \mathbb{R}^2$ such that $f(t_0, x_0) \neq 0$. Then by the continuity of f, there is a $\delta > 0$ and an $\epsilon > 0$ such that $|f|^2(t,x) \geq \epsilon$ for all $(t,x) \in B_\delta(t_0, x_0)$. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be such that $\phi = 1$ in this ball, $\phi \geq 0$ and define $\phi^i = \phi f_i$. Then

$$0 = \int f_i \phi^i dxdt = \int \phi |f|^2 dxdt \geq \pi\delta^2\epsilon > 0.$$

We have reached the desired contradiction. □

Note that (91) is an equation of the form for which we have proved local existence. The right hand side defines the non-linearity $F(u, \partial u)$. By the continuation criteria we have developed, the only obstruction to global existence is that u or ∂u become unbounded. If we can prove that these objects cannot become unbounded in finite time, we have proved global existence. Let us consider a special target.

LEMMA 10. *Define a Riemannian metric on $\mathbb{R}^2$ by $g_{11} = 1$, $g_{12} = g_{21} = 0$ and $g_{22}(x) = e^{2x^1}$. Then the only non-zero Christoffel symbols are given by*

$$\Gamma_{22}^1 = -e^{2x^1}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = 1.$$

Remark. For those familiar with Riemannian geometry, let us note that the metric g is the hyperbolic metric.

Proof. Note that $g^{11} = 1$, $g^{22} = e^{-2x^1}$ and that $g^{ij} = 0$ for $i \neq j$. Let us consider the case when $l = 1$ in (92). Then m has to equal 1 since g^{ij} is diagonal. No matter what k and j are, g_{k1} and g_{j1} are constant so that the two first terms on the right hand side of (92) are zero. The third term is only non-zero if $j = k = 2$. We then obtain

$$\Gamma^1_{22} = -e^{2x^1},$$

since $g_{22} = e^{2x^1}$. Assume $l = 2$. Then m has to equal 2. Since g_{ij} is independent of x^2, the last term in (92) is zero. The first two terms can only be non-zero if one of j, k is 1 and the other is 2. We obtain

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} e^{-2x^1} 2e^{2x^1} = 1.$$

This proves the lemma. □

Denote the coordinates of the wave map $f^1 = P$ and $f^2 = Q$. Let us write down the equation (91) in this case. Set $l = 1$ in (91). Then k, j both have to equal 2, since the Christoffel symbols are zero otherwise by Lemma 10. By Lemma 10, we obtain

$$P_{tt} - P_{xx} = e^{2P}(Q_t^2 - Q_x^2).$$

Set $l = 2$ in (91). The right hand side is a sum of two terms; $k = 1, j = 2$ and $k = 2, j = 1$. We obtain

$$Q_{tt} - Q_{xx} = -2(P_t Q_t - P_x Q_x).$$

Let us prove global existence for solutions to these equations.

THEOREM 20. *Consider a solution to*

$$P_{tt} - P_{xx} = e^{2P}(Q_t^2 - Q_x^2) \tag{94}$$

$$Q_{tt} - Q_{xx} = -2(P_t Q_t - P_x Q_x) \tag{95}$$

with initial data

$$(P, Q)(0) = f, \quad (P_t, Q_t)(0) = g \tag{96}$$

for some $f, g \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$. Then the maximal existence interval is $(-\infty, \infty)$.

Proof. Due to Corollary 4, all we need to prove is that P, Q and their first derivatives cannot become unbounded in a finite time. Let us define the objects $\mathcal{A}_+$ and $\mathcal{A}_-$ by

$$\mathcal{A}_\pm = \frac{1}{2}\left[(P_t \pm P_x)^2 + e^{2P}(Q_t \pm Q_x)^2\right].$$

Let us compute

$$(\partial_t \mp \partial_x)\frac{1}{2}(P_t \pm P_x)^2 = (P_t \pm P_x)(P_{tt} - P_{xx}) = (P_t \pm P_x)e^{2P}(Q_t^2 - Q_x^2)$$

and

$$\begin{aligned}(\partial_t \mp \partial_x)\frac{1}{2}e^{2P}(Q_t \pm Q_x)^2 &= (P_t \mp P_x)e^{2P}(Q_t \pm Q_x)^2 \\ &\quad + e^{2P}(Q_t \pm Q_x)(Q_{tt} - Q_{xx}) \\ &= (P_t \mp P_x)e^{2P}(Q_t \pm Q_x)^2 \\ &\quad + e^{2P}(Q_t \pm Q_x)[-2P_t Q_t + 2P_x Q_x].\end{aligned}$$

By adding these two expressions, one obtains 0 as a result. In other words,

$$(\partial_t \mp \partial_x)\mathcal{A}_\pm = 0. \tag{97}$$

Similarly to the derivation of the solution to the inhomogeneous wave equation, it is natural to consider the objects

$$h_\pm(s) = \mathcal{A}_\pm(s, x_0 \mp s)$$

for some fixed x_0. Due to (97), we obtain

$$\frac{dh_\pm}{ds} = [(\partial_t \mp \partial_x)\mathcal{A}_\pm](s, x_0 \mp s) = 0.$$

In other words, the $h_\pm$ are constant. Taking the supremum over x_0, we obtain

$$\|\mathcal{A}_\pm(t,\cdot)\|_{C_b(\mathbb{R})} = \|\mathcal{A}_\pm(0,\cdot)\|_{C_b(\mathbb{R})}$$

for all t belonging to the maximal existence interval (t_-, t_+) of the solution. Since the initial data have compact support, the right hand side is finite. Note that

$$P_t^2 + P_x^2 + e^{2P}(Q_t^2 + Q_x^2) = \mathcal{A}_+ + \mathcal{A}_- \leq \|\mathcal{A}_+(t,\cdot)\|_{C_b(\mathbb{R})} + \|\mathcal{A}_-(t,\cdot)\|_{C_b(\mathbb{R})}. \tag{98}$$

We conclude that the left hand side is bounded by a constant given by the initial data. Consequently P_t and P_x are bounded. Integrating this bound, we conclude that P cannot become unbounded in a finite time. Combining this observation with (98), we conclude that Q_t and Q_x cannot become unbounded in a finite time. Integrating the control of Q_t, we conclude that Q cannot become unbounded in finite time. Consequently, P, Q and their first derivatives cannot become unbounded in a finite time. The theorem follows. □

It is possible to improve the above result and prove that there is global existence for $f, g \in C^\infty(\mathbb{R}, \mathbb{R}^2)$. In other words, the condition that the initial data have compact support is not necessary. Consider (94)-(96) for such initial data. Let $(t_0, x_0) \in \mathbb{R}^2$. We wish to define the solution in a neighbourhood of this point. Let $I = [x_0 - |t_0| - 1, x_0 + |t_0| + 1]$. Let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ for $x \in I$. Let $f_1 = \phi f$ and $g_1 = \phi g$. Then there is a global solution (P_1, Q_1) to (94)-(96) with f and g replaced by f_1 and g_1. We then define the solution in the interior of the square determined by the corners $(|t_0|+1, x_0)$, $(0, x_0 - |t_0| - 1)$, $(-|t_0|-1, x_0)$ and $(0, x_0 + |t_0| + 1)$ to equal (P_1, Q_1). By uniqueness, Theorem 14, if two such squares have non-empty intersection, then the corresponding solutions have to agree on the intersection. This means that the above definition makes sense and that we get a solution to (94)-(96) on all of $\mathbb{R}^2$. In fact, it is possible to generalize the result to a finite degree of differentiability as well. We leave the details to the interested reader.

The key ingredient to prove global existence is (97). This is the identity that leads to control of the solution and its first derivatives. The definition of $\mathcal{A}_\pm$ might seem a bit arbitrary, but in fact it comes from the wave map structure.

Let f be a solution to (91). Define

$$\mathcal{A}_\pm(t,x) = \frac{1}{2} g_{ij}[f(t,x)][f_t^i \pm f_x^i](t,x)[f_t^j \pm f_x^j](t,x).$$

The interested reader is encouraged to prove that

$$(\partial_t \mp \partial_x)\mathcal{A}_\pm = 0.$$

As a consequence of this,

$$\|\mathcal{A}_{\pm}(t,\cdot)\|_{C_b(\mathbb{R})} \tag{99}$$

are conserved quantities. This is a good starting point for proving global existence, but it is not in general enough. It is not clear how we should obtain control of the first derivatives with respect to the standard Euclidean norm by controlling (99). Due to a paper of Gu Chao-Hao, 1980, On the Cauchy Problem for Harmonic Maps Defined on Two-Dimensional Minkowski Space, *Comm. Pure Appl. Math.*, 727-737, there are geometric conditions that ensure global existence. For those familiar with Riemannian geometry, we mention that the condition is that of completeness of the target manifold.

Exercise. Consider a Riemannian metric on $\mathbb{R}^3$ defined by $g_{11} = 1$, $g_{22} = \exp(2x^1)$, $g_{33} = \exp(2x^2)$ and $g_{ij} = 0$ for $i \neq j$. Prove that there is global existence of solutions to (91) given this metric and smooth initial data with compact support.

2. A wave map equation arising in General Relativity

As we have already mentioned, when considering General Relativity, one often considers solutions that satisfy symmetry conditions. We have already given examples of equations one obtains by dropping the isotropy condition. In this section, we give an example where the condition of homogeneity has also been relaxed. There are symmetry assumption under which Einstein's equations reduce to the following system of equations:

$$P_{tt} + \frac{1}{t}P_t - P_{xx} = e^{2P}(Q_t^2 - Q_x^2) \tag{100}$$

$$Q_{tt} + \frac{1}{t}Q_t - Q_{xx} = -2(P_tQ_t - P_xQ_x). \tag{101}$$

Note the similarity of these equations with (94)-(95). In fact, these equations can also be viewed as wave map equations, but that requires a reformulation that is not of any greater interest here. The purpose of this section is to prove global existence of solutions to these equations. Since there is a $1/t$ appearing in the equations, global existence here means existence for $t \in (0,\infty)$. Furthermore, the reasonable initial value problem for (100)-(101) is to specify initial data for some time $t_0 > 0$. The tools for proving local and global existence of solutions to these equations are Theorem 19 and Corollary 4.

Exercise. Consider a solution to (100)-(101) given smooth initial data with compact support at some time $t_0 > 0$. Let the maximal existence interval be (t_-, t_+). Define $\mathcal{B}_+$ and $\mathcal{B}_-$ by

$$\mathcal{B}_{\pm} = \frac{t}{2}\left[(P_t \pm P_x)^2 + e^{2P}(Q_t \pm Q_x)^2\right].$$

Prove that

$$(\partial_t \mp \partial_x)\mathcal{B}_{\pm} = \frac{1}{2}\left[-P_t^2 + P_x^2 + e^{2P}(-Q_t^2 + Q_x^2)\right].$$

Prove that this implies that $(t_-, t_+) = (0,\infty)$.

The sort of solutions to (100)-(101) that one is interested in in General Relativity are such that are periodic in the x coordinate. To prove global existence of solutions with such initial data, one simply proceeds as in the argument presented after the proof of Theorem 20.

3. Energy

So far we have considered quantities of the form (59). These quantities are very powerful tools when analyzing $1+1$-dimensional wave equations, but they are not available in higher dimensions. Assume that u is a solution of the linear wave equation corresponding to initial data with compact support. Then u has compact support in x for every fixed t. Consequently, it makes sense to define

$$H(t) = \frac{1}{2}\int_{-\infty}^{\infty}[u_t^2 + u_x^2](t,x)dx. \tag{102}$$

Let us differentiate this object. Under the assumptions we have made on u, there is no problem in differentiating under the integral sign. We obtain

$$\frac{dH}{dt} = \int_{-\infty}^{\infty}[u_t u_{tt} + u_x u_{xt}]dx = \int_{-\infty}^{\infty}[u_t u_{tt} - u_{xx}u_t]dx = \int_{-\infty}^{\infty}[u_{tt} - u_{xx}]u_t dx = 0.$$

The crucial step is the second equality. We integrate the term $u_x u_{tx}$ by parts. By this trick, we obtain $u_{tt} - u_{xx}$ which is given by the equation. We see that if u solves (52) and we assume that u and F both have compact support in x during any finite time interval, then

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} u_t F dx.$$

We shall refer to the object defined in (102) as the *energy* of the solution. In the case of the linear wave equation the energy is conserved. For the inhomogeneous wave equation this is no longer true. The higher dimensional analogues of (102) are essential tools in proving local existence and uniqueness in higher dimensions; they play a role similar to the objects based on the quantity (59).

The energy H gives us control over the integral of certain quantities. Is it possible to obtain control over the function itself given bounds on the integral of the function and its derivatives?

PROPOSITION 12. *Assume $\phi \in C_0^1(\mathbb{R})$. Then*

$$|\phi(x)| \le \left(\int_{-\infty}^{\infty}[\phi^2 + (\phi')^2](s)ds\right)^{1/2} \tag{103}$$

for all $x \in \mathbb{R}$.

Proof. Since ϕ has compact support, there is an M such that $\phi(x) = 0$ for $|x| \ge M$. Consequently

$$\phi^2(x) = \int_{-M}^{x}\partial_s[\phi^2](s)ds = \int_{-M}^{x}2\phi(s)\phi'(s)ds \le \int_{-M}^{x}[\phi^2 + (\phi')^2](s)ds, \tag{104}$$

where we have used the fact that $ab \le (a^2+b^2)/2$, which in its turn follows from

$$0 \le (a-b)^2 = a^2 + b^2 - 2ab.$$

Note that if we take the square root of the left hand side of (104), we obtain the left hand side of (103) and the square root of the right hand side of (104) is bounded by the right hand side of (103). $\square$

Note that we can rewrite (103) in the following way:

$$\|\phi\|_{C_b(\mathbb{R})} \leq \left(\int_{-\infty}^{\infty} [\phi^2 + (\phi')^2](s)ds\right)^{1/2} \tag{105}$$

for all $\phi \in C_0^1(\mathbb{R})$. The inequality (105) is the first example of a so called *Sobolev inequality*. In the analysis of higher dimensional wave equations these sorts of inequalities play a very important role and we shall see that they are one crucial ingredient to proving local existence.

Let us consider the non-linear wave equation

$$\begin{cases} u_{tt} - u_{xx} = -u^k \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases} \tag{106}$$

where $f, g \in C_0^\infty(\mathbb{R})$ and k is an odd integer. Note that since there is a constant $C < \infty$ such that $f(x)$ and $g(x)$ are zero for $|x| \geq C$, $u(t,x)$ is zero for $|x| \geq C + |t|$. This is a consequence of uniqueness, Theorem 14. Let us define

$$\hat{H}(t) = \frac{1}{2}\int_{-\infty}^{\infty} \left[u_t^2 + u_x^2 + \frac{2}{k+1}u^{k+1}\right](t,x)dx. \tag{107}$$

Note that since k is odd, $k+1$ is even, so that all the terms in the integrand are non-negative. Again, we are allowed to differentiate under the integral sign in order to obtain,

$$\frac{d\hat{H}}{dt} = \int_{-\infty}^{\infty} [u_{tt} - u_{xx} + u^k]u_t dx = 0.$$

In other words, $\hat{H}$ is a conserved quantity. This is all that is needed to prove global existence of solutions to (106).

THEOREM 21. *Let k be an odd integer and $f, g \in C_0^\infty(\mathbb{R})$. Then there is a unique $u \in C^\infty(\mathbb{R}^2)$ solving (106).*

Proof. We know that there is a local solution to the equation, and due to Corollary 3, all we need to show is that u cannot blow up in finite time. Due to (103), all we need to prove is that

$$\int_{-\infty}^{\infty} [u^2 + u_x^2](t,x)dx$$

does not blow up in finite time. We know that $\hat{H}$ is a conserved quantity, and consequently there is a constant $C < \infty$ such that

$$\int_{-\infty}^{\infty} [u_t^2 + u_x^2](t,x)dx \leq C.$$

All that remains to be proved is thus that

$$F(t) = \int_{-\infty}^{\infty} u^2(t,x)dx$$

cannot blow up in finite time. Let us compute

$$\left|\frac{dF}{dt}\right| = 2\left|\int_{-\infty}^{\infty} uu_t dx\right| \leq \int_{-\infty}^{\infty} [u^2 + u_t^2]dx \leq C + F.$$

Letting $G = F + C$, we conclude that as long as the solution is not identically zero, $G > 0$ and

$$\left|\frac{dG}{dt}\right| \leq G.$$

Integrating this inequality, we obtain the conclusion that G, and therefore F, cannot blow up in finite time. The theorem follows. □

When considering (106), we demanded that k be odd. Why? We already pointed out that all the terms in the integrand of the right hand side of (107) are non-negative due to the fact k is odd. If k is even, $\hat{H}$ will also be conserved, but it is not possible to draw any conclusions from that. In fact, consider the ODE

$$u_{tt} = -u^k,$$

where $k \geq 2$ is an even integer. If we let $u(0) < 0$ and $u_t(0) < 0$, then the solution blows up in finite time. Similarly to the proof of Proposition 10, one can then construct smooth initial data with compact support such that the solution to (106) blows up in finite time. Similarly, if we consider the equation

$$u_{tt} - u_{xx} = u^k$$

for some integer $k \geq 2$, then the solution will in general blow up in finite time.

The global existence result given by Theorem 21 is a special case of a more general result.

Exercise. Consider the equation

$$\begin{cases} u_{tt} - u_{xx} = -F(u) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x), \end{cases} \tag{108}$$

where $F \in C^\infty(\mathbb{R})$ is a function with the properties that $F(0) = 0$ and there is a function $G \in C^\infty(\mathbb{R})$ such that $G' = F$, $G(u) \geq 0$ for all u and $G(0) = 0$. Then for any $f, g \in C_0^\infty(\mathbb{R})$, there is a $u \in C^\infty(\mathbb{R}^2)$ solving (108).

Note that Theorem 21 is the special case of this result when k is odd and $F(u) = u^k$, $G(u) = u^{k+1}/(k+1)$. Note also that the restriction that the initial data have compact support can be removed by an argument similar to the one presented after the proof of Theorem 20.

4. Asymptotic behaviour

So far, we have only discussed the question of global existence. What about the asymptotics? It is possible to say something about the asympotic behaviour of solutions to (100)-(101) for instance, but that would require a substantial amount of time. Let us consider the special case of (100)-(101) when $Q = 0$. One is then left with a linear equation

$$\begin{cases} P_{tt} + \frac{1}{t}P_t - P_{xx} = 0 \\ P(0,x) = f(x) \\ P_t(0,x) = g(x). \end{cases} \tag{109}$$

Note that in order to prove existence of solutions to this equation, we can appeal to Theorem 19. This equation can of course be solved by separation of variables, but we shall use the methods developed in this chapter to analyze the asymptotics.

As was mentioned before, the case of interest for General Relativity is when P is periodic in x, with period 2π. Define

$$H(t) = \frac{1}{2}\int_0^{2\pi} (P_t^2 + P_x^2)(t,x)dx. \tag{110}$$

Using the standard integration by parts trick and the fact that P is 2π-periodic, one can compute that

$$\frac{dH}{dt} = -\frac{1}{t}\int_0^{2\pi} P_t^2 dx. \tag{111}$$

From this we conclude that H decays to the future. However, it is not clear that it decays to zero. This is rather similar to the following ODE situation. Let us make (a rather long) digression to explain it.

Consider the ODE

$$\ddot{x} + 2a\dot{x} + b^2 x = 0 \tag{112}$$

where $a > 0$ and $b^2 > a^2$. Of course we know how to solve this equation, but the idea is to develop methods that do not depend on our ability to solve it. We know that the solution in this case has to decay to zero as e^{-at}. Let us try to prove this without solving the equation. Define the quantity

$$H = \frac{1}{2}[\dot{x}^2 + b^2 x^2].$$

Compute that

$$\frac{dH}{dt} = -2a\dot{x}^2.$$

Again, we obtain the conclusion that H decays, but it is not clear that H converges to zero, even though we know that it should converge to zero exponentially. The idea is then to introduce a correction term

$$\Gamma = ax\dot{x}.$$

Why should one want to introduce such a quantity? The first reason is that

$$\frac{d\Gamma}{dt} = a\dot{x}^2 - 2a^2 x\dot{x} - ab^2 x^2,$$

so that

$$\frac{d(H+\Gamma)}{dt} = -2a(H+\Gamma).$$

In other words, $H + \Gamma$ has to decay as e^{-2at}, which is very promising. The second property is that

$$|\Gamma| = \left|\frac{a}{b}\right| |bx\dot{x}| \leq \left|\frac{a}{b}\right| \frac{1}{2}(\dot{x}^2 + b^2 x^2) = \left|\frac{a}{b}\right| H. \tag{113}$$

Note that $|a/b| < 1$ so that (113) implies

$$\left(1 - \left|\frac{a}{b}\right|\right) H \leq H + \Gamma \leq \left(1 + \left|\frac{a}{b}\right|\right) H. \tag{114}$$

This means that H can be bounded by a positive constant times $H + \Gamma$, which in its turn decays as e^{-2at}. We conclude that x and $\dot{x}$ decay like e^{-at}, which is the optimal result. So what's the point of all this? We could have solved the equation to start with and we would have obtained the same conclusion. Consider the equation

$$\ddot{x} + 2a\dot{x} + b^2 x = f(x, \dot{x}), \tag{115}$$

where f is smooth and

$$|f(x,\dot{x})| \leq C[|x|^2 + |\dot{x}|^2]^{\alpha/2} \tag{116}$$

for some $\alpha > 1$ and some constant C. Define H and Γ as above. Then one can compute

$$\frac{d(H+\Gamma)}{dt} = -2a(H+\Gamma) + \dot{x}f(x,\dot{x}) + axf(x,\dot{x}). \tag{117}$$

Combining (114) with (116) and the definition of H, we conclude that there is a positive constant c_0 such that

$$|\dot{x}f(x,\dot{x}) + axf(x,\dot{x})| \leq c_0(H+\Gamma)^\gamma,$$

where $\gamma = (\alpha+1)/2$. Note that $\gamma > 1$. Combining this estimate with (117), and using the notation

$$\hat{H} = H + \Gamma,$$

we obtain

$$\frac{d\hat{H}}{dt} \leq -2a\hat{H} + c_0\hat{H}^\gamma = [-2a + c_0\hat{H}^{\gamma-1}]\hat{H}. \tag{118}$$

If $\hat{H}(t_0) = 0$ for some t_0, then $\hat{H}(t) = 0$ for all t, so let us assume this is not the case. Assume $\hat{H}$ satisfies the bound

$$-2a + c_0\hat{H}^{\gamma-1}(t) \leq -c_1 \tag{119}$$

for some $c_1 > 0$ and $t = t_0$. Due to (118), we conclude that $d\hat{H}/dt$ is strictly negative at t_0. One can conclude that $\hat{H}$ decays to the future so that (119) holds for all $t \geq t_0$. Combining this observation with (118), we obtain

$$\frac{d\hat{H}}{dt} \leq -c_1\hat{H}.$$

As a consequence $\hat{H}$ decays as $e^{-c_1 t}$. This means that $\hat{H}^{\gamma-1}$ is an integrable function, so that (118) implies

$$\hat{H}(t) \leq \hat{H}(t_0)\exp\left[-2a(t-t_0) + \int_{t_0}^t c_0\hat{H}^{\gamma-1}(s)ds\right] \leq K\exp(-2at)$$

for some constant K. In other words, solutions to the equation (115) behave in the same way as solutions to (112) if the initial data are small enough, where smallness is defined by (119). The conclusion of the above argument is that it sometimes makes sense to develop methods that are based on decay of energy (H) and not on our ability to solve the equation. In a more general setting, we are usually not able to solve the equation, but arguments concerning decay of energy might very well generalize. Let us point out that the above argument was brought to our attention by Vincent Moncrief.

Let us return to the equation (109). Let H be defined by (110) and note that we have (111). We are in a similar situation to the one in the ODE case. Let us see if we can define a similar correction. The most naive analogue would be

$$\Gamma = \frac{1}{2t}\int_0^{2\pi} PP_t dx.$$

The analogy here is that in (109), $1/t$ corresponds to $2a$ in (112), whereas x and $\dot{x}$ correspond to P and P_t respectively. Since H only depends on t, we of course also need to integrate. This leads us to our definition of Γ. There is however a problem with this definition. In analogy with the ODE case, we want $|\Gamma|$ to be bounded by H at the very least. However, if we add a constant to P, it still solves the equation

$$P_{tt} + \frac{1}{t}P_t - P_{xx} = 0, \tag{120}$$

and the energy H is unchanged, but if we assume that the average of P_t is non-zero, Γ does change. In other words, given the above definition of Γ, it is not possible to have the inequality $|\Gamma| < H$ in all generality. Since the problem is related to constant translations in P, it is natural to subtract the average of P. Let us define

$$\langle P\rangle(t) = \frac{1}{2\pi}\int_0^{2\pi} P(t,x)dx.$$

Furthermore

$$\Gamma = \frac{1}{2t}\int_0^{2\pi}(P - \langle P\rangle)P_t dx.$$

In order to prove that using the above definition of Γ, one does get the desired bound, it is useful to have the bound

$$\int_0^{2\pi} f^2(s)ds \le \int_0^{2\pi}(f')^2(s)ds, \tag{121}$$

which holds for all f that are smooth, 2π periodic and such that the average is zero. To prove this inequality, note that for such f, we have

$$f(s) = \sum_{n\in\mathbb{Z}} a_n e^{ins},$$

where $a_0 = 0$. Thus

$$\int_0^{2\pi} f^2(s)ds = 2\pi\sum_{n\in\mathbb{Z}}|a_n|^2 \le 2\pi\sum_{n\in\mathbb{Z}} n^2|a_n|^2 = \int_0^{2\pi}(f')^2(s)ds.$$

Since $P - \langle P\rangle$ clearly has zero average, is smooth and 2π-periodic, we obtain

$$\int_0^{2\pi}(P - \langle P\rangle)^2 dx \le \int_0^{2\pi} P_x^2 dx.$$

Using $ab \le (a^2 + b^2)/2$ and our definition of Γ, we thus obtain

$$|\Gamma| \le \frac{1}{4t}\int_0^{2\pi}[(P - \langle P\rangle)^2 + P_t^2]dx \le \frac{1}{4t}\int_0^{2\pi}[P_x^2 + P_t^2]dx = \frac{1}{2t}H.$$

For $t \ge 1$, there is thus no problem in bounding $|\Gamma|$ in terms of H. The question is then if we get something nice by differentiating Γ. Compute

$$\begin{aligned}
\frac{d\Gamma}{dt} &= -\frac{1}{2t^2}\int_0^{2\pi}(P - \langle P\rangle)P_t dx + \frac{1}{2t}\int_0^{2\pi}(P_t - \langle P_t\rangle)P_t dx \\
&\quad + \frac{1}{2t}\int_0^{2\pi}(P - \langle P\rangle)P_{tt}dx \\
&= -\frac{1}{t}\Gamma + \frac{1}{2t}\int_0^{2\pi}P_t^2 dx - \frac{\pi}{t}\langle P_t\rangle^2 + \frac{1}{2t}\int_0^{2\pi}(P - \langle P\rangle)\left(-\frac{1}{t}P_t + P_{xx}\right)dx \\
&= -\frac{1}{t}\Gamma + \frac{1}{2t}\int_0^{2\pi}P_t^2 dx - \frac{\pi}{t}\langle P_t\rangle^2 - \frac{1}{t}\Gamma - \frac{1}{2t}\int_0^{2\pi}P_x^2 dx,
\end{aligned}$$

where we integrated by parts in the last equality. Observe that the point of the definition of Γ is that when differentiating, we obtain the last term; what's missing in (111) is a term involving the integral of P_x^2. Combining this equality with (111), we obtain

$$\frac{d(H+\Gamma)}{dt} = -\frac{1}{t}(H+\Gamma) - \frac{1}{t}\Gamma - \frac{\pi}{t}\langle P_t\rangle^2. \tag{122}$$

Note that the last term is negative. Furthermore, since $|\Gamma| \leq H/2$ for $t \geq 1$, we have

$$\frac{1}{2}H \leq H + \Gamma \leq \frac{3}{2}H$$

for $t \geq 1$. Letting $E = H + \Gamma$, we then obtain

$$|\Gamma| \leq \frac{1}{t}E$$

for $t \geq 1$. Adding these observations to (122), we obtain

$$\frac{dE}{dt} \leq -\frac{1}{t}E + \frac{1}{t^2}E = -\left(\frac{1}{t} - \frac{1}{t^2}\right)E.$$

If H is identically zero, the solution is constant, so let us assume this is not the case. Then neither H nor E are ever zero. Consequently, we can divide this inequality by E and integrate for $t \geq 1$ in order to obtain

$$\ln\frac{E(t)}{E(1)} \leq -\ln t + 1 - \frac{1}{t} \leq -\ln t + 1,$$

which implies

$$E(t) \leq \frac{e}{t}E(1). \tag{123}$$

In other words E and thus H decay as $1/t$. Note that if we differentiate the equation (120) with respect to x k times, the equation remains the same. In other words, $\partial_x^k P$ also satisfies (120). Consequently we also get decay of the form (123) for the higher derivatives. Thus there is for every k a real constant C_k such that for $t \geq 1$,

$$\int_0^{2\pi} [(\partial_x^k \partial_t P)^2 + (\partial_x^{k+1} P)^2] dx \leq \frac{C_k}{t}.$$

In order to turn this decay into information concerning the function itself, we need an inequality of the form (105). The functions we are interested in estimating do however not have compact support. Let us consider a smooth, 2π-periodic function with zero average. Then there must be an $x_0 \in [0, 2\pi)$ such that $f(x_0) = 0$ (in order for the average to be zero). Consequently, for $x \in [0, 2\pi)$, we obtain

$$f^2(x) = \int_{x_0}^x 2f'(s)f(s)ds \leq \int_0^{2\pi} [f^2(s) + (f')^2(s)]ds \leq 2\int_0^{2\pi} (f')^2(s)ds,$$

where we have used (121) in the last step. Consequently

$$\|f\|_{C_b(\mathbb{R})} \leq \sqrt{2}\left(\int_0^{2\pi} (f')^2(s)ds\right).$$

In our situation, this means that

$$\|P - \langle P\rangle\|_{C_b(\mathbb{R})} \leq \frac{C}{t^{1/2}} \tag{124}$$

for some constant C and $t \geq 1$. Since $\langle P \rangle$ satisfies the equation

$$\langle P_{tt} \rangle + \frac{1}{t} \langle P_t \rangle = 0,$$

there are constants α and β such that

$$\langle P \rangle = \alpha \ln t + \beta.$$

Combining this with (124), we obtain

$$P = \alpha \ln t + \beta + O(t^{-1/2}).$$

By the comments made earlier, one can also make statements concerning the derivatives of P.

The point of the above discussion is not really the conclusions, since it would have been possible to obtain them using separation of variables methods. The point is rather to illustrate that there are methods based on energy quantities such as (110) and which do not depend on our ability to solve the equation. Similarly to the ODE case, these methods can be generalized to the non-linear case assuming the energy is small initially. In fact, it is possible to obtain conclusions for solutions to (100)-(101) assuming the initial energy is small using this sort of method.

CHAPTER 6

Local existence for $n+1$-dimensional wave equations

By now we have illustrated how to prove local existence in several different situations. The only difference with proving local existence in the $n+1$-dimensional case as opposed to the $1+1$-dimensional case is the function spaces used, plus some small technical complications. Our first task is thus to develop suitable function spaces. In the $1+1$ dimensional case we had tools such as the quantity (59). This is no longer available in the $n+1$-dimensional case. On the other hand, if we have a solution to the wave equation

$$\Box u = 0,$$

then the quantity

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n}[u_t^2 + |\nabla u|^2](t,x)dx, \tag{125}$$

which we shall refer to as the energy, is conserved, assuming the initial data for u have compact support for instance. In this definition

$$\nabla u = (\partial_1 u, ..., \partial_n u).$$

In order to prove that E is conserved, let us compute

$$\frac{dE}{dt} = \int_{\mathbb{R}^n}[u_t u_{tt} + \nabla u \cdot \nabla u_t]dx = \int_{\mathbb{R}^n}[u_{tt} - \Delta u]u_t dx = 0.$$

In the 1+1-dimensional case, the fact that $\mathcal{E}$ defined in (59) was a conserved quantity for the linear wave equation led us to use the C_d^k spaces as suitable spaces for the initial data. Furthermore, it led us to consider spaces of the form $C([-\epsilon,\epsilon], C_d^k(\mathbb{R}))$ when proving convergence of the iteration. In the $n+1$-dimensional case, the natural norm comes from the energy (125) and its analogues obtained by differentiating u. Thus the natural norm that presents itself is not the $C_b^k(\mathbb{R}^n)$ norm, but rather something of the form

$$\|u\|_{H^k(\mathbb{R}^n)} = \left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n}|\partial^\alpha u|^2 dx\right)^{1/2}. \tag{126}$$

That this object defines a norm is not completely clear at this point, but we shall prove it in the section on Sobolev spaces. However, we need the corresponding space of functions to be complete in order to get convergence of an iteration. Furthermore, we need to relate this norm to the function itself; i.e. we need analogues of the inequality (105) in n dimensions. These questions will be the subject of the first three sections.

1. The Fourier transform

The sources for this section are Peter Gilkey's book *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem* and Lars Hörmander's book *The Analysis of Linear Partial Differential Operators I.*

Before defining the Fourier transform and writing down its basic properties, it is natural to define the set of Schwartz functions.

DEFINITION 21. The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is the subset of $C^\infty(\mathbb{R}^n, \mathbb{C})$ (i.e. smooth, complex valued functions) such that for every pair of multiindices α and β, there is a real constant $C_{\alpha,\beta}$ such that

$$|x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta} \tag{127}$$

for all $x \in \mathbb{R}^n$.

Note that in the above definition, we use the notation

$$x^\alpha = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$$

for $x \in \mathbb{R}^n$ and a multiindex α. Note that $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is a subset of $\mathcal{S}(\mathbb{R}^n)$. The function spaces do however not coincide. An example of a function which is in $\mathcal{S}(\mathbb{R}^n)$ but not in $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is $\exp(-|x|^2)$. Let us define the Fourier transform.

DEFINITION 22. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define the Fourier transform of f, $\hat{f}$, by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx. \tag{128}$$

Since f is a Schwartz function, there is for every k a constant C_k such that

$$|f(x)| \leq C_k (1 + |x|^2)^{-k},$$

cf. (127). Consequently (128) makes sense. We leave it to the reader to prove that if $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f}$ is a smooth function and one can differentiate under the integral sign. As a consequence,

$$\partial_\xi^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^n} (-ix)^\alpha e^{-ix\cdot\xi} f(x) dx. \tag{129}$$

By integration by parts, we also obtain

$$\xi^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^n} i^{|\alpha|} \partial_x^\alpha \left(e^{-ix\cdot\xi}\right) f(x) dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} (-i)^{|\alpha|} \partial_x^\alpha f(x) dx. \tag{130}$$

Combining these two observations, we conclude that $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. Our next goal is to prove that it is possible to invert the Fourier transform. In fact, we wish to prove that for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi. \tag{131}$$

As a preparation, let us prove the following lemma.

LEMMA 11. *Let f_0 be defined by*

$$f_0(x) = \exp\left(-\frac{1}{2}|x|^2\right). \tag{132}$$

Then

$$\hat{f}_0(\xi) = (2\pi)^{n/2}\exp\left(-\frac{1}{2}|\xi|^2\right).$$

Proof. Note first that

$$\int_{\mathbb{R}^n} f_0(x)dx = (2\pi)^{n/2}. \tag{133}$$

Compute

$$\begin{aligned}\hat{f}_0(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\exp\left(-\frac{1}{2}|x|^2\right)dx\\ &= \exp\left(-\frac{1}{2}|\xi|^2\right)\int_{\mathbb{R}^n}\exp\left[-\frac{1}{2}(x+i\xi)\cdot(x+i\xi)\right]dx\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^n}\exp\left[-\frac{1}{2}(x+i\xi)\cdot(x+i\xi)\right]dx\\ &= \prod_{j=1}^{n}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(x^j+i\xi^j)\cdot(x^j+i\xi^j)\right]dx^j.\end{aligned}$$

Each factor on the right hand side is an integral in the complex plane, and by standard methods of complex analysis, we are allowed to shift the contour $t+i\xi^j$, $t\in\mathbb{R}$, to the real axis. By (133), we obtain the result. □

Due to Lemma 11, we conclude that (131) holds for $f=f_0$ (in order to obtain this conclusion we have used the fact that $f_0(x)=f_0(-x)$). Let us prove (131) in all generality.

THEOREM 22. *For all $f\in\mathcal{S}(\mathbb{R}^n)$, we have (131).*

Proof. Assume first that $f(0)=0$. Then

$$f(x) = \int_0^1 \frac{d}{dt}[f(tx)]dt = \sum_j x^j\int_0^1(\partial_j f)(tx)dt = \sum_j x^j g_j(x)$$

for some $g_j\in C^\infty(\mathbb{R}^n,\mathbb{C})$, $j=1,...,n$. Let $\phi\in C_0^\infty(\mathbb{R}^n)$ equal 1 on $\bar{B}_1(0)$. We can then write

$$f(x) = \phi(x)f(x)+[1-\phi(x)]f(x) = \sum_j x^j\phi(x)g_j(x)+\sum_j x^j\frac{x^j[1-\phi(x)]f}{|x|^2}.$$

Since the first term has compact support, it is in $\mathcal{S}(\mathbb{R}^n)$. The last term is also in $\mathcal{S}(\mathbb{R}^n)$ since $f\in\mathcal{S}(\mathbb{R}^n)$ and $1-\phi$ is identically zero in $\bar{B}_1(0)$. In fact, the above shows that there are $h_j\in\mathcal{S}(\mathbb{R}^n)$, $j=1,...,n$, such that

$$f = \sum_j x^j h_j.$$

By Fourier transforming this equality and using (129), we obtain

$$\hat{f}(\xi) = \sum_j i\partial_{\xi^j}\hat{h}_j.$$

Evaluating the right hand side of (131) at 0, we have

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{f}(\xi)d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n}\sum_j i\partial_{\xi^j}\hat{h}_j(\xi)d\xi = 0.$$

Since the left hand side of (131) vanishes at 0 by assumption, we conclude that (131) holds at $x = 0$ for all functions f such that $f(0) = 0$. Let $f \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary and decompose

$$f = f(0)f_0 + (f - f(0)f_0),$$

where f_0 is defined in (132). Since the second term vanishes at zero and since (131) holds for f_0, as was noted before the statement of the theorem, we obtain (131) for $x = 0$ and arbitrary $f \in \mathcal{S}(\mathbb{R}^n)$. In order to prove the equality for arbitrary x, let $x_0 \in \mathbb{R}^n$ and let $g(x) = f(x + x_0)$. By a change of variables, one can then compute that

$$\hat{g}(\xi) = e^{ix_0\cdot\xi}\hat{f}(\xi).$$

Consequently,

$$f(x_0) = g(0) = (2\pi)^{-n}\int_{\mathbb{R}^n}\hat{g}(\xi)d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix_0\cdot\xi}\hat{f}(\xi)d\xi,$$

which proves (131) in all generality. □

The Fourier transform is a very powerful tool that can be used in several different contexts. Here, we shall however only make very crude use of it. The main reason for discussing the Fourier transform is that we are interested in function spaces where the norm is given by (126). As we have seen, differentiation of the function corresponds to multiplication by powers of ξ on the Fourier side. This property of the Fourier transform makes it possible to give a very simple characterization of the norm (126) on the Fourier side. Before we can write down this characterization, we do however need to make a few more observations. Let us define the *convolution* of two elements $f, g \in \mathcal{S}(\mathbb{R}^n)$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

By a change of variables, one obtains $f * g = g * f$. There is a natural relation between convolution and the Fourier transform. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and consider

$$\begin{aligned}
\hat{f}(\xi)\hat{g}(\xi) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)e^{-iy\cdot\xi}g(y)dxdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi}f(x-y)e^{-iy\cdot\xi}g(y)dxdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x-y)g(y)dxdy.
\end{aligned}$$

Let us change the order of integration in the last expression. Note that this requires justification, which is provided by measure and integration theory due to the fact that the integral is absolutely convergent. We obtain

$$\hat{f}\hat{g} = \widehat{f * g}.$$

From this one can obtain

$$\widehat{fg} = (2\pi)^{-n}\hat{f} * \hat{g}.$$

We also have

$$\int_{\mathbb{R}^n} \hat{f} h dx = \int_{\mathbb{R}^n} f \hat{h} dx, \tag{134}$$

since both integrals equal

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) h(\xi) e^{-ix\cdot\xi} d\xi dx,$$

after a change of the order of integration. Let us apply (134) to h and f, where

$$h(x) = (2\pi)^{-n} \bar{\check{g}}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \bar{g}(\xi) d\xi.$$

Due to the Fourier inversion formula, we conclude that $\hat{h} = \bar{g}$. By applying (134), we obtain

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f} \bar{\hat{g}} dx = \int_{\mathbb{R}^n} f \bar{g} dx. \tag{135}$$

This identity is referred to as *Parseval's formula*, and it is a very useful tool. Let us for instance apply it with $f = g = \partial^\alpha u$, where $u \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\widehat{\partial^\alpha u} = i^{|\alpha|} \xi^\alpha \hat{u},$$

due to (130), we obtain

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\partial^\alpha u(x)|^2 dx.$$

By comparing with (126), we see the use of this expression. In fact, let $u \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\|u\|^2_{H^k(\mathbb{R}^n)} = (2\pi)^{-n} \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi.$$

It is convenient to give a somewhat different definition of the norm $\|\cdot\|_{H^k(\mathbb{R}^n)}$ on the Fourier side. We shall need the following preliminary result.

LEMMA 12. *For every positive integer k, there are positive real constants $c_{1,k}, c_{2,k}$ such that*

$$c_{1,k}(1 + |\xi|^2)^k \le \sum_{|\alpha| \le k} \xi^{2\alpha} \le c_{2,k}(1 + |\xi|^2)^k. \tag{136}$$

Proof. In order to prove the lemma, we only need to prove that the two functions

$$(1 + |\xi|^2)^{-k} \sum_{|\alpha| \le k} \xi^{2\alpha}, \quad (1 + |\xi|^2)^k \left(\sum_{|\alpha| \le k} \xi^{2\alpha} \right)^{-1} \tag{137}$$

are bounded. For $|\xi| \le 1$, both expressions are bounded since they are continuous functions. Furthermore, for $|\xi| \ge 1$, we have

$$(1 + |\xi|^2)^{-k} \sum_{|\alpha| \le k} \xi^{2\alpha} \le |\xi|^{-2k} \sum_{|\alpha| \le k} \xi^{2\alpha} = \sum_{|\alpha| \le k} \left(\frac{\xi}{|\xi|} \right)^{2\alpha} |\xi|^{2|\alpha| - 2k} \le \sum_{|\alpha| \le k} \left(\frac{\xi}{|\xi|} \right)^{2\alpha}.$$

This expression is also bounded, since $|\xi^j|/|\xi| \leq 1$. Consequently, the first expression in (137) is bounded. Consider the second expression for $|\xi| \geq 1$. We have

$$(1+|\xi|^2)^k \left(\sum_{|\alpha|\leq k} \xi^{2\alpha}\right)^{-1} \leq 2^k|\xi|^{2k} \left(\sum_{|\alpha|=k} \xi^{2\alpha}\right)^{-1} = 2^k \left[\sum_{|\alpha|=k} \left(\frac{\xi}{|\xi|}\right)^{2\alpha}\right]^{-1}.$$

In order to obtain the desired result, we need to prove that

$$\sum_{|\alpha|=k} \left(\frac{\xi}{|\xi|}\right)^{2\alpha}$$

is bounded from below. Since

$$\sum_{j=1}^{n} \frac{(\xi^j)^2}{|\xi|^2} = 1,$$

there must be a j such that

$$\frac{|\xi^j|}{|\xi|} \geq \frac{1}{\sqrt{n}}.$$

If α is the multiindex whose j:th component is k, we then obtain

$$\left(\frac{\xi}{|\xi|}\right)^{2\alpha} = \left(\frac{\xi^j}{|\xi|}\right)^{2k} \geq n^{-k}.$$

Consequently

$$\sum_{|\alpha|=k} \left(\frac{\xi}{|\xi|}\right)^{2\alpha} \geq n^{-k}$$

and the lemma follows. □

We are now in a position to define a norm which is equivalent to (126) (we shall prove below that (138) defines a norm).

DEFINITION 23. Define for $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}$,

$$|u|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{1/2}. \tag{138}$$

Note that if $s = k$ is a non-negative integer, there are positive constants $C_{i,k}$, $i = 1, 2$ such that

$$C_{1,k}|u|_{H^k(\mathbb{R}^n)} \leq \|u\|_{H^k(\mathbb{R}^n)} \leq C_{2,k}|u|_{H^k(\mathbb{R}^n)} \tag{139}$$

due to Lemma 12. When two norms satisfy this type of inequality, we say that they are *equivalent*. Due to this equivalence, it is clear that the index s has something to do with the degree of differentiability of the function. On the other hand, the definition (138) makes sense for any real number s. This makes it natural to speak of for instance half a derivative. In fact, one can use the Fourier transform to define non-integer powers of certain differential operators, though that will not be of any major interest in this course.

Before we can state the result that motivates the entire section from our point of view, we need the following.

DEFINITION 24. Define

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^2(x)dx\right)^{1/2}$$

for $f \in C(\mathbb{R})$ such that the right hand side is finite.

LEMMA 13. *Let $f, g \in C(\mathbb{R}^n)$ be such that their square is integrable. Then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{140}$$

Proof. By assumption $\|f\|_{L^2(\mathbb{R}^n)}$ and $\|g\|_{L^2(\mathbb{R}^n)}$ are finite. If one of them is zero, then the corresponding function has to be zero, and then the inequality follows trivially. Assume therefore that $\|f\|_{L^2(\mathbb{R}^n)}$ and $\|g\|_{L^2(\mathbb{R}^n)}$ are both positive. Define

$$f_1 = \frac{f}{\|f\|_{L^2(\mathbb{R}^n)}}, \quad g_1 = \frac{f}{\|g\|_{L^2(\mathbb{R}^n)}}.$$

Then $\|f_1\|_{L^2(\mathbb{R}^n)} = 1$ and $\|g_1\|_{L^2(\mathbb{R}^n)} = 1$. By the inequality $ab \leq (a^2 + b^2)/2$, we obtain

$$\int_{\mathbb{R}^n} |f_1(x)g_1(x)|dx \leq \frac{1}{2}\int_{\mathbb{R}^n} [f_1^2(x) + g_1^2(x)]dx = \frac{1}{2}[\|f_1\|^2_{L^2(\mathbb{R}^n)} + \|g_1\|^2_{L^2(\mathbb{R}^n)}] = 1.$$

Multiplying this inequality with $\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$, we obtain the conclusion of the lemma. □

COROLLARY 5. *Let $f, g \in C(\mathbb{R}^n)$ be such that their square is integrable. Then*

$$\|f + g\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}. \tag{141}$$

Proof. Let us estimate

$$\begin{aligned}
\|f + g\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |f + g|^2 dx \leq \int_{\mathbb{R}^n} [|f|^2 + 2|fg| + |g|^2]dx \\
&\leq \|f\|^2_{L^2(\mathbb{R}^n)} + 2\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} + \|g\|^2_{L^2(\mathbb{R}^n)} \\
&= (\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)})^2,
\end{aligned}$$

where we used (140) in the second to last step. We conclude that (141) holds. □

So far it is not so clear why we should be interested in any of these things. However, as we mentioned in the introduction, norms of the form (126) are forced upon us by the wave equation. On the other hand, at some stage we need to control the sup norm of the derivatives of the function. The purpose of this section is to bridge the gap.

THEOREM 23. *Let k be a non-negative integer and assume that $s > k + n/2$. Then there is a constant C, depending on k, n and s such that for all $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|f\|_{C^k_b(\mathbb{R}^n,\mathbb{C})} \leq C|f|_{H^s(\mathbb{R}^n)}. \tag{142}$$

Remark. In the statement of this theorem and below, we shall use the letter C to denote any constant. In other words, what C actually is may change from line to line. This inequality is an example of a Sobolev inequality.

Proof. Let us first consider the case $k=0$. Due to (131), we have

$$\begin{aligned}|f(x)| &\leq (2\pi)^{-n}\int_{\mathbb{R}^n}|\hat{f}(\xi)|d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}|\hat{f}(\xi)|d\xi \\ &\leq (2\pi)^{-n}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{s}|\hat{f}(\xi)|^2d\xi\right)^{1/2} \\ &= (2\pi)^{-n}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi\right)^{1/2}|f|_{H^s(\mathbb{R}^n)},\end{aligned}$$

where we have used (140) in the second to last step. Since $(1+|\xi|^2)^{-s}$ is integrable for $s>n/2$, we obtain the desired result. Let α be any multiindex. Then if $s-|\alpha|>n/2$, we obtain

$$\|\partial^\alpha f\|_{C_b(\mathbb{R},\mathbb{C})} \leq C|\partial^\alpha f|_{H^{s-|\alpha|}(\mathbb{R}^n)} \leq C|f|_{H^s(\mathbb{R}^n)},$$

where we have used the inequality (143) below. Adding these inequalities for all α such that $|\alpha|\leq k$, we obtain the statement of the theorem. □

LEMMA 14. *Let α be a multiindex and $s\in\mathbb{R}$. Then there is a constant C depending on α and s such that for all $f\in\mathcal{S}(\mathbb{R}^n)$,*

$$|\partial^\alpha f|_{H^{s-|\alpha|}(\mathbb{R}^n)} \leq C|f|_{H^s(\mathbb{R}^n)}. \tag{143}$$

Proof. Note that there is a constant C such that

$$|\xi^\alpha|^2(1+|\xi|^2)^{s-|\alpha|} \leq C(1+|\xi|^2)^s.$$

This follows from (136). Consequently

$$|\partial^\alpha f|^2_{H^{s-|\alpha|}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|\xi|^2)^{s-|\alpha|}|\xi^\alpha\hat{f}(\xi)|^2d\xi \leq C|f|^2_{H^s(\mathbb{R}^n)},$$

and the lemma follows. □

COROLLARY 6. *Let k and m be non-negative integers such that $m>k+n/2$. Then there is a constant C depending on n, k and m such that for all $f\in\mathcal{S}(\mathbb{R}^n)$,*

$$\|f\|_{C^k_b(\mathbb{R}^n,\mathbb{C})} \leq C\|f\|_{H^m(\mathbb{R}^n)}.$$

Proof. The statement follows from (142) and (139). □

2. Elements of measure and integration theory

As we have already mentioned, "norms" of the form (126) are forced upon us by the equation. In order to get an iteration that converges, we need to provide a class of functions such that this class, with the norm (126) constitutes a Banach space. We begin this section by illustrating that this class of functions must have certain peculiarities.

Let us consider the special case of (126) when $k=0$ and $n=1$. Consider the sequence of functions

$$f_n(x) = \begin{cases} x^n & \text{if } x\in[-1,1] \\ x^{-n} & \text{if } x\notin[-1,1]. \end{cases}$$

For $n = 1, 2, ...$, this defines a sequence of continuous, square integrable functions. Compute

$$\|f_n\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |f_n(x)|^2 dx = \frac{2}{2n+1} + \frac{2}{2n-1}.$$

We conclude that $\|f_n\|_{L^2(\mathbb{R})}$ converges to zero. This leads to several problems. If one considers the pointwise convergence of f_n, one sees that for $x \notin \{-1, 1\}$, $f_n(x)$ converges to zero, for $x = 1$, $f_n(x)$ converges to 1 and finally, for $x = -1$, $f_n(x)$ does not converge at all. It is thus natural to think of the limit function f as being 0 for $x \notin \{-1, 1\}$, as being 1 for $x = 1$ and as being undefined for $x = -1$. On the other hand $\|f_n\|_{L^2(\mathbb{R})}$ converges to zero, so the limit should be zero. Thus, if we want a class of functions such that $\|\cdot\|_{L^2(\mathbb{R})}$ defines a norm on this class, which turns the space into a Banach space, then the limit function has to be thought of as being zero. The second problem is that even though the sequence of functions we started with was continuous, the pointwise limit certainly is not. So what is the resolution to these problems? First of all, one has to give up the continuity of the limit; that is not reasonable to ask. Secondly, one has to give up the idea to consider spaces of functions. Instead, one has to consider spaces of equivalence classes of funtions, where f is equivalent to g if the set on which they differ has "zero measure". For practical calculations, one can however think of an equivalence class as being one function, at least when integrating, since sets of measure zero do not influence the result.

We shall not give a detailed account of measure and integration theory here, but we wish to provide the reader with enough information that at least the definition of the $L^2(\mathbb{R}^n)$-functions becomes clear. The presentation given in this section is inspired by Avner Friedman's *Foundations of Modern Analysis*. The first question we are confronted with is how to define the volume, or measure, of as general subsets of $\mathbb{R}^n$ as possible. We know what the measure of a cube or a ball is, but how far can we genaralize this concept? Say that we can assign a measure to two sets A and B, then we should reasonably be able to assign a measure to the sets

$$A \cup B, \quad A \cap B, \quad A - B,$$

where

$$A - B = \{x \in A : x \notin B\}.$$

More optimistically, if A_n, $n \geq 1$ is a countable collection of sets to which we can assign a measure, it should be possible to assign a measure to

$$\bigcup_{n \geq 1} A_n \quad \text{and} \quad \bigcap_{n \geq 1} A_n.$$

We are led to the following definition.

Definition 25. Let X be a set. A class $\mathcal{A}$ of subsets of X is called a *σ-algebra* if the following conditions are fulfilled:

- $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.
- If $A,\ B \in \mathcal{A}$, then $A - B \in \mathcal{A}$.
- For any sequence of sets $A_n \in \mathcal{A}$, $n \geq 1$,

$$\bigcup_{n \geq 1} A_n \in \mathcal{A}. \tag{144}$$

Remark. The way to think about a σ-algebra is as a collection of sets to which one can assign a measure. In the above definition, $\emptyset$ denotes the empty set

If we apply (144) with $A_1 = A$, $A_2 = B$ and $A_n = \emptyset$ for $n \geq 3$, we can conclude that $A,\ B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. Let us denote the complement of a set A by A^c. In other words, $A^c = X - A$. Note that if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$. If $A,\ B \in \mathcal{A}$ then $A \cap B = A - B^c \in \mathcal{A}$. Finally, if $A_n \in \mathcal{A}$ for $n \geq 1$, then

$$\bigcap_{n\geq 1} A_n = X - \bigcup_{n\geq 1} A_n^c \in \mathcal{A}.$$

The next step is to define what we mean by a measure. First of all, it should be defined on a σ-algebra. It should also be non-negative, even though we allow it to equal ∞ for some sets. We shall say that a function which is allowed to equal plus or minus infinity as well as any real number an *extended real-valued* function. We expect the empty set to have measure zero and finally the measure of the union of two disjoint sets should be the sum of the measure of the sets. In fact, we are going to ask a little bit more. Let $\mathcal{A}$ be a σ-algebra and let μ be a function defined on it. Then we say that μ is *completely additive* on $\mathcal{A}$ if

$$\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

whenever the $A_n \in \mathcal{A}$, $n \geq 1$ are all disjoint.

DEFINITION 26. A *measure* is an extended real-valued set function μ having the following properties:

- The domain of μ is a σ-algebra $\mathcal{A}$.
- μ is non-negative on $\mathcal{A}$.
- μ is completely additive on $\mathcal{A}$.
- $\mu(\emptyset) = 0$.

From the definition one can draw several conclusions. Say for instance that $A \subseteq B$, where $A,\ B \in \mathcal{A}$. Then $B = (B - A) \cup A$ is a union of disjoint sets. Applying the complete additivity of the measure with $A_1 = A$, $A_2 = B - A$ and $A_n = \emptyset$ for $n \geq 3$, we obtain

$$\mu(B) = \mu(A) + \mu(B - A).$$

Since $\mu(B - A) \geq 0$, we obtain the conclusion that for any $A,\ B \in \mathcal{A}$ such that $A \subseteq B$,

$$\mu(A) \leq \mu(B).$$

We shall take the following fact from measure and integration theory for granted. There is a σ-algebra $\mathcal{A}_n$ of subsets of $\mathbb{R}^n$ which contains all the open and all the closed subsets of $\mathbb{R}^n$ and a measure μ_n defined on $\mathcal{A}_n$ such that:

- If $a^i \leq b^i$, $i = 1, ..., n$ are real numbers and A is defined by
$$A = \{x \in \mathbb{R}^n : a^i \leq x^i \leq b^i\},$$
then
$$\mu_n(A) = \prod_{i=1}^{n} (b^i - a^i).$$

- The measure μ_n, with domain $\mathcal{A}_n$, is *complete*. In other words, if $A \subseteq B$, $B \in \mathcal{A}_n$ and $\mu(B) = 0$, then $A \in \mathcal{A}_n$.

This measure is referred to as the *Lebesgue measure.*

The next step is to generalize the concept of integration. In order to do so, we need to define a suitable class of functions.

DEFINITION 27. Let $X \in \mathcal{A}_n$ and let $f : X \to \mathbb{R}$. We say that f is a (Lebesgue) *measurable* function on X if for every open $U \subseteq \mathbb{R}$,

$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$

is measurable, in other words if $f^{-1}(U) \in \mathcal{A}_n$.

Remark. A function with values in $\mathbb{R}^m$ is said to be measurable if all the components are.

One can prove that if f and g are measurable functions, then so are $f \pm g$, fg and af for any $a \in \mathbb{R}$.

Let us turn to the definition of the integral. If A is a measurable set, let

$$\chi_A = \begin{cases} 1 & \text{if } \ x \in A \\ 0 & \text{if } \ x \notin A. \end{cases}$$

Clearly χ_A is a measurable function, since the only possibilities for $\chi_A^{-1}(U)$ are A, X and $\emptyset$. The natural definition of the integral of χ_A is

$$\int \chi_A d\mu_n = \mu_n(A).$$

Let us define a more general class of functions. A function f is called a *simple function* if there is a finite number of mutually disjoint measurable sets $E_1, ..., E_k$ and real numbers $\alpha_1, ..., \alpha_k$ such that

$$f = \sum_{i=1}^{k} \alpha_i \chi_{E_i}.$$

It is also clear how to define the integral of f:

$$\int f d\mu_n = \sum_{i=1}^{k} \alpha_i \mu_n(E_i).$$

Of course, in order for this to make sense, $\mu_n(E_i)$ has to be finite if $\alpha_i \neq 0$. If this is the case, we say that the simple function f is *integrable.* Note that, to make sure that this definition makes sense, we need to check that two different representations of the same simple function lead to the same result for the integral. This can be done. If f is an integrable simple function, we also define the integral of f over a measurable set E by

$$\int_E f d\mu_n = \int \chi_E f d\mu_n.$$

In order to define the concept of integrability more generally, we need to introduce some terminology.

DEFINITION 28. A sequence $\{f_m\}$ of integrable simple functions is said to be a *Cauchy sequence in the mean* if for every $\epsilon > 0$ there is an N such that $l, m \geq N$ implies

$$\int |f_l - f_m| d\mu_n \leq \epsilon.$$

DEFINITION 29. An extended real valued measurable function f is said to be *integrable* if there is a sequence $\{f_m\}$ of integrable simple functions such that

- $\{f_m\}$ is a Cauchy sequence in the mean.
- The set A of points x for which $f_m(x)$ does not converge to $f(x)$ has zero measure.

If f is integrable, we define the integral by

$$\int f d\mu_n = \lim_{m\to\infty} \int f_m d\mu_n.$$

This definition requires some justification. First of all, note that the limit exists, since

$$\left| \int f_l d\mu_n - \int f_m d\mu_n \right| \leq \int |f_l - f_m| d\mu_n \to 0$$

as $l, m \to \infty$. Secondly, one needs to prove that the definition does not depend on the chosen sequence of simple functions. This can be done, but we shall not write down the argument here. If f is integrable we shall refer to the above limit as the Lebesgue integral of f. Finally, let us note that if f is continuous and $|f|$ is Riemann integrable, then the Riemann integral of f coincides with the Lebesgue integral.

We are now in a position to define the L^2 spaces.

DEFINITION 30. Denote the class of Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $|f|^2$ is integrable by $\mathcal{L}^2(\mathbb{R}^n)$.

One can prove that if $f, g \in \mathcal{L}^2(\mathbb{R}^n)$, then $f + g \in \mathcal{L}^2(\mathbb{R}^n)$. Furthermore, the norm $\|\cdot\|_{L^2(\mathbb{R}^n)}$ can be extended to make sense for $f \in \mathcal{L}^2(\mathbb{R}^n)$:

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\int |f|^2 d\mu_n \right)^{1/2}.$$

We also have the inequality (141). One property of the Lebesgue integral is that if f is measurable and the set A on which f is non-zero has the property that $\mu_n(A) = 0$, then

$$\int |f| d\mu_n = 0.$$

Consequently, if $f \in \mathcal{L}^2(\mathbb{R}^n)$ has the property that the set on which it is non-zero has measure zero, then $\|f\|_{L^2(\mathbb{R}^n)} = 0$. In other words, in order to turn $\|\cdot\|_{L^2(\mathbb{R}^n)}$ into a norm, we need to think of functions that are only non-zero on a set of measure zero as being zero. Let us introduce the following equivalence relation on functions in $\mathcal{L}^2(\mathbb{R}^n)$: we say that f and g are *equivalent* and write $f \sim g$, if the set A on which $f \neq g$ has the property that $\mu_n(A) = 0$. Note that this relation is *reflexive*:

$$f \sim f,$$

symmetric:
$$f \sim g \Rightarrow g \sim f,$$
and *transitive*:
$$f \sim g \quad \text{and} \quad g \sim h \Rightarrow f \sim h.$$
Any relation having the three properties of being reflexive, symmetric and transitive is called an *equivalence relation*. Given $f \in \mathcal{L}^2(\mathbb{R}^n)$, we denote its *equivalence class* by
$$[f] = \{g \in \mathcal{L}^2(\mathbb{R}^n) : g \sim f\}.$$
Due to the fact that $\sim$ is an equivalence relation, $g \in [f]$ implies $[g] = [f]$.

DEFINITION 31. Define $L^2(\mathbb{R}^n)$ to be the set of equivalence classes $[f]$ of functions $f \in \mathcal{L}^2(\mathbb{R}^n)$. In other words,
$$L^2(\mathbb{R}^n) = \{[f] : f \in \mathcal{L}^2(\mathbb{R}^n)\}.$$
We define the norm $\|\cdot\|_{L^2(\mathbb{R}^n)}$ on $L^2(\mathbb{R}^n)$ by
$$\|[f]\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

This definition requires some justification. First of all, we need to prove that the norm is well defined. Say that $[f] = [g]$. Then $f = g$ except for a set of zero measure, and consequently $\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}$. Secondly, we need to prove that $\|\cdot\|_{L^2(\mathbb{R}^n)}$ defines a norm. Since we have (141) and since it is clear that $\|a[f]\|_{L^2(\mathbb{R}^n)} = |a| \cdot \|[f]\|_{L^2(\mathbb{R}^n)}$, the only thing that remains to be proved is that $\|[f]\|_{L^2(\mathbb{R}^n)} = 0$ implies $[f] = 0$. However, it is a result of measure and integration theory that if $\|f\|_{L^2(\mathbb{R}^n)} = 0$, then $f = 0$ except for a set of measure zero, so that $[f] = 0$. The main result is the following.

THEOREM 24. *The space $L^2(\mathbb{R}^n)$ with norm $\|\cdot\|_{L^2(\mathbb{R}^n)}$ is a Banach space.*

For the sake of notational simplicity, we shall from now on write f when we actually mean $[f]$.

3. Sobolev spaces

We have introduced several objects that we have referred to as norms, cf. (126) and (138). We of course have to prove that these formulas actually define norms. Furthermore, we would like to have function spaces that together with these norms constitute Banach spaces.

LEMMA 15. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$, $s \in \mathbb{R}$ and let k be a non-negative integer. Then*
$$\|f+g\|_{H^k(\mathbb{R}^n)} \leq \|f\|_{H^k(\mathbb{R}^n)} + \|g\|_{H^k(\mathbb{R}^n)}, \tag{145}$$
$$|f+g|_{H^s(\mathbb{R}^n)} \leq |f|_{H^s(\mathbb{R}^n)} + |g|_{H^s(\mathbb{R}^n)}, \tag{146}$$
$$\|\lambda f\|_{H^k(\mathbb{R}^n)} = |\lambda| \cdot \|f\|_{H^k(\mathbb{R}^n)},$$
$$|\lambda f|_{H^s(\mathbb{R}^n)} = |\lambda| \cdot |f|_{H^s(\mathbb{R}^n)}.$$

Remark. Note that there is one important property of the norm that we do not prove in this lemma. We need to know that $\|f\|_{H^k(\mathbb{R}^n)} = 0$ implies $f = 0$. If $f \in \mathcal{S}(\mathbb{R}^n)$ this is clearly true, but it turns out that if we give $\mathcal{S}(\mathbb{R}^n)$ the norm $\|\cdot\|_{H^k(\mathbb{R}^n)}$, then the resulting space is not complete. Since we want to have a Banach space, we consequently have to consider a larger space of functions than merely the Schwartz functions, and it turns out that in this larger space, the question of whether $\|f\|_{H^k(\mathbb{R}^n)} = 0$ implies $f = 0$ or not is more delicate. We shall therefore consider it separately below.

Proof. The only assertions of the lemma that are not obvious are (145) and (146). Consider

$$\begin{aligned}
\|f+g\|_{H^k(\mathbb{R}^n)} &= \left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n}|\partial^\alpha(f+g)|^2dx\right)^{1/2}\\
&= \left(\sum_{|\alpha|\leq k}\|\partial^\alpha f+\partial^\alpha g\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}\\
&\leq \left(\sum_{|\alpha|\leq k}\left[\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}+\|\partial^\alpha g\|_{L^2(\mathbb{R}^n)}\right]^2\right)^{1/2}\\
&\leq \left(\sum_{|\alpha|\leq k}\|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}+\left(\sum_{|\alpha|\leq k}\|\partial^\alpha g\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}\\
&= \|f\|_{H^k(\mathbb{R}^n)}+\|g\|_{H^k(\mathbb{R}^n)},
\end{aligned}$$

where we have used (141), and in the second to last step, the inequality

$$\left(\sum_{|\alpha|\leq k}[|a_\alpha|+|b_\alpha|]^2\right)^{1/2}\leq\left(\sum_{|\alpha|\leq k}|a_\alpha|^2\right)^{1/2}+\left(\sum_{|\alpha|\leq k}|b_\alpha|^2\right)^{1/2}$$

where a_α and b_α are complex numbers for multiindices α such that $|\alpha|\leq k$. This inequality in its turn is simply a reformulation of the fact that for any $x,y\in\mathbb{R}^m$, we have

$$|x+y|\leq|x|+|y|.$$

Let us turn to (146). We have

$$\begin{aligned}
\|f+g\|_{H^s(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{f}+\hat{g}|^2d\xi\right)^{1/2}\\
&= \|(1+|\xi|^2)^{s/2}\hat{f}+(1+|\xi|^2)^{s/2}\hat{g}\|_{L^2(\mathbb{R}^n)}\\
&\leq \|(1+|\xi|^2)^{s/2}\hat{f}\|_{L^2(\mathbb{R}^n)}+\|(1+|\xi|^2)^{s/2}\hat{g}\|_{L^2(\mathbb{R}^n)}\\
&= \|f\|_{H^s(\mathbb{R}^n)}+\|g\|_{H^s(\mathbb{R}^n)},
\end{aligned}$$

where we have used (141). The lemma follows. □

We defined the L^2-"functions" in Definition 31. Note that we can view $\mathcal{S}(\mathbb{R}^n)$ as a linear subspace of $L^2(\mathbb{R}^n)$ and that $\|\cdot\|_{H^k(\mathbb{R}^n)}$ is defined on $\mathcal{S}(\mathbb{R}^n)$. Furthermore,

$$\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}\leq\|f\|_{H^k(\mathbb{R}^n)}$$

for any $f\in\mathcal{S}(\mathbb{R}^n)$ and $|\alpha|\leq k$. Consequently, if $\{f_m\}$ is a Cauchy sequence with respect to $\|\cdot\|_{H^k(\mathbb{R}^n)}$, i.e. if for every $\epsilon>0$ there is an M such that $m,l\geq M$ implies

$$\|f_m-f_l\|_{H^k(\mathbb{R}^n)}\leq\epsilon,$$

then $\{\partial^\alpha f_m\}$ is a Cauchy sequence with respect to $\|\cdot\|_{L^2(\mathbb{R}^n)}$ for $|\alpha|\le k$. By the completeness result, Theorem 24, we conclude that there is a function $f^\alpha\in L^2(\mathbb{R}^n)$ such that $\partial^\alpha f_m\to f^\alpha$ with respect to $\|\cdot\|_{L^2(\mathbb{R}^n)}$. We shall also use the notation $f=f^0$. We wish to think of f^α as being $\partial^\alpha f$, but it is not so clear that this is allowed, since f need not be differentiable or even continuous. One can however generalize the concept of differentiability in the following way. Let $\phi\in\mathcal{S}(\mathbb{R}^n)$. Then, for any $g\in\mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n}\phi\partial^\alpha g dx=(-1)^{|\alpha|}\int_{\mathbb{R}^n}g\partial^\alpha\phi dx$$

by partial integration. Since $f^\alpha\in L^2(\mathbb{R}^n)$ and $\phi\in L^2(\mathbb{R}^n)$, $f^\alpha\phi$ is integrable. Furthermore, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\phi f^\alpha dx-\int_{\mathbb{R}^n}\phi\partial^\alpha f_m dx\right| &\le \int_{\mathbb{R}^n}|(f^\alpha-\partial^\alpha f_m)\phi|dx\\ &\le \|f^\alpha-\partial^\alpha f_m\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)}\to 0.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^n}f_m\partial^\alpha\phi dx\to\int_{\mathbb{R}^n}f\partial^\alpha\phi dx.$$

We conclude that

$$\int_{\mathbb{R}^n}\phi f^\alpha dx=(-1)^{|\alpha|}\int_{\mathbb{R}^n}f\partial^\alpha\phi dx.$$

This observation leads to a natural generalization of the concept of differentiability.

DEFINITION 32. Let k be a non-negative integer. We say that a function $f\in L^2(\mathbb{R}^n)$ is k times L^2-*weakly differentiable* if for every multiindex α such that $|\alpha|\le k$, there is a function $f^\alpha\in L^2(\mathbb{R}^n)$ such that for every $\phi\in\mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\phi f^\alpha dx=(-1)^{|\alpha|}\int_{\mathbb{R}^n}f\partial^\alpha\phi dx. \tag{147}$$

We refer to f^α as the α:th weak derivative of f.

Remark. One can generalize this definition in the following way. We say that f is a *locally integrable function* if f is measurable and if $\chi_K f$ is integrable for every compact set K. We say that f is k times *weakly differentiable* if for every multiindex α such that $|\alpha|\le k$ there is a locally integrable function f^α such that (147) holds for all $\phi\in C_0^\infty(\mathbb{R}^n,\mathbb{C})$. This generalization will however not be of any interest to us here.

The definition requires some justification, since it is not clear that the weak derivative f^α is well defined. This can however be proved by measure and integration theory.

Note that if f is a k times L^2-weakly differentiable function, we can define $\|f\|_{H^k(\mathbb{R}^n)}$ by

$$\|f\|_{H^k(\mathbb{R}^n)}=\left(\sum_{|\alpha|\le k}\int_{\mathbb{R}^n}|f^\alpha|^2dx\right)^{1/2}.$$

By the above observations, if we have a sequence $\{f_m\}$ of functions in $\mathcal{S}(\mathbb{R}^n)$ which is a Cauchy sequence with respect to $\|\cdot\|_{H^k(\mathbb{R}^n)}$, then there is a k times weakly differentiable function f such that $\|f - f_m\|_{H^k(\mathbb{R}^n)}$ converges to zero. This suggests the following function space.

DEFINITION 33. Let $H^k(\mathbb{R}^n)$ be the set of k times L^2-weakly differentiable functions f such that there is a sequence $f_m \in \mathcal{S}(\mathbb{R}^n)$ with $\|f - f_m\|_{H^k(\mathbb{R}^n)} \to 0$.

Remark. One can in fact prove that $H^k(\mathbb{R}^n)$ is simply the set of k times L^2-weakly differentiable functions. We shall however not do so here.

THEOREM 25. *The space $H^k(\mathbb{R}^n)$ with norm $\|\cdot\|_{H^k(\mathbb{R}^n)}$ is a Banach space.*

Proof. We have already proved that $\|\cdot\|_{H^k(\mathbb{R}^n)}$ is a norm. What remains to be proved is that we have a Banach space. Let $\{f_l\}$ be a Cauchy sequence of functions in $H^k(\mathbb{R}^n)$. By definition there is, for each l, a sequence of functions in $\mathcal{S}(\mathbb{R}^n)$ converging to f_l. Thus, for every l, there is a $g_l \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f_l - g_l\|_{H^k(\mathbb{R}^n)} \leq 1/l$. Then $\{g_l\}$ is a Cauchy sequence. To prove this, let $\epsilon > 0$. Then there is an L such that $1/L < \epsilon/3$ and $m, l \geq L$ implies $\|f_l - f_m\|_{H^k(\mathbb{R}^n)} < \epsilon/3$. Consequently, if $l, m \geq L$,

$$\begin{aligned} \|g_l - g_m\|_{H^k(\mathbb{R}^n)} &\leq \|g_l - f_l\|_{H^k(\mathbb{R}^n)} + \|f_l - f_m\|_{H^k(\mathbb{R}^n)} + \|f_m - g_m\|_{H^k(\mathbb{R}^n)} \\ &< \frac{1}{l} + \frac{\epsilon}{3} + \frac{1}{m} < \epsilon. \end{aligned}$$

By arguments presented prior to the statement of the theorem, there is a $g \in H^k(\mathbb{R}^n)$ such that $\|g - g_l\|_{H^k(\mathbb{R}^n)} \to 0$. We need to prove that $\|g - f_l\|_{H^k(\mathbb{R}^n)} \to 0$. However

$$\|g - f_l\|_{H^k(\mathbb{R}^n)} \leq \|g - g_l\|_{H^k(\mathbb{R}^n)} + \|g_l - f_l\|_{H^k(\mathbb{R}^n)} \to 0.$$

Thus for every Cauchy sequence $\{f_l\}$ in $H^k(\mathbb{R}^n)$, there is an $f \in H^k(\mathbb{R}^n)$ such that $\|f - f_l\|_{H^k(\mathbb{R}^n)} \to 0$. ☐

We shall need to know that Corollary 6 holds for $f \in H^k(\mathbb{R}^n)$.

THEOREM 26. *Let k and m be non-negative integers such that $m > k + n/2$. Then there is a constant C depending on n, k and m such that for all $f \in H^m(\mathbb{R}^n)$*

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C\|f\|_{H^m(\mathbb{R}^n)}. \tag{148}$$

Remark. This statement should be interpreted in the following way. An element of $H^m(\mathbb{R}^n)$ is strictly speaking an equivalence class of functions. The statement is that in this equivalence class, there is one function which is in $C_b^k(\mathbb{R}^n, \mathbb{C})$, and for this function, we have the inequality (148). Note that if there are two functions f_i, $i = 1, 2$ which are in the same equivalence class and which are both continuous, then they have to coincide, since if two continuous functions differ, they have to differ on a set of positive measure.

Proof. Let $f \in H^m(\mathbb{R}^n)$. This means that there is a sequence $\{f_l\}$ of functions in $\mathcal{S}(\mathbb{R}^n)$ such that $\|f_l - f\|_{H^m(\mathbb{R}^n)} \to 0$. We are allowed to apply (148) to the functions f_l due to Corollary 6. In particular, we conclude that $\{f_l\}$ is a Cauchy sequence in $C_b^k(\mathbb{R}^n, \mathbb{C})$. Thus there is an $g \in C_b^k(\mathbb{R}^n, \mathbb{C})$ such that f_l converges to g with respect to $\|\cdot\|_{C_b^k(\mathbb{R}^n,\mathbb{C})}$. Of course, we would like to say that $f = g$, but this requires some justification. We know that $f_l \to f$ with respect to $\|\cdot\|_{H^m(\mathbb{R}^n)}$. In particular $f_l \to f$ with respect to $\|\cdot\|_{L^2(\mathbb{R}^n)}$. By a result of measure and integration theory, there is then a sequence f_{l_j} such that $f_{l_j}(x) \to f(x)$ except for a set of points of zero measure. Since we know that f_{l_j} converges to g uniformly, we conclude that $f = g$ except for a set of zero measure. The theorem follows. $\square$

4. The linear wave equation in $n+1$-dimensions

In this section, we follow quite closely the presentation given in *Lectures on nonlinear wave equations* by Christopher D. Sogge. We shall only consider the case $n = 3$. The reader interested in the general case is referred to Sogge's book. The reason for the restriction is that it is in the interest of brevity; to consider the general case does not require that much more of an extra effort. For $h \in C^2(\mathbb{R}^3)$, let us define

$$(A_r h)(x) = \frac{1}{4\pi}\int_{S^2} h(x+ry)d\sigma(y), \tag{149}$$

where $d\sigma$ is the standard measure on the 2-sphere. Let us compute

$$\begin{aligned} \partial_r(A_r h)(x) &= \frac{1}{4\pi}\int_{S^2}(\partial_j h)(x+ry)y^j d\sigma(y) \\ &= \frac{r}{4\pi}\int_{|y|<1}(\Delta h)(x+ry)dy \\ &= \frac{1}{4\pi r^2}\Delta\int_{|x-y|<r} h(y)dy. \end{aligned}$$

Here and below the Laplace operator Δ is taken with respect to x. Note that

$$\frac{1}{4\pi}\int_{|x-y|<r} h(y)dy = \int_0^r \rho^2 (A_\rho h)(x)d\rho$$

so that

$$\partial_r[A_r h(x)] = r^{-2}\Delta\int_0^r \rho^2(A_\rho h)(x)d\rho.$$

Multiplying with r^2 and differentiating with respect to r, we obtain

$$\partial_r\{r^2\partial_r[(A_r h)(x)]\} = \Delta[r^2 A_r h](x).$$

Letting $H(r,x) = (A_r h)(x)$, we obtain

$$\Delta H = r^{-1}\partial_r^2(rH). \tag{150}$$

Let us now suppose that $u \in C^2(\mathbb{R}^4)$ solves

$$u_{tt} - \Delta u = F \tag{151}$$

and define

$$U(r;t,x) = [A_r u(t,\cdot)](x) = \frac{1}{4\pi}\int_{S^2} u(t,x+ry)d\sigma(y).$$

Due to (150), we obtain

$$\Delta U = r^{-1}\partial_r^2(rU).$$

Furthermore, using (151),

$$\begin{aligned} \Delta U &= \frac{1}{4\pi}\int_{S^2}\Delta u(t,x+ry)d\sigma(y) \\ &= \partial_t^2 U - [A_r F(t,\cdot)](x). \end{aligned}$$

Fixing x and defining $v(t,r) = rU(r;t,x)$ and

$$G(t,r) = r[A_r F(t,\cdot)](x),$$

we thus obtain

$$v_{tt} - v_{rr} = G.$$

If $u(0,x) = f$ and $u_t(0,x) = g$, then the initial data for v are

$$v(0,r) = r(A_r f)(x), \quad (\partial_t v)(0,r) = r(A_r g)(x).$$

Using (53), we obtain

$$\begin{aligned} v(t,r) &= \frac{1}{2}\{[(r+t)A_{r+t}f](x) + [(r-t)A_{r-t}f](x)\} + \frac{1}{2}\int_{r-t}^{r+t} \rho(A_\rho g)(x)d\rho \\ &+\frac{1}{2}\int_0^t \left[\int_{r+s-t}^{r+t-s} G(s,v)dv\right] ds. \end{aligned}$$

Since $v = rU$ and $A_r f$ and $A_r g$ are even functions of r, we obtain

$$\begin{aligned} U(r;t,x) &= \frac{1}{2r}\{[(r+t)A_{r+t}f](x) - [(t-r)A_{t-r}f](x)\} + \frac{1}{2r}\int_{t-r}^{r+t} \rho(A_\rho g)(x)d\rho \\ &+\frac{1}{2r}\int_0^t \left[\int_{r+s-t}^{r+t-s} G(s,v)dv\right] ds. \end{aligned}$$

Note that $U(0;t,x) = u(t,x)$. The natural thing to do is thus to take the limit $r \to 0$ in the above expression. The first two terms converge to

$$\partial_t[tA_t f](x) + t(A_t g)(x).$$

What remains to be considered is thus the limit of

$$\frac{1}{2r}\int_0^t \left[\int_{r+s-t}^{r+t-s} G(s,v)dv\right] ds$$

as $r \to 0$. Consider

$$\begin{aligned} \frac{1}{2r}\int_{r+s-t}^{r+t-s} G(s,v)dv &= \frac{1}{2r}\int_{r+s-t}^{r+t-s} v[A_v F(s,\cdot)](x)dv \\ &= \frac{1}{2r}\int_{-r+t-s}^{r+t-s} v[A_v F(s,\cdot)](x)dv, \end{aligned}$$

due to the fact that $A_v F$ is an even function of v. Taking the limit, we thus obtain

$$(t-s)[A_{t-s}F(s,\cdot)](x).$$

Consequently

$$\frac{1}{2r}\int_0^t \left[\int_{r+s-t}^{r+t-s} G(s,v)dv\right] ds \to \int_0^t (t-s)[A_{t-s}F(s,\cdot)](x)ds.$$

To conclude, we thus obtain

$$u(t,x) = \partial_t[tA_t f](x) + t(A_t g)(x) + \int_0^t (t-s)[A_{t-s}F(s,\cdot)](x)ds.$$

We can rewrite this as

$$\begin{aligned} (152) \qquad u(t,x) &= \frac{1}{4\pi t^2}\int_{|x-y|=t} [tg(y) + f(y) - (\partial_j f)(y)(x^j - y^j)]d\sigma(y) \\ &+\frac{1}{4\pi}\int_0^t \int_{S^2} (t-s)F[s, x+(t-s)y]d\sigma(y)ds. \end{aligned}$$

Note that this proves uniqueness of solutions to

$$\begin{cases} u_{tt} - \Delta u = F \\ u(0,x) = f(x) \\ u_t(0,x) = g(x). \end{cases} \tag{153}$$

Furthermore, it proves that given $f \in C^3(\mathbb{R}^3)$, $g \in C^2(\mathbb{R}^3)$ and $F \in C^2(\mathbb{R}^4)$, there is a $C^2(\mathbb{R}^4)$ solution to (153).

Due to the formula (152) it is clear that if we want to determine the solution to (153) at some point (t,x) with $t > 0$, all we need to know is what f and g are on the sphere $S_{t,x} = \{y : |x-y| = t\}$ and what F is on the cone with base $S_{t,x}$ and vertex (t,x). The formula (152) also has the following interesting consequence.

PROPOSITION 13. *Consider a solution to (153) with $F = 0$ in $3+1$-dimensions. Assume that $f, g \in C_0^\infty(\mathbb{R}^3)$ and that $f(x) = g(x) = 0$ for $|x| \geq R$. Then there is a constant C_R only depending on R such that for all t,*

$$|u(t,x)| \leq C_R(1+t^2)^{-1/2}\left[\|f\|_{C_b(\mathbb{R}^3)} + \sum_j \|\partial_j f\|_{C_b(\mathbb{R}^3)} + \|g\|_{C_b(\mathbb{R}^3)}\right].$$

Proof. Consider first the case $0 < t \leq 2(R+1)$. The case $t < 0$ can be obtained by a simple time reversion. Then

$$\begin{aligned} |u(t,x)| &\leq \frac{1}{4\pi t^2}\int_{|x-y|=t}[t|g(y)| + |f(y)| + |(\partial_j f)(y)(x^j - y^j)|]d\sigma(y) \\ &\leq \left[\|f\|_{C_b(\mathbb{R}^3)} + t\sum_j \|\partial_j f\|_{C_b(\mathbb{R}^3)} + t\|g\|_{C_b(\mathbb{R}^3)}\right]. \end{aligned}$$

Since $t \leq 2(R+1)$, we can in this case use

$$C_R = 2(R+1)[1 + 4(R+1)^2]^{1/2}.$$

Let us estimate

$$\int_{|x-y|=t} |f(y)|d\sigma(y) \leq \|f\|_{C_b(\mathbb{R}^3)}\int_{|x-y|=t,|y|\leq R} d\sigma(y).$$

Consider the last factor. We want to maximize the area of the intersection of a sphere of radius t and a ball of radius R, where $t \geq 2(R+1)$. If the center of the sphere is inside the ball, the intersection is empty, so let us assume this is not the case. Then, let us place the origin at the center of the sphere and let the direction from the origin to the center of the ball define the z-axis. If we let θ measure the angle to the z-axis, then any point p which lies in the intersection between the sphere and the ball must have $\sin\theta \leq R/t$. Let $\theta_{\max}$ correspond to the maximum angle. The maximum area of the intersection is then bounded by

$$\begin{aligned} t^2\int_0^{2\pi} d\varphi \int_0^{\theta_{\max}} \sin\theta d\theta &= 2\pi t^2(1 - \cos\theta_{\max}) = 2\pi t^2\left[1 - \left(1 - \frac{R^2}{t^2}\right)^{1/2}\right] \\ &= 2\pi R^2\left[1 + \left(1 - \frac{R^2}{t^2}\right)^{1/2}\right]^{-1} \leq 2\pi R^2. \end{aligned}$$

The argument for the other terms is similar, and we obtain the desired conclusion. □

Note that as a consequence of this result, a solution to the free wave equation has to decay like $1/t$. In $1+1$-dimensions there is no decay, but the larger the dimension, the higher the decay. In fact, in $n+1$-dimensions the decay is of the form $t^{-(n-1)/2}$, cf. Sogge's book.

5. Energy, weak solutions

The basic tool in the proof of local existence will be the energy, defined by

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3}[u_t^2 + |\nabla u|^2]dx,$$

assuming for example that u has compact support in x on any finite time interval. Similarly to the argument in $1+1$-dimension, we can compute

$$\frac{dH}{dt} = \int_{\mathbb{R}^3} u_t F dx,$$

assuming u is a solution of (151). Note that this equality immediately gives a different proof of uniqueness of solutions to (153). In fact, assume we have two C^2-solutions u_i, $i = 1, 2$ to (153) with compact support in x for any finite time interval. Then the difference $u = u_1 - u_2$ satisfies (153) with $F = 0$ and $f = g = 0$. If we let H be the energy corresponding to u, we thus obtain $H(0) = 0$ and $dH/dt = 0$. Thus $H = 0$ so that $u_1 = u_2$.

Note that if we look at the homogeneous wave equation, then the energy H is preserved, and similarly the energies for any number of derivatives of u are preserved. In other words, if we define

$$H_k[u] = \frac{1}{2}\sum_{|\alpha|\le k}\int_{\mathbb{R}^3}[(\partial^\alpha\partial_t u)^2 + |\nabla\partial^\alpha u|^2]dx, \tag{154}$$

then H_k is preserved for a solution to the homogeneous wave equation. The natural spaces for the initial data that then present themselves are

$$f \in H^{k+1}(\mathbb{R}^3), \quad g \in H^k(\mathbb{R}^3)$$

(here and below we take it to be understood that f and g are real valued) and the natural space in which an iteration should converge would be

$$C\{[0,T], H^{k+1}(\mathbb{R}^3)\}, \quad C\{[0,T], H^k(\mathbb{R}^3)\},$$

the former for u and the latter for $\partial_t u$. The question then arises, what does it mean for a function to belong to this space? How can one relate membership in this class with classical differentiability?

LEMMA 16. *Assume $u \in C\{[0,T], H^k(\mathbb{R}^3)\}$ where $k > 3/2$. Then u is continuous, i.e. $u \in C\{[0,T]\times\mathbb{R}^3\}$.*

Proof. Since $u(t) \in H^k(\mathbb{R}^3)$ and $k > 3/2$, Theorem 26 yields the conclusion that there is a unique continuous function coinciding with $u(t)$. Thus we can think of u as a function of $3+1$ variables. Let $(t_k, x_k) \to (t, x)$, where $0 \le t, t_k \le T$. Estimate

$$|u(t,x) - u(t_k,x_k)| \le |u(t,x) - u(t,x_k)| + |u(t,x_k) - u(t_k,x_k)|.$$

The first term tends to zero as k tends to infinity, due to the fact that $u(t, \cdot)$ is continuous, cf. Theorem 26. The second term can be estimated by

$$C\|u(t) - u(t_k)\|_{H^k(\mathbb{R}^3)}$$

due to Theorem 26, but since $u \in C\{[0,T], H^k(\mathbb{R}^3)\}$, this also converges to zero. The lemma follows. $\square$

As a consequence, if

$$u \in C\{(T_-, T_+), H^{k+1}(\mathbb{R}^3)\}, \quad u_t \in C\{(T_-, T_+), H^k(\mathbb{R}^3)\}, \tag{155}$$

where $k > 3/2$, then $u \in C^1\{(T_-, T_+) \times \mathbb{R}^3\}$. We shall be interested in solutions to

$$u_{tt} - \Delta u = F(u, \partial u) \tag{156}$$

satisfying (155). Note that it is not completely clear what should be meant by this. Since u is C^1, the right hand side is well defined, but the left hand side is not. A function $u \in C^1\{(T_-, T_+) \times \mathbb{R}^3\}$ is said to be a *weak solution* of (156) if for every $\phi \in C_0^\infty[(T_-, T_+) \times \mathbb{R}^3]$,

$$\int_{(T_-, T_+)\times\mathbb{R}^3} [\phi_{tt} - \Delta\phi] u dt dx = \int_{(T_-, T_+)\times\mathbb{R}^3} F(u, \partial u)\phi dt dx. \tag{157}$$

Note that a weak solution which is also C^2 is a solution in the classical sense.

6. Iteration

The goal is to prove local existence of solutions to

$$\begin{cases} u_{tt} - \Delta u = F(u, \partial u) \\ u(0, \cdot) = f \\ u_t(0, \cdot) = g \end{cases} \tag{158}$$

where $f \in H^{k+1}(\mathbb{R}^3)$ and $g \in H^k(\mathbb{R}^3)$. In order to do so, we set up an iteration. For the sake of convenience, we wish to have smooth iterates, and consequently, we consider the iteration

$$\begin{cases} \partial_t^2 u_0 - \Delta u_0 = 0 \\ u_0(0, \cdot) = f_0 \\ \partial_t u_0(0, \cdot) = g_0 \end{cases} \tag{159}$$

and, for $l \geq 0$,

$$\begin{cases} \partial_t^2 u_{l+1} - \Delta u_{l+1} = F(u_l, \partial u_l) \\ u_l(0, \cdot) = f_l \\ \partial_t u_l(0, \cdot) = g_l. \end{cases} \tag{160}$$

Here, the sequences $\{f_l\}$ and $\{g_l\}$ have been chosen such that $f_l, g_l \in \mathcal{S}(\mathbb{R}^3)$, $f_l \to f$ with respect to $H^{k+1}(\mathbb{R}^3)$ and $g_l \to g$ with respect to $H^k(\mathbb{R}^3)$. The first task we have to perform is to prove that the iterates belong to the right spaces. By the equation prior to (152), the solution to (153) can be written

$$u(t, x) = \partial_t[tA_t f](x) + t(A_t g)(x) + \int_0^t (t - s)[A_{t-s}F(s, \cdot)](x) ds,$$

where A_t is defined in (149). Using this equation and the fact that the initial data are in $\mathcal{S}(\mathbb{R}^3)$, one can prove that

$$u_l, \partial_t u_l \in C[\mathbb{R}, H^m(\mathbb{R}^3)]$$

for all non-negative integers m. Note however that this does require some work.

7. Local existence

Prior to proving local existence, let us prove a result that will be of interest in the following. Note the similarity with Grönwall's lemma.

LEMMA 17. *Let $f, g \in C([0,T])$ be non-negative functions and assume that there is a constant $C \geq 0$ such that*

$$f(t) \leq C + \int_0^t g(s) f^{1/2}(s) ds \tag{161}$$

for all $t \in [0,T]$. Then, for all $t \in [0,T]$,

$$f^{1/2}(t) \leq C^{1/2} + \frac{1}{2}\int_0^t g(s) ds. \tag{162}$$

Remark. There is of course a similar result in the opposite time direction.

Proof. Define

$$h = C + \epsilon + \int_0^t g(s) f^{1/2}(s) ds$$

for some $\epsilon > 0$. Then h is continuously differentiable and strictly positive. We have

$$\frac{dh}{dt} = g f^{1/2} \leq g h^{1/2}.$$

Since h is positive we can divide by $h^{1/2}$ and integrate in order to obtain

$$2[h^{1/2}(t) - h^{1/2}(0)] \leq \int_0^t g(s) ds,$$

so that

$$h^{1/2}(t) \leq (C+\epsilon)^{1/2} + \frac{1}{2}\int_0^t g(s) ds.$$

Using (161) and letting ϵ tend to zero, we obtain (162). □

THEOREM 27. *Let $f \in H^{k+1}(\mathbb{R}^3)$ and $g \in H^k(\mathbb{R}^3)$. Assume furthermore that $F(0,0) = 0$ and that $k > 3$. Then there is an $\epsilon > 0$, depending on F, the H^{k+1}-norm of f, the H^k-norm of g and k, and a unique $C^{k-1}[(-\epsilon,\epsilon) \times \mathbb{R}^3]$ solution to the equation (158) such that*

$$u \in C\{(-\epsilon,\epsilon), H^{k+1}(\mathbb{R}^3)\}, \quad u_t \in C\{(-\epsilon,\epsilon), H^k(\mathbb{R}^3)\}. \tag{163}$$

Proof. Similarly to the earlier local existence proofs, we start by proving that we have *rough control* of the iterates. We wish to prove that there is an $\epsilon > 0$ and a C such that

$$H_k[u_l](t) \leq C$$

for all l and $|t| \leq \epsilon$. Note that we can choose the sequences $\{f_l\}$ and $\{g_l\}$ to be such that

$$\|f_l\|_{H^{k+1}(\mathbb{R}^3)} \leq 2\|f\|_{H^{k+1}(\mathbb{R}^3)}, \quad \|g_l\|_{H^k(\mathbb{R}^3)} \leq 2\|g\|_{H^k(\mathbb{R}^3)}. \tag{164}$$

For the sake of convenience, let us introduce the following convention. When we say that a constant depends on the *initial data*, we mean that it depends on $\|f\|_{H^{k+1}(\mathbb{R}^3)}$ and $\|g\|_{H^k(\mathbb{R}^3)}$. When we say that a constant depends on *the data*, we mean that it depends on the initial data, on F and on k. Since u_0 is a solution to the homogeneous wave equation,

$$H_k[u_0](t) = H_k[u_0](0) \leq C_1,$$

where C_1 only depends on the initial data and is such that

$$H_k[u_l](0) \leq C_1 \tag{165}$$

for all l. Let us make the *inductive assumption* that

$$H_k[u_l](t) \leq C_1 + 1 \tag{166}$$

for $|t| \leq \epsilon$. For $l = 0$ this is clearly true. Let us note some consequences of this assumption. Since $k > 3/2$, we get a bound (depending on the initial data) on

$$\sum_{j=1}^{3} \|\partial_j u_l(t, \cdot)\|_{C_b(\mathbb{R}^3)}, \quad \|\partial_t u_l(t, \cdot)\|_{C_b(\mathbb{R}^3)}.$$

Note that we do not obtain a bound on u_l in L^2 directly. However,

$$\left|\partial_t \int_{\mathbb{R}^3} u_l^2 dx\right| = 2\left|\int_{\mathbb{R}^3} u_l \partial_t u_l dx\right| \leq 2\left(\int_{\mathbb{R}^3} u_l^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^3} (\partial_t u_l)^2 dx\right)^{1/2},$$

where we used Lemma 13 in the last step. Letting

$$a_l = \int_{\mathbb{R}^3} u_l^2 dx, \quad b_l = \left(\int_{\mathbb{R}^3} (\partial_t u_l)^2 dx\right)^{1/2}$$

and integrating this inequality, we obtain

$$a_l(t) \leq a_l(0) + 2\left|\int_0^t b_l(s) a_l^{1/2}(s) ds\right|.$$

We can then apply Lemma 17 in order to obtain

$$a_l^{1/2}(t) \leq a_l^{1/2}(0) + \left|\int_0^t b_l(s) ds\right|.$$

If we assume that $\epsilon \leq 1$, we can use the inductive assumption together with (164) in order to get an estimate for

$$a_l(t) = \int_{\mathbb{R}^3} u_l^2(t, x) dx$$

for $|t| \leq \epsilon$ which only depends on the initial data. To conclude

$$\|u_l(t, \cdot)\|_{H^{k+1}(\mathbb{R}^3)} + \|\partial_t u_l(t, \cdot)\|_{H^k(\mathbb{R}^3)} \leq C, \tag{167}$$

for $|t| \leq \epsilon$, where C only depends on the initial data. Note that as a consequence, we also have a bound on $\|u_l(t, \cdot)\|_{C_b(\mathbb{R}^3)}$. Thus we have a bound on

$$\|[(\partial^\beta F)(u_l, \partial u_l)](t, \cdot)\|_{C_b(\mathbb{R}^3)}$$

for $|\beta| \leq k$ which only depends on the data.

We need to prove (166) with l replaced by $l + 1$. Let us consider

$$\frac{dH_k[u_{l+1}]}{dt} = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} \partial^\alpha F(u_l, \partial u_l) \partial^\alpha \partial_t u_{l+1} dx.$$

By Schwartz inequality,

$$\left|\sum_{|\alpha|\leq k} a_\alpha b_\alpha\right| \leq \left(\sum_{|\alpha|\leq k} |a_\alpha|^2\right)^{1/2} \left(\sum_{|\alpha|\leq k} |b_\alpha|^2\right)^{1/2}.$$

Combining this observation with Lemma 13, we obtain
(168)

$$\left|\frac{dH_k[u_{l+1}]}{dt}\right| \leq \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} [\partial^\alpha F(u_l, \partial u_l)]^2 dx\right)^{1/2} \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} (\partial^\alpha \partial_t u_{l+1})^2 dx\right)^{1/2}.$$

The last factor can be estimated by a constant times $H_k^{1/2}[u_{l+1}]$. We need to estimate the first factor. Let us consider the case $|\alpha| > 0$. Note that

$$\partial^\alpha F(u_l, \partial u_l)$$

can be written as a sum of terms that consist of a constant times an expression of the form

$$(\partial^\beta F)(u_l, \partial u_l)\partial^{\gamma_1} u_l \cdots \partial^{\gamma_i} u_l \partial^{\delta_1} \partial_{j_1} u_l \cdots \partial^{\delta_k} \partial_{j_k} u_l \partial^{\zeta_1} \partial_t u_l \cdots \partial^{\zeta_m} \partial_t u_l, \tag{169}$$

where $\gamma_1 + ... + \gamma_i + \delta_1 + ... + \delta_k + \zeta_1 + ... + \zeta_m = \alpha$ and $|\gamma_p|$, $|\delta_q|$, $|\zeta_r| > 0$. We have already noted that the first factor, involving F, is bounded. Consider the remaining factors. Note that if $k > 1/2 + |\gamma_p|$, then

$$||\partial^{\gamma_p} u_l||_{C_b(\mathbb{R}^3)}$$

is bounded by a constant depending on the initial data due to the induction hypothesis and Theorem 26. Similarly, if $k > 3/2 + |\delta_q|$ or $k > 3/2 + |\zeta_r|$, then

$$||\partial^{\delta_q} \partial_{j_q} u_l||_{C_b(\mathbb{R}^3)}, \quad ||\partial^{\zeta_r} \partial_t u_l||_{C_b(\mathbb{R}^3)}$$

are bounded by a constant depending on the initial data respectively, due to the induction hypothesis and Theorem 26. Let us prove that for $k > 3$, there can only be one factor in an expression of the form (169) that cannot be estimated by a constant depending on the data. Assume that there are two factors that cannot be so estimated. We then have to have

$$k \leq 3/2 + |\delta_q|, \quad k \leq 3/2 + |\zeta_r|,$$

$$k \leq 1/2 + |\gamma_p|, \quad k \leq 3/2 + |\zeta_r|,$$

or

$$k \leq 3/2 + |\delta_q|, \quad k \leq 1/2 + |\gamma_r|.$$

Since $|\gamma_1| + ... + |\gamma_i| + |\delta_1| + ... + |\delta_k| + |\zeta_1| + ... + |\zeta_m| \leq k$, we obtain,

$$2k \leq 3 + k.$$

In other words, $k \leq 3$. Since we are assuming $k > 3$, we have a contradiction. In an expression of the form (169), there can thus at most be one term that cannot be estimated in the sup-norm. Say that $\partial^{\zeta_r}\partial_t u_l$ cannot be estimated in the sup norm. Then

$$\left(\int_{\mathbb{R}^3} |(\partial^\beta F)(u_l, \partial u_l)\partial^{\gamma_1} u_l \cdots \partial^{\gamma_i} u_l \partial^{\delta_1} \partial_{j_1} u_l \cdots \partial^{\delta_k} \partial_{j_k} u_l \partial^{\zeta_1} \partial_t u_l \cdots \partial^{\zeta_m} \partial_t u_l|^2 dx\right)^{1/2}$$

can be estimated by

$$C\left(\int_{\mathbb{R}^3}|\partial^{\zeta_r}\partial_t u_l|^2 dx\right)^{1/2} \leq C,$$

where C depends on the data and the last inequality is due to the fact that the integral can be bounded by a constant times $H_k^{1/2}[u_l]$. If all the terms in an expression of the form (169) can be bounded in the sup-norm, we still retain one term inside the integral, and we obtain a similar conclusion. In order to estimate

$$\int_{\mathbb{R}^3}|F(u_l,\partial u_l)|^2 dx,$$

we use the fact that $F(0,0)=0$ to derive an estimate

$$|F(u_l,\partial u_l)| \leq C[|u_l|+|\partial u_l|],$$

where C is allowed to depend the data. Using the fact that we control u_l and ∂u_l in L^2, we obtain the conclusion that

$$\int_{\mathbb{R}^3}|F(u_l,\partial u_l)|^2 dx \leq C,$$

where C depends on the data. To conclude,

$$\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^3}[\partial^\alpha F(u_l,\partial u_l)]^2 dx\right)^{1/2} \leq C,$$

where C depends on the data. Inserting this information into (168), we obtain

$$\left|\frac{dH_k[u_{l+1}]}{dt}\right| \leq CH_k^{1/2}[u_{l+1}].$$

Due to Lemma 17, we obtain

$$H_k^{1/2}[u_{l+1}](t) \leq H_k^{1/2}[u_{l+1}](0)+\frac{1}{2}C|t|.$$

Due to (165) we obtain, for ϵ small enough (the bound on ϵ depending only on the data), the inductive assumption (166) with l replaced by $l+1$.

Let us turn to convergence. Similarly to the above, we have

$$\left|\frac{dH_k[\hat{u}_l]}{dt}\right| \leq \left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^3}(\partial^\alpha \hat{F}_l)^2 dx\right)^{1/2}\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^3}(\partial^\alpha\partial_t\hat{u}_l)^2 dx\right)^{1/2}, \tag{170}$$

where

$$\hat{u}_l = u_{l+1}-u_l, \quad \hat{F}_l = F(u_l,\partial u_l)-F(u_{l-1},\partial u_{l-1}).$$

In order to estimate the L^2-norm of $\hat{F}_l$, we rewrite it in a way similar to (85). Let us consider for instance

$$\begin{aligned} &G(u_l,u_{l-1},\partial u_l,\partial u_{l-1})\partial_t\hat{u}_{l-1} \\ &= \int_0^1 \partial_{z^2}F[\tau u_l+(1-\tau)u_{l-1},\tau\partial u_l+(1-\tau)\partial u_{l-1}]d\tau\cdot\partial_t\hat{u}_{l-1}. \end{aligned}$$

We have

$$\begin{aligned}&\partial^\alpha[G(u_l,u_{l-1},\partial u_l,\partial u_{l-1})\partial_t\hat{u}_{l-1}]\\&\quad=\sum_{\beta+\gamma=\alpha}c_{\beta,\gamma}\partial^\beta[G(u_l,u_{l-1},\partial u_l,\partial u_{l-1})]\partial^\gamma\partial_t\hat{u}_{l-1}.\end{aligned}$$

Consider one term in the sum. There are two cases to consider. If $|\beta|<k-3/2$, then we can estimate

$$\|\partial^\beta[G(u_l,u_{l-1},\partial u_l,\partial u_{l-1})\|_{C_b(\mathbb{R}^3)}\leq C,$$

where C depends on the data. In order to prove this, one uses the bound (167) together with an argument which is similar to the one given above in order to bound $\partial^\alpha F(u_l,\partial u_l)$ in L^2. Furthermore,

$$\|\partial^\gamma\partial_t\hat{u}_{l-1}\|_{L^2(\mathbb{R}^3)}\leq 2^{1/2}H_k^{1/2}[\hat{u}_{l-1}].$$

If $|\beta|\geq k-3/2$, then we have to have $|\gamma|<k-3/2$, since $k>3$ and $|\beta|+|\gamma|=|\alpha|\leq k$. In that case, we have

$$\|\partial^\beta[G(u_l,u_{l-1},\partial u_l,\partial u_{l-1})]\|_{L^2(\mathbb{R}^3)}\leq C,$$

and

$$\|\partial^\gamma\partial_t\hat{u}_{l-1}\|_{C_b(\mathbb{R}^3)}\leq CH_k^{1/2}[\hat{u}_{l-1}].$$

Regardless of which case actually occurs, we thus have a bound

$$CH_k^{1/2}[\hat{u}_{l-1}],$$

where C depends on the data. When considering terms of the form

$$\int_0^1\partial_{z^i}F[\tau u_l+(1-\tau)u_{l-1},\tau\partial u_l+(1-\tau)\partial u_{l-1}]d\tau\cdot\partial_j\hat{u}_{l-1},$$

we get a similar estimate. However, when we consider

$$\int_0^1\partial_{z^1}F[\tau u_l+(1-\tau)u_{l-1},\tau\partial u_l+(1-\tau)\partial u_{l-1}]d\tau\cdot\hat{u}_{l-1},$$

the situation is somewhat different. If at least one derivative hits $\hat{u}_{l-1}$, we get a similar estimate, but if not, we have an estimate of the form

$$C\left[\|\hat{u}_{l-1}\|_{L^2(\mathbb{R}^3)}+H_k^{1/2}[\hat{u}_{l-1}]\right]^{1/2}.$$

To conclude, we thus have

$$\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^3}(\partial^\alpha\hat{F}_l)^2dx\right)^{1/2}\leq C\left[H_k^{1/2}[\hat{u}_{l-1}]+\|\hat{u}_{l-1}\|_{L^2(\mathbb{R}^3)}\right].$$

Inserting this information in (170), and using the notation

$$e_l=\sup_{|t|\leq\epsilon}H_k^{1/2}[\hat{u}_{l-1}](t)+\sup_{|t|\leq\epsilon}\hat{u}_{l-1}(t,\cdot)\|_{L^2(\mathbb{R}^3)}$$

we obtain, for $|t|\leq\epsilon$,

$$\left|\frac{dH_k[\hat{u}_l]}{dt}\right|\leq Ce_lH_k[\hat{u}_l]^{1/2}.$$

Applying Lemma 17, we obtain

$$H_k^{1/2}[\hat{u}_l](t)\leq H_k^{1/2}[\hat{u}_l](0)+\frac{1}{2}Ce_l\epsilon \tag{171}$$

for $|t| \leq \epsilon$. In order to prove convergence, we also need to know something about the L^2-norm of $\hat{u}_l$. Since

$$\left| \frac{d}{dt} \int_{\mathbb{R}^3} \hat{u}_l^2 dx \right| \leq 2^{3/2} H_k^{1/2}[\hat{u}_l] \|\hat{u}_l\|_{L^2(\mathbb{R}^3)},$$

we can apply Lemma 17 in order to obtain

$$\|\hat{u}_l(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq \|\hat{u}_l(0,\cdot)\|_{L^2(\mathbb{R}^3)} + 2^{1/2} \left| \int_0^t H_k^{1/2}[\hat{u}_l](t) dt \right|.$$

Assuming $\epsilon < 1/2$ and combining this inequality with (171), we obtain

$$e_{l+1} \leq \|\hat{u}_l(0,\cdot)\|_{L^2(\mathbb{R}^3)} + 2H_k^{1/2}[\hat{u}_l](0) + Ce_l\epsilon, \tag{172}$$

where C only depends on the data. Let us make the inductive assumption that there is a constant $C_0 > 1$, depending on the data, such that

$$e_l \leq C_0 2^{-l}. \tag{173}$$

Note that for $l = 1$ this is true, assuming C_0 is big enough. By a suitable choice of approximating sequence, we can assume that

$$\|\hat{u}_l(0,\cdot)\|_{L^2(\mathbb{R}^3)} + 2H_k^{1/2}[\hat{u}_l](0) \leq 2^{-l-2}.$$

Combining this inequality with (172) and (173), we obtain, assuming $C\epsilon \leq 1/4$,

$$e_{l+1} \leq 2^{-l-2} + C_0 2^{-l-2} \leq C_0 2^{-l-1}.$$

This proves that the induction hypothesis (173) is true for all l. Similarly to the proof of convergence for $1+1$ non-linear wave equations, we conclude that u_l is a Cauchy sequence in

$$C\{[-\epsilon,\epsilon], H^{k+1}(\mathbb{R}^3)\},$$

and that $\partial_t u_l$ is a Cauchy sequence in

$$C\{[-\epsilon,\epsilon], H^{k}(\mathbb{R}^3)\}.$$

Using Lemma 16 and its proof, we conclude that u_l, $\partial_i u_l$, $\partial_t u_l$, $\partial_i \partial_j u_l$ and $\partial_i \partial_t u_l$ converge in $C\{[-\epsilon,\epsilon] \times \mathbb{R}^3\}$. Using the equation and an argument similar to the end of the proof of local existence for $1+1$ non-linear wave equations, one can conclude that $\partial_t^2 u_l$ is also a Cauchy sequence in this space. Consequently, we obtain a C^2 solution to the equation which is also in the right spaces (163). Similarly to the $1+1$-dimensional case, we obtain C^{k-1} by a little more work. In order to prove uniqueness, one can proceed in a fashion similar to the proof of convergence, and then use Grönwall's lemma. □

Let us make some remarks concerning the result. Note that it is disappointing because it does not give local existence of smooth solutions right away. This is however a problem which can be dealt with similarly to how we dealt with the same problem in $1+1$ dimensions. The argument can be generalized to $n+1$ dimensions; all we need to do is to change the condition on k to $k > n$. The reason for this condition is the following. Assume that $f, g \in H^k(\mathbb{R}^n)$ for $k > n$ and say that we want to estimate

$$\|\partial^\alpha (fg)\|_{L^2(\mathbb{R}^n)}$$

for some α such that $|\alpha| \leq k$. Since

$$\partial^\alpha (fg) = \sum_{\beta+\gamma=\alpha} c_{\beta,\gamma} \partial^\beta f \partial^\gamma g,$$

where $|\beta|+|\gamma|=|\alpha|$, we obtain an estimate if we can take out one of the factors in the sup norm and estimate the other in L^2. The natural factor to take out in the sup norm is the one with the least number of derivatives. Say that $|\gamma|\le|\beta|$. Then $|\gamma|\le k/2$. Since $k>n$, we have $k>|\gamma|+n/2$. Due to (148), we have

$$\|g\|_{C^{|\gamma|}(\mathbb{R}^n)}\le C\|g\|_{H^k(\mathbb{R}^n)},$$

so that

$$\|\partial^\beta f\partial^\gamma g\|_{L^2(\mathbb{R}^n)}\le C\|f\|_{H^k(\mathbb{R}^n)}\|g\|_{H^k(\mathbb{R}^n)}.$$

Consequently,

$$\|fg\|_{H^k(\mathbb{R}^n)}\le C\|f\|_{H^k(\mathbb{R}^n)}\|g\|_{H^k(\mathbb{R}^n)}.$$

As a consequence we see that if we multiply two functions in $H^k(\mathbb{R}^n)$ for $k>n$, we get a function in the same space. For low k, for instance $k=0$, we certainly do not get this conclusion. The reason why these sorts of observations are important is that the crucial step in obtaining the rough control and in proving convergence is to estimate, in L^2, derivatives of the non-linear function of the unknown. One is then forced to estimate the L^2-norm of the product of derivatives of functions that are in certain H^k-spaces. Above, we have illustrated a very primitive way of estimating the derivative of products of functions. If we were able to acheive better estimates, then we would be able to prove local existence with lower regularity conditions on the initial data. One can then of course ask the question why this would be of any interest. Proving local existence with a lower degree of regularity is perhaps not of any greater interest if one is only interested in classical solutions, but the point is that reducing the regularity condition on the initial data is intimately connected with improving the continuation criterion, as we shall see below. The fact is that it is possible to prove the following result. The reader interested in a proof is referred to e.g. Lars Hörmander's book *Non-linear Hyperbolic Differential Equations*.

THEOREM 28. *If $u,v\in C_b(\mathbb{R}^n)$, $\partial^\alpha u,\partial^\alpha v\in L^2(\mathbb{R}^n)$ for all $|\alpha|=m$, then $\partial^\alpha(uv)\in L^2(\mathbb{R}^n)$ for all $|\alpha|=m$, and*

$$\sum_{|\alpha|=m}\|\partial^\alpha(uv)\|_{L^2(\mathbb{R}^n)}\le C_m\sum_{|\alpha|=m}(\|\partial^\alpha u\|_{L^2(\mathbb{R}^n)}\|v\|_{C_b(\mathbb{R}^n)}+\|\partial^\alpha v\|_{L^2(\mathbb{R}^n)}\|u\|_{C_b(\mathbb{R}^n)}).$$

Note that this theorem can be used to improve the estimates we wrote down above. In fact, if $f,g\in H^k(\mathbb{R}^n)$ with $k>n/2$, then $f,g\in C_b(\mathbb{R}^n)$ due to (148), so that we can apply the theorem in order to conclude that $fg\in H^k(\mathbb{R}^n)$. Using this sort of estimate, one can prove that if F is smooth and $F(0,0)=0$, then, for $k>n/2$,

$$\|F(u,\partial u)(t,\cdot)\|_{H^k(\mathbb{R}^n)}\le C[\|u(t,\cdot)\|_{H^{k+1}(\mathbb{R}^n)}+\|\partial_t u(t,\cdot)\|_{H^k(\mathbb{R}^n)}]\tag{174}$$

where the constant C depends on F, $\|u(t,\cdot)\|_{C_b^1(\mathbb{R}^n)}$ and $\|\partial_t u(t,\cdot)\|_{C_b(\mathbb{R}^n)}$. In the proof of local existence we used a similar estimate, the only difference being that we only obtained the estimate for $k>n$ and the constant depended upon higher norms of u. Due to the improved estimate (174), one can prove the following result.

THEOREM 29. *Let $f\in H^{k+1}(\mathbb{R}^n)$ and $g\in H^k(\mathbb{R}^n)$. Assume furthermore that $F(0,0)=0$ and that $k>n/2$. Then there is an $\epsilon>0$, depending on F, the H^{k+1}-norm of f, the H^k-norm of g and k, and a unique weak $C^1[(-\epsilon,\epsilon)\times\mathbb{R}^n]$ solution to the equation (158) such that*

$$u\in C\{(-\epsilon,\epsilon),H^{k+1}(\mathbb{R}^n)\},\quad u_t\in C\{(-\epsilon,\epsilon),H^k(\mathbb{R}^n)\}.$$

Let us turn to the question of proving local existence of smooth solutions and to providing a continuation criterion. In doing so, we shall use estimates of the form (174) even though we shall not prove them. One can of course derive continuation criteria without these estimates, but they are not very powerful.

THEOREM 30. *Let $f \in H^{k+1}(\mathbb{R}^n)$ and $g \in H^k(\mathbb{R}^n)$ for some $k > n/2$. Let F be a smooth function such that $F(0,0) = 0$. Assume we have a solution to (158) on the time interval (T_-, T_+) satisfying*

$$u \in C\{(T_-, T_+), H^{k+1}(\mathbb{R}^n)\}, \quad u_t \in C\{(T_-, T_+), H^k(\mathbb{R}^n)\}.$$

If there is a real constant c_0 such that

$$\|u(t,\cdot)\|_{C_b^1(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|_{C_b(\mathbb{R}^n)} \leq c_0$$

on $[0, T_+)$ where $T_+ < \infty$, then there is a constant C_k depending on F, c_0, T_+, $\|f\|_{H^{k+1}(\mathbb{R}^n)}$ and $\|g\|_{H^k(\mathbb{R}^n)}$ such that

$$\|u(t,\cdot)\|_{H^{k+1}(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|_{H^k(\mathbb{R}^n)} \leq C_k$$

for all $t \in [0, T_+)$. The statement concerning T_- is similar.

Proof. Estimate

$$\left|\frac{dH_k[u]}{dt}\right| \leq \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} [\partial^\alpha F(u, \partial u)]^2 dx\right)^{1/2} \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} (\partial^\alpha \partial_t u)^2 dx\right)^{1/2}. \tag{175}$$

Note that, strictly speaking, we are not allowed to carry out this argument; we cannot assume we have enough regularity to differentiate $H_k[u]$. It is however possible to prove an integral version of this estimate. Note that if we set up an iteration as in the proof of local existence, we have the following version of (175),

$$\begin{aligned} H_k[u_{l+1}](t) &\leq H_k[u_{l+1}](0) \\ &+ \left|\int_0^t \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} [\partial^\alpha F(u_l, \partial u_l)]^2 dx\right)^{1/2} \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} (\partial^\alpha \partial_t u_{l+1})^2 dx\right)^{1/2} dt\right|. \end{aligned}$$

Since all the different objects appearing in this inequality converge as $l \to \infty$, assuming t is within the interval in which the iteration converges, we obtain

$$\begin{aligned} H_k[u](t) &\leq H_k[u](0) \\ &+ \left|\int_0^t \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} [\partial^\alpha F(u, \partial u)]^2 dx\right)^{1/2} \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^3} (\partial^\alpha \partial_t u)^2 dx\right)^{1/2} dt\right|, \end{aligned}$$

for t inside the interval in which the iteration converges. Since this estimate has nice additive properties, we obtain it for all t for which a solution exists. Using an estimate of the form (174), we obtain

$$H_k[u](t) \leq H_k[u](0) + C\left|\int_0^t [\|u(s,\cdot)\|_{H^{k+1}(\mathbb{R}^n)} + \|\partial_t u(s,\cdot)\|_{H^k(\mathbb{R}^n)}] H_k^{1/2}[u](s) ds\right|,$$

where the constant C depends on F and c_0. In order to get an estimate for $\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, we note that, similarly to the above, one can prove the estimate

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \leq \|u(0,\cdot)\|^2_{L^2(\mathbb{R}^n)} + 2\left|\int_0^t \|u(s,\cdot)\|_{L^2(\mathbb{R}^n)}\|\partial_t u(s,\cdot)\|_{L^2(\mathbb{R}^n)} ds\right|.$$

Letting

$$E_k[u] = \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + H_k[u](t)$$

and adding up the last two estimates, we obtain

$$E_k[u](t) \leq E_k[u](0) + C\left|\int_0^t E_k[u](s)ds\right|,$$

where the constant C depends on F and c_0. Using this inequality together with Grönwall's lemma, we obtain the conclusion of the theorem. □

Similarly to the $1+1$ dimensional case, this leads to local existence of smooth solutions. Since the argument is essentially the same, we omit the proof.

COROLLARY 7. *Let F be smooth and have the property that $F(0,0) = 0$ and let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Let $u \in C^2[(T_-, T_+) \times \mathbb{R}^n]$ be a solution to (158) satisfying*

$$u \in C\{(T_-, T_+), H^{k+1}(\mathbb{R}^n)\}, \quad u_t \in C\{(T_-, T_+), H^k(\mathbb{R}^n)\}$$

for some $k > n/2$, where (T_-, T_+) is the maximal existence interval. Then $u \in C^\infty[(T_-, T_+) \times \mathbb{R}^n]$ and either $T_+ = \infty$ or

$$\|u(t,\cdot)\|_{C^1_b(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|_{C_b(\mathbb{R}^n)}$$

is unbounded on $[0, T_+)$. The statement concerning T_- is similar.

Similarly to the $1+1$ dimensional case, we can improve the results in the case that F only depends on u.

THEOREM 31. *Let $f \in H^{k+1}(\mathbb{R}^n)$ and $g \in H^k(\mathbb{R}^n)$ for some $k > n/2$. Let F be a smooth function such that $F(0) = 0$. Assume we have a solution to*

$$\begin{cases} u_{tt} - \Delta u = F(u) \\ u(0,\cdot) = f \\ u_t(0,\cdot) = g \end{cases} \tag{176}$$

on the time interval (T_-, T_+) satisfying

$$u \in C\{(T_-, T_+), H^{k+1}(\mathbb{R}^n)\}, \quad u_t \in C\{(T_-, T_+), H^k(\mathbb{R}^n)\}.$$

If there is a real constant c_0 such that

$$\|u(t,\cdot)\|_{C_b(\mathbb{R}^n)} \leq c_0$$

on $[0, T_+)$ where $T_+ < \infty$, then there is a constant C_k depending on F, c_0, T_+, $\|f\|_{H^{k+1}(\mathbb{R}^n)}$ and $\|g\|_{H^k(\mathbb{R}^n)}$ such that

$$\|u(t,\cdot)\|_{H^{k+1}(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|_{H^k(\mathbb{R}^n)} \leq C_k$$

for some $t \in [0, T_+)$. The statement concerning T_- is similar.

Remark. In this case, one can in fact improve the local existence result to $k > n/2 - 1$.

Furthermore, we have the following result.

COROLLARY 8. *Let F be smooth and have the property that $F(0) = 0$ and let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Let $u \in C^2[(T_-, T_+) \times \mathbb{R}^n]$ be a solution to (176), satisfying*

$$u \in C\{(T_-, T_+), H^{k+1}(\mathbb{R}^n)\}, \quad u_t \in C\{(T_-, T_+), H^k(\mathbb{R}^n)\},$$

for all $k > n/2$, where (T_-, T_+) is the maximal existence interval. Then $u \in C^\infty[(T_-, T_+) \times \mathbb{R}^n]$ and either $T_+ = \infty$ or $\|u(t, \cdot)\|_{C_b(\mathbb{R}^n)}$ is unbounded on $[0, T_+)$. The statement concerning T_- is similar.

Let us end this chapter with a comment on uniqueness. To prove that, given initial data, one obtains a unique solution is not very difficult; one simply uses the sort of results that were used in the proof of local existence and Grönwall's lemma. However, it is of some interest to prove that if the initial data coincide in some finite ball $B_r(x_0)$, then the solutions have to coincide in

$$\Lambda_{r,x_0} = \{(t, x) \in \mathbb{R}^4 : |x - x_0| < r - |t|\}.$$

In fact, it is possible to prove such results, but we shall not do so here.

CHAPTER 7

Global existence in $n+1$-dimensions

Similarly to the situation in $1+1$-dimensions, it is not possible to prove any particularly general theorems concerning global existence in $n+1$ dimensions. Consequently, one has to be satisfied with considering special cases. Let us start by considering equations of the form

$$\begin{cases} u_{tt} - \Delta u = F(u) \\ u(0,\cdot) = f \\ u_t(0,\cdot) = g \end{cases} \tag{177}$$

The question we are interested in answering is, for which F do we get global existence for arbitrary smooth initial data with compact support? It is not so difficult to prove that if there are solutions to the ODE $u'' = F(u)$ that blow up in finite time, then there are smooth initial data with compact support such that the corresponding solution to (177) blows up in finite time. In order to prove this, one does however need to use the uniqueness result we stated, but did not prove, at the very end of the last chapter. It is then very tempting to conjecture that if there is no solution to the ODE $u'' = F(u)$ that blows up in finite time, then there is global existence for arbitrary initial data to (177). As far as we are aware, there are no counter examples to this conjecture, but there are special cases of it that by themselves have turned out to be very difficult to solve. In the following we shall only consider $3+1$ dimensions. Note that in some cases, this is certainly a restriction. In particular, when considering equations of the form (177), we have the formula (152), and there are essential aspects of it that do not generalize to arbitrary dimensions. Finally, let us note that in this chapter, we do not strive for any degree of optimality in the statements; we simply wish to give some examples of when it is possible to prove global existence without any greater effort.

1. General conditions on the non-linearity

Let us try to find some general conditions on the non-linearity that lead to global existence. Let us start by an estimate on F.

PROPOSITION 14. *Let F be a smooth function satisfying $F(0) = 0$ and assume that there is a constant C such that*

$$|F(u)| \leq C(1+u^2)^{1/2} \tag{178}$$

for all u. Then the existence interval for solutions to (177) with $f, g \in \mathcal{S}(\mathbb{R}^3)$ is $\mathbb{R}$.

Proof. Say that we have a solution on the interval $[0, T_+)$, where $T_+ < \infty$. We wish to prove that there is a bound on $\|u(t,\cdot)\|_{C_b(\mathbb{R}^3)}$ for $t \in [0, T_+)$ depending only on F, the initial data and T_+. Such a bound would immediately imply global existence due to Corollary 8 (the result in the opposite time direction is obtained by a simple time reversal). Consider (152). We have

$$\begin{aligned}\|u(t,\cdot)\|_{C_b(\mathbb{R}^3)} &\leq (1+T_+^2)^{1/2}[\|f\|_{C_b^1(\mathbb{R}^3)} + \|g\|_{C_b(\mathbb{R}^3)}] \\ &\quad +(1+T_+^2)^{1/2}\left|\int_0^t \|F(s,\cdot)\|_{C_b(\mathbb{R}^3)}ds\right| \\ &\leq C + C\left|\int_0^t \|(1+u^2)^{1/2}(s,\cdot)\|_{C_b(\mathbb{R}^3)}ds\right|,\end{aligned}$$

where the constants depend on the initial data, F and T_+. Note however that

$$\|(1+u^2)^{1/2}(t,\cdot)\|_{C_b(\mathbb{R}^3)} \leq 1 + \|u(t,\cdot)\|_{C_b(\mathbb{R}^3)}.$$

Letting $h(t) = \|(1+u^2)^{1/2}(t,\cdot)\|_{C_b(\mathbb{R}^3)}$, we thus obtain

$$h(t) \leq C + C\left|\int_0^t h(s)ds\right|.$$

By Grönwall's lemma, we obtain an estimate for h, and thus for u, depending only on the initial data, F and T_+. The proposition follows. □.

Note that as a consequence, any solution to (177), where

$$F(u) = (1+u^2)^{1/2}\sin(e^u - 1),$$

corresponding to smooth initial data with compact support exists globally in time. Similarly to the $1+1$ dimensional case, the condition that the initial data have compact support can be removed by a standard argument. If we relax the estimate (178) to, say,

$$|F(u)| \leq C(1+u^2)^\gamma$$

for some $\gamma > 1/2$, then we do not obtain global existence, since it is not difficult to prove that there are solutions to the ODE

$$u'' = (1+u^2)^\gamma - 1$$

that blow up in finite time assuming $\gamma > 1/2$. If we want a condition on F which is simply a crude estimate, we can consequently not do much better than (178). However, if we take the sign of the non-linearity into account, we can obtain different statements.

PROPOSITION 15. *Let F be a smooth function satisfying $F(0) = 0$ and assume that $F(u) \leq 0$ for $u \geq 0$ and that there is a constant C such that $|F(u)| \leq C$ for $u \leq 0$. Then the existence interval for solutions to (177) with $f, g \in \mathcal{S}(\mathbb{R}^3)$ is $\mathbb{R}$.*

Proof. The strategy of the proof is similar to that of the previous proposition. Assume we have a solution on $[0, T_+)$ and consider (152). We obtain

$$u(t,x) \leq C + \frac{1}{4\pi}\int_0^t \int_{S^2}(t-s)[F(u)][s, x+(t-s)y]d\sigma(y)ds,$$

where C is a constant depending on the initial data and T_+. Since F is bounded from above, the second term is also bounded by a constant, depending on F and T_+. We conclude that u is bounded from above. Since $|F(u)| \leq C$ for $u \leq 0$, we conclude that $|F(u)|$ is bounded in the interval $[0, T_+)$. Inserting this information into (152), we obtain the conclusion that $|u(t,x)|$ is bounded on $[0, T_+)$. □

Note that as a consequence, if

$$F(u) = -e^u + 1,$$

then we obtain global existence of solutions to (177) for arbitrary initial data. On the other hand, this function certainly does not satisfy the inequality (178).

2. Inequalities

The purpose of the present section is to prepare for the study of (177) when $F(u) = \pm u^k$. Note that if k is positive and even or if $F(u) = u^k$ with $k \geq 2$, then solutions typically blow up in finite time. In the case of $1+1$ dimensions, we were however able to prove that if $F(u) = -u^k$ with k positive and odd, then there is global existence. We are now interested in seeing if it is possible to prove something similar in $3+1$ dimensions. In order to do so, we do however need to develop some tools. The presentation given below is by and large taken from *Elliptic Partial Differential Equations of Second Order* by D. Gilbarg and N. Trudinger.

Let $1 \leq p < \infty$. We define the space $L^p(\mathbb{R}^n)$ similarly to the space L^2; it is the space of equivalence classes of measurable functions (two functions being equivalent if the set on which they differ has Lebesgue measure zero) such that their absolute value raised to the power p is integrable. One can prove that the L^p functions can be made into a Banach space. We shall most of the time not be very careful but rather think of elements of L^p as being functions. For $u \in L^p(\mathbb{R}^n)$, we define

$$\|u\|_p = \left(\int_{\mathbb{R}^n} |u|^p d\mu \right)^{1/p}.$$

In order to prove that this defines a norm, we need some preparations. Let us prove that if p and q are positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then *Young's inequality*,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{179}$$

holds for all non-negative a, b. If either a or b are zero, then the inequality holds trivially. If we let $t = a/b^{q-1}$, then (179) is the same as

$$t \leq \frac{1}{q} + \frac{t^p}{p}.$$

However, the function

$$\frac{t^{-1}}{q} + \frac{t^{p-1}}{p}$$

tends to infinity as $t \to 0+$ and as $t \to \infty$. Furthermore it has a unique minimum for $t = 1$. This proves (179). In fact, Young's inequality can be generalized in the following way. Assume $p_1, ..., p_k$ are positive numbers such that

$$\frac{1}{p_1} + ... + \frac{1}{p_k} = 1. \tag{180}$$

Then if $a_1, ..., a_k$ are non-negative numbers, we have

$$a_1 \cdots a_k \leq \frac{a_1^{p_1}}{p_1} + ... + \frac{a_k^{p_k}}{p_k}. \tag{181}$$

We know that (181) holds for $k = 2$. Assume it holds for some $k \geq 2$, and let us prove that it holds for $k+1$. Given $p_1, ..., p_{k+1}$ satisfying (180) with k replaced by $k+1$, let us define $r_i = p_i$ for $i = 1, ..., k-1$ and

$$r_k = \frac{p_k p_{k+1}}{p_k + p_{k+1}}.$$

Then $r_1, ..., r_k$ are positive numbers satisfying a condition of the form (180). Consequently

$$a_1 \cdots a_{k+1} \leq \frac{a_1^{p_1}}{p_1} + ... + \frac{a_{k-1}^{p_{k-1}}}{p_{k-1}} + \frac{(a_k a_{k+1})^{r_k}}{r_k}.$$

However, if we let $p = p_k/r_k$ and $q = p_{k+1}/r_k$, we can apply (179) in order to obtain

$$(a_k a_{k+1})^{r_k} \leq \frac{a_k^{r_k p}}{p} + \frac{a_{k+1}^{r_k q}}{q} = r_k \left(\frac{a_k^{p_k}}{p_k} + \frac{a_{k+1}^{p_{k+1}}}{p_{k+1}} \right).$$

This completes the induction. As a consequence, we obtain the following result.

LEMMA 18. *Let $p_1, ..., p_k$ be positive numbers such that (180) holds. Assume $u_i \in L^{p_i}(\mathbb{R}^n)$ for $i = 1, ..., k$. Then $u_1 \cdots u_k \in L^1(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |u_1 \cdots u_k| dx \leq \|u_1\|_{p_1} \cdots \|u_k\|_{p_k}. \tag{182}$$

Proof. Note that if $\|u_i\|_{p_i} = 0$ for some i, then the right and the left hand side of (182) equal zero. We can consequently assume $\|u_i\|_{p_i} > 0$. Define $v_i = u_i/\|u_i\|_{p_i}$. Due to (181) we have

$$\int_{\mathbb{R}^n} |v_1 \cdots v_k| dx \leq \int_{\mathbb{R}^n} \left(\frac{|v_1|^{p_1}}{p_1} + ... + \frac{|v_k|^{p_k}}{p_k} \right) dx = 1,$$

where we have used (180) and the fact that $\|v_i\|_{p_i} = 1$. Multiplying this inequality with $\|u_1\|_{p_1} \cdots \|u_k\|_{p_k}$, we obtain the desired result. $\square$

LEMMA 19. *Assume $1 \leq p < \infty$ and that $f, g \in L^p(\mathbb{R}^n)$. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{183}$$

Remark. This is the only non-trivial aspect of proving that $\|\cdot\|_p$ defines a norm on the space of L^p-functions.

Proof. If $p = 1$ the result holds trivially. Otherwise, note that

$$\int_{\mathbb{R}^n} |f+g|^p dx \leq \int_{\mathbb{R}^n} |f||f+g|^{p-1} dx + \int_{\mathbb{R}^n} |g||f+g|^{p-1} dx. \tag{184}$$

Let $q = p/(p-1)$ and note that $1/p + 1/q = 1$. We can thus apply (182) with $p_1 = p$ and $p_2 = q$ in order to obtain

$$\int_{\mathbb{R}^n} |f||f+g|^{p-1} dx \leq \|f\|_p \|f+g\|_p^{p-1},$$

and similarly if we replace f by g. Inserting these observations into (184), we obtain

$$\int_{\mathbb{R}^n} |f+g|^p dx \leq (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}.$$

If $\|f+g\|_p > 0$, we can divide this inequality with $\|f+g\|_p^{p-1}$ in order to obtain the conclusion. If $\|f+g\|_p = 0$, the conclusion holds trivially. □

PROPOSITION 16. *Let p and n be given with $1 \leq p < n$. There is a constant C, depending on p and n such that for all $u \in C_0^1(\mathbb{R}^n)$,*

$$\|u\|_{np/(n-p)} \leq C\|\nabla u\|_p. \tag{185}$$

Proof. For any $1 \leq i \leq n$, we have

$$|u(x)| \leq \int_{-\infty}^{x^i} |\nabla_i u| dx^i \leq \int_{-\infty}^{\infty} |\nabla_i u| dx^i.$$

Thus

$$|u(x)|^{n/(n-1)} \leq \left(\prod_{i=1}^n \int_{-\infty}^{\infty} |\nabla_i u| dx^i \right)^{1/(n-1)}.$$

Let us integrate this inequality with respect to x^1. Note that there are only $n-1$ factors that depend on x^1. Consequently, we can apply (182) to these $n-1$ factors using $p_1 = ... = p_{n-1} = n-1$ in order to obtain

$$\int_{-\infty}^{\infty} |u(x)|^{n/(n-1)} dx^1 \leq \left(\int_{-\infty}^{\infty} |\nabla_1 u(x)| dx^1 \prod_{i=2}^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla_i u(x)| dx^1 dx^i \right)^{1/(n-1)}.$$

We can then integrate with respect to $x^2, ..., x^n$ and use the same argument. In the end we obtain, after raising the inequality to $(n-1)/n$,

$$\|u\|_{n/(n-1)} \leq \left(\prod_{i=1}^n \int_{\mathbb{R}^n} |\nabla_i u| dx \right)^{1/n} \leq \frac{1}{n} \int_{\mathbb{R}^n} \sum_{i=1}^n |\nabla_i u| dx \leq \frac{1}{\sqrt{n}} \|\nabla u\|_1, \tag{186}$$

where we have used (181) with $p_1 = ... = p_n = n$ in the second step and Schwartz inequality in the third step. We have thus established (185) for $p = 1$. Let us apply this inequality to $|u|^\gamma$ for some $\gamma > 1$. Note that if $u \in C_0^1(\mathbb{R}^n)$, then $|u|^\gamma$ is in the same space. Applying (186) with $p = 1$ to $|u|^\gamma$ we obtain

$$\|\, |u|^\gamma \|_{n/(n-1)} \leq \frac{\gamma}{\sqrt{n}} \| |u|^{\gamma-1} |\nabla u| \|_1 \leq \frac{\gamma}{\sqrt{n}} \|\, |u|^{\gamma-1} \|_q \|\nabla u\|_p,$$

where $1/q = 1 - 1/p$. Choosing

$$\gamma = \frac{(n-1)p}{n-p},$$

we obtain

$$\|u\|_{np/(n-p)}^\gamma \leq \frac{\gamma}{\sqrt{n}} \|u\|_{np/(n-p)}^{\gamma-1} \|\nabla u\|_p.$$

If $\|u\|_{np/(n-p)} = 0$, then the inequality holds trivially. Otherwise, we divide by $\|u\|_{np/(n-p)}^{\gamma-1}$. The theorem follows. □

3. Power type non-linearities

Let us consider (177) with $F(u) = -u^k$ where k is an odd non-negative integer. By Proposition 14, we have global existence for $k = 1$. The interesting cases are thus when $k = 3, 5, \ldots$. Let us assume that the initial data are smooth with compact support. Then the same will be true for $u(t,\cdot)$ for any t. This follows from the uniqueness result we mentioned at the end of the last chapter, but it can also be obtained simply by looking at the proof of local existence. If the initial data are zero outside of the ball of radius R, then the solution at t is zero outside the ball of radius $R + |t|$. Note also that if we let

$$H = \frac{1}{2}\int_{\mathbb{R}^3}\left[u_t^2 + |\nabla u|^2 + \frac{2}{k+1}u^{k+1}\right]dx,$$

then H is preserved for a solution to (177) with $F(u) = -u^k$.

PROPOSITION 17. *Consider a solution to (177) with $F(u) = -u^3$ and $f, g \in C_0^\infty(\mathbb{R}^3)$. Then the maximal existence interval is $\mathbb{R}$.*

Proof. Due to the fact that H is preserved, $\|\nabla u(t,\cdot)\|_2$ is bounded. By applying (185) with $p = 2$ and $n = 3$, we obtain the conclusion that $\|u(t,\cdot)\|_6$ is bounded by a constant that only depends on the initial data. Let

$$H_1 = \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}[(\partial_j\partial_t u)^2 + |\nabla\partial_j u|^2]dx.$$

Then

$$\frac{dH_1}{dt} = \sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_j(u_{tt} - \Delta u)\partial_j\partial_t u dx = -3\sum_{j=1}^{3}\int_{\mathbb{R}^3}u^2\partial_j u\partial_j\partial_t u dx,$$

so that

$$\left|\frac{dH_1}{dt}\right| \leq 3\sum_{j=1}^{3}\int_{\mathbb{R}^3}u^2|\partial_j u\partial_j\partial_t u|dx.$$

Let us apply (182) to the last integral. Note that $1/3 + 1/6 + 1/2 = 1$. We obtain

$$\left|\frac{dH_1}{dt}\right| \leq 3\sum_{j=1}^{3}\|u^2\|_3\|\partial_j u\|_6\|\partial_j\partial_t u\|_2. \tag{187}$$

Note here that $\|u^2(t,\cdot)\|_3 = \|u(t,\cdot)\|_6^2$ is bounded due to the observations made in the beginning of the proof. Furthermore, by (185), we obtain

$$\|\partial_j u\|_6 \leq C\|\nabla\partial_j u\|_2 \leq CH_1^{1/2}.$$

Combining these observations with (187) and the fact that

$$\|\partial_j\partial_t u\|_2 \leq CH_1^{1/2},$$

we obtain

$$\left|\frac{dH_1}{dt}\right| \leq CH_1.$$

This proves that H_1 cannot blow up in finite time. Using the fact that $\|\partial_t u(t,\cdot)\|_2$ is bounded, one can prove that $\|u(t,\cdot)\|_2$ cannot blow up in finite time. Consequently, $\|u(t,\cdot)\|_{H^2(\mathbb{R}^3)}$ cannot blow up in finite time. By (148), we conclude that $\|u(t,\cdot)\|_{C_b(\mathbb{R}^3)}$ cannot blow up in finite time. This proves global existence. □

It is also possible to prove global existence of solutions to (177) when $F(u) = -u^k$ and $k = 5$, but the argument is much more complicated. We refer the interested reader to Sogge's book. Concerning the cases $k = 7, 9, \ldots$ essentially nothing is known for large data.

4. Global existence for small data

Let us consider the equation

$$\begin{cases} u_{tt} - \Delta u = F(u) \\ u(0,\cdot) = \epsilon f \\ u_t(0,\cdot) = \epsilon g, \end{cases} \tag{188}$$

where $f, g \in C_0^\infty(\mathbb{R}^3)$. The question we are interested in is if there is an $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$ solutions to (188) exist globally. When asking this question, we first fix f and g and then we vary ϵ.

PROPOSITION 18. *Let $F \in C^\infty(\mathbb{R})$ satisfy*

$$|F(u)| \leq C_0 |u|^k$$

for some $k \geq 4$. Let $f, g \in C_0^\infty(\mathbb{R}^3)$ be given. Then there is an $\epsilon_0 > 0$ depending on C_0, k, f and g such that for $\epsilon \leq \epsilon_0$ the existence interval for solutions to (188) is $\mathbb{R}$. Furthermore, there is a constant C, depending on f and g such that for all t

$$\|u(t,\cdot)\|_{C_b(\mathbb{R}^3)} \leq C\epsilon(1+t^2)^{-1/2}.$$

Proof. Due to (152) a solution to (188) satisfies

$$\begin{aligned} u(t,x) &= \frac{1}{4\pi t^2}\int_{|x-y|=t}[t\epsilon g(y) + \epsilon f(y) - \epsilon(\partial_j f)(y)(x^j - y^j)]d\sigma(y) \\ &\quad + \frac{1}{4\pi}\int_0^t\int_{S^2}(t-s)[F(u)][s, x+(t-s)y]d\sigma(y)ds. \end{aligned} \tag{189}$$

The first term is the solution to the homogeneous wave equation corresponding to initial data ϵf, ϵg. Let us call it $u_{0,\epsilon}$. Due to Proposition 13 there is a constant C_1, depending on f and g such that for all t,

$$|u_{0,\epsilon}(t,x)| \leq C_1\epsilon(1+t^2)^{-1/2}. \tag{190}$$

Let us prove that for any $T > 0$, there is an $\epsilon_1 > 0$, depending on f, g and T, such that if $\epsilon \leq \epsilon_1$, then the solution to (188) exists on $[0,T]$ and satisfies

$$|u(t,x)| \leq (C_1+1)\epsilon(1+t^2)^{-1/2} \tag{191}$$

for all $x \in \mathbb{R}^3$ and $t \in [0,T]$. Let $\mathcal{A}$ be the subset of $[0,T]$ defined by the property that $s \in \mathcal{A}$ if and only if s belongs to the maximal existence interval of solutions to (188) and (191) holds for all $x \in \mathbb{R}^3$ and for all $t \in [0,s]$. We wish to prove that for ϵ small enough, $\mathcal{A} = [0,T]$. In order to do so, we have to prove four things: that $\mathcal{A}$ is connected, that it is closed, that it is open and that it is non-empty. That it is connected and non-empty follows by definition. Let us prove that it is closed.

Let $s_k \to s$ with $s_k \in \mathcal{A}$. Assuming T_+ is the maximal existence time, we have to have $s \leq T_+$ since $s_k < T_+$ for all k. Furthermore, since (191) is fulfilled in $[0, s_k]$ for all k, we conclude that it is satisfied in $[0, s)$. Due to Corollary 8, we conclude that $s < T_+$ so that the solution to (188) exists on $[0, s]$. By continuity (191) holds on $[0, s]$. Thus $\mathcal{A}$ is closed. All that remains to prove is that $\mathcal{A}$ is open. Assume $s \in \mathcal{A}$. Since the maximal existence interval is open, we conclude that the solution to (177) exists in some open neighbourhood of s. What remains to be proved is that (191) holds in some neighbourhood of s with respect to $[0, T]$. Consider the last term in (189). For $t \in [0, s]$, we have

$$\frac{1}{4\pi}\left|\int_0^t \int_{S^2} (t-s)[F(u)][s, x+(t-s)y]d\sigma(y)ds\right| \leq T^2 C_0 (C_1+1)^k \epsilon^k.$$

Let us demand that ϵ satisfy

$$T^2 C_0 (C_1+1)^k (1+T^2)^{1/2} \epsilon^{k-1} \leq \frac{1}{2}.$$

Combining this estimate with (189) and (190) we obtain the conclusion that

$$|u(t,x)| \leq \left(C_1 + \frac{1}{2}\right) \epsilon (1+t^2)^{-1/2}$$

for $t \in [0, s]$. By continuity, we conclude that there is an open neighbourhood of s such that (191) holds.

Let us assume that ϵ is small enough that we have (191) for $x \in \mathbb{R}^3$ and $t \in [0, T]$, where $T = 8(R+1)$ and R is such that $f(x) = g(x) = 0$ for $|x| \geq R$. In order to prove that we have future global existence, let us define a set $\mathcal{B}$ similarly to $\mathcal{A}$ defined in the first half of the proof. Let $\mathcal{B}$ be the subset of $[0, \infty)$ defined by the property that $s \in \mathcal{B}$ if and only if s belongs to the maximal existence interval of solutions to (188) and (191) holds for all $t \in [0, s]$ and $x \in \mathbb{R}^3$. Similarly to the first half of the proof, $\mathcal{B}$ is connected, non-empty and closed. What remains to be proved is that it is open with respect to the topology induced on $[0, \infty)$. Since we know that $[0, T] \subseteq \mathcal{B}$, let us assume $t \geq T$ and let us prove that there is an open neighbourhood of t contained in $\mathcal{B}$. Let us consider the second term in (189). Let us divide the integral with respect to s into two parts; $0 \leq s \leq t/4$ and $t/4 \leq s \leq t$. Our first task is thus to estimate

$$\frac{1}{4\pi}\int_0^{t/4} \int_{S^2} (t-s)[F(u)][s, x+(t-s)y]d\sigma(y)ds.$$

Note that

$$\left|\int_{S^2} (t-s)[F(u)][s, x+(t-s)y]d\sigma(y)\right| \leq \frac{1}{t-s}\int_{|x-y|=t-s} C_0 |u(s,y)|^k d\sigma(y).$$

Note that $t - s \geq 3t/4$, since $s \leq t/4$. Furthermore $u(s, y) = 0$ for $|y| \geq R + s$. Consequently, there is a constant C, depending on C_0, C_1 and k such that

$$\begin{aligned} & \frac{1}{t-s}\int_{|x-y|=t-s} C_0 |u(s,y)|^k d\sigma(y) \\ \leq \quad & C\epsilon^k (1+t^2)^{-1/2} (1+s^2)^{-k/2} \int_{|x-y|=t-s, |y| \leq R+s} d\sigma(y). \end{aligned}$$

Here we have used the fact that there is a constant C such that

$$\frac{1}{t-s} \leq C(1+t^2)^{-1/2}$$

when $s \leq t/4$ and $t \geq T$. Note that since $t \geq T$ and $s \leq t/4$, we have

$$2(R+s+1) \leq t-s.$$

Consequently, we can use the argument presented at the end of the proof of Proposition 13 in order to conclude that

$$\int_{|x-y|=t-s, |y| \leq R+s} d\sigma(y) \leq 2\pi (R+s)^2.$$

Adding the pieces together, we obtain

$$\left| \int_{S^2} (t-s)[F(u)][s, x+(t-s)y] d\sigma(y) \right| \leq 2\pi C \epsilon^k (1+t^2)^{-1/2}(1+s^2)^{-k/2}(R+s)^2.$$

For $k \geq 4$ it is clear that this quantity is integrable with respect to s. We obtain

$$\left| \frac{1}{4\pi} \int_0^{t/4} \int_{S^2} (t-s)[F(u)][s, x+(t-s)y] d\sigma(y) ds \right| \leq C_a \epsilon^k (1+t^2)^{-1/2}.$$

What remains to be considered is the integral

$$\frac{1}{4\pi} \int_{t/4}^{t} \int_{S^2} (t-s)[F(u)][s, x+(t-s)y] d\sigma(y) ds.$$

However,

$$\frac{1}{4\pi} \left| \int_{S^2} (t-s)[F(u)][s, x+(t-s)y] d\sigma(y) ds \right| \leq C \epsilon^k |t| (1+s^2)^{-k/2},$$

for some constant C depending on C_1, C_0 and k. Thus

$$\frac{1}{4\pi} \left| \int_{t/4}^{t} \int_{S^2} (t-s)[F(u)][s, x+(t-s)y] d\sigma(y) ds \right| \leq C_b \epsilon^k (1+t^2)^{-1/2},$$

since

$$\left| \int_{t/4}^{t} |t| (1+s^2)^{-(k-1)/2} ds \right|$$

is bounded. To conclude

$$\frac{1}{4\pi} \left| \int_0^{t} \int_{S^2} (t-s)[F(u)][s, x+(t-s)y] d\sigma(y) ds \right| \leq (C_a + C_b) \epsilon^k (1+t^2)^{-1/2}.$$

If we assume that ϵ is small enough that

$$(C_a + C_b)\epsilon^{k-1} \leq \frac{1}{2},$$

we obtain the conclusion that

$$|u(t,x)| \leq \left(C_1 + \frac{1}{2} \right) \epsilon (1+t^2)^{-1/2}$$

for all x. Since u has compact support in x for any finite time interval, we obtain the conclusion that there is a neighbourhood of t such that for all τ in this neighbourhood and all x, we have

$$|u(\tau, x)| \leq (C_1 + 1)\epsilon (1+\tau^2)^{-1/2}.$$

We have thus proven that $\mathcal{B}$ is open. Consequently $\mathcal{B} = [0, \infty)$ and the result follows in the future time direction. In order to get the conclusion in the opposite time direction, it suffices to make a simple time reversal. □

As a consequence of the above result, proving global existence for small data is not a problem if we are interested in equations of the form

$$u_{tt} - \Delta u = \pm u^k$$

and $k \geq 4$, even though we know that for even k or the wrong sign, solutions to these equations typically blow up. Note that the condition of fixed compact suppport is important. Consider the ODE

$$u_{tt} = u^4.$$

If $u_t(0)$ and $u(0)$ are both positive, then the corresponding solution will blow up in finite time regardless of how small $u_t(0)$ and $u(0)$ are.

5. Observations concerning the Einstein equations

The pupose of this section is to try to connect the material we have presented so far with the Einstein equations. It will not be possible for us to do so in any detail; our goal is rather to give the reader a feeling for the type of arguments that are involved when considering the question of global existence. In the beginning of this section we shall presuppose some knowledge of Lorentz geometry. If we consider the Einstein equations in vacuum, they can be written in the form

$$R_{\mu\nu} = 0,$$

where $R_{\mu\nu}$ is the Ricci tensor associated with a Lorentz metric g. If we write out these equations in coordinates, they are

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)} + g^{\alpha\beta}g^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}] = 0. \tag{192}$$

Here Greek indices run from 0 to 3,

$$\begin{aligned}\Gamma_{\alpha\mu\beta} &= \frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}), \quad \Gamma_\mu = g^{\alpha\beta}\Gamma_{\alpha\mu\beta},\\ \nabla_{(\mu}\Gamma_{\nu)} &= \frac{1}{2}(\partial_\mu\Gamma_\nu + \partial_\nu\Gamma_\mu) - g^{\alpha\beta}\Gamma_{\mu\alpha\nu}\Gamma_\beta,\end{aligned}$$

and we sum over all repeated indices. The first term in (192),

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu},$$

looks promising; it can be thought of as a wave operator acting on the metric. The problem is that we also have the term $\nabla_{(\mu}\Gamma_{\nu)}$. This term involves second derivatives of the metric and makes it impossible to view (192) as a non-linear wave equation for the metric. This problem is fundamental; it is due to the diffeomorphism invariance of the equations. Say that we specify initial data for (192) at $t = 0$ for $x \in \mathbb{R}^3$ and say that we have a solution $g_{\mu\nu}$ to this equation on an interval $[-T, T]$. Let ϕ be a diffeomorphism of $\mathbb{R}^4$ which is the identity close to $t = 0$ and let $\hat{g} = \phi^* g$ be the metric obtained by pulling back $g_{\mu\nu}$ by ϕ. Since

$$0 = \phi^*\mathrm{Ric}[g] = \mathrm{Ric}[\phi^* g],$$

we obtain the conclusion that $\hat{g}$ also satisfies (192). Since $\hat{g} \neq g$ in general, it is not possible to get uniqueness of solutions to (192) given initial data, and consequently (192) cannot be a non-linear wave equation. However, the diffeomorphism freedom, which causes the problem, also gives us an opportunity; we can impose extra conditions on the metric in order to obtain a unique solution. When we do so we do need to keep one thing in mind; the extra conditions we impose can lead to complications that do not reflect any pathology of the geometry. In other words, the solution to the equation we are considering might blow up, even though space time has no singularities. In the pioneering work of Yvonne Choquet-Bruhat, she showed that one can make the choice $\Gamma_\mu = 0$ consistent. The idea of the argument is to consider (192) with $\Gamma_\mu = 0$. One then obtains a non-linear wave equation for which there is local existence and uniqueness. Then one proves that the metric thus constructed has $\Gamma_\mu = 0$, given that the initial data have been set up properly. Thus one obtains a solution to the equation $\mathrm{Ric} = 0$. The condition $\Gamma_\mu = 0$ is referred to as *wave coordinates gauge*. There is one complication which appears when considering the Einstein equations that we have not mentioned. The initial data cannot be imposed freely but rather have to satisfy so called *constraint equations*. These equations by themselves have a complicated structure and they are far from understood. We shall however ignore this complication here.

In wave coordinates gauge, the Einstein vacuum equations take the form

$$-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + g^{\alpha\beta}g^{\gamma\delta}[\Gamma_{\alpha\gamma\mu}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\delta}] = 0. \tag{193}$$

Since the $\Gamma_{\alpha\mu\beta}$ are just a sum of first derivatives of the metric, the essential structure of this equation is

$$\Box_g g + F(g, \partial g) = 0,$$

where F is quadratic in the first derivatives of g. The first sort of question one might want to ask is the following. Say that we start with initial data close to those of Minkowski space, do we get a solution which is global in time (in some suitable geometric sense) and behaves in a way similar to Minkowski space? We have to be careful when specifying what we mean by small data in the case of (193). First of all, g should be a Lorentz metric, and we want it to be close to the Minkowski metric. Thus g itself should not be small. Instead, $g_{\mu\nu} - \eta_{\mu\nu}$ should be small. Furthermore, it is a consequence of the so called *positive mass theorem* that if the difference between the perturbed initial data and those of Minkowski space decays rapidly, in particular if the difference has compact support, then the perturbed initial data actually have to coincide with those of Minkowski space. As a toy problem, we shall however consider

$$\begin{cases} u_{tt} - \Delta u = F(u, \partial u) \\ u(0,\cdot) = \epsilon f \\ u_t(0,\cdot) = \epsilon g, \end{cases} \tag{194}$$

where F is quadratic in the first derivatives of u and $f, g \in C_0^\infty(\mathbb{R}^3)$. Reasonably, this should be a simpler problem. If we consider this problem in $n+1$-dimensions with $n \geq 4$ and for an F which only depends on ∂u, then global existence is ensured for ϵ small enough, cf. Sogge's book. In three dimensions the problem is however more subtle. The reason is that one crucial aspect of the problem is the decay of solutions to the linear wave equation, and the decay is better the larger the dimension is. In fact, if we let $F(\partial u) = u_t^2$, then solutions to (194) blow up in finite time in $3+1$ dimensions for all non-trivial compactly supported data, cf. John, F. *Blow-up for quasi-linear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), 29-51.

Due to this example, it seems one has to believe in miracles in order to believe that it is possible to prove global existence for initial data close to those of Minkowski space. However, there is a condition, called the *null condition*, on the non-linearity F that can improve the situation. Before we discuss this condition, let us write down the Klainerman Sobolev inequalities. The starting point is the observation that if we define

$$L_0 = t\partial_t + \sum_{i=1}^n x^i\partial_i, \quad \Omega_{ij} = x^j\partial_i - x^i\partial_j, \quad \Omega_{0i} = t\partial_i + x^i\partial_t,$$

where Latin indices range from 1 to n and $i<j$ in the definition of Ω_{ij}, then

$$[\Box, \Omega_{\mu\nu}] = 0, \quad [\Box, L_0] = 2\Box,$$

where Greek indices range from 0 to n. Let Λ_i, $i = 1, ..., m = n^2/2 + 3n/2 + 2$ denote the vectorfields

$$\partial_0, ..., \partial_n, L_0, \Omega_{01}, ..., \Omega_{n-1n}$$

and let us use the notation

$$\Lambda^\alpha = \Lambda_1^{\alpha_1} \cdots \Lambda_m^{\alpha_m}.$$

The following result is referred to as the Klainerman-Sobolev inequalities:

THEOREM 32. *Let $u \in C^\infty(\mathbb{R}^{n+1})$ be such that for a fixed t, it vanishes when $|x|$ is large. Then, if $t>0$,*

$$(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}|u(t,x)| \leq C \sum_{|\alpha|\leq(n+2)/2} \|\Lambda^\alpha u(t,\cdot)\|_2.$$

The reader interested in a proof is referred to Sogge's book. A primitive but interesting use of this inequality is the following. Say that we have a solution u to the homogeneous wave equation. Then due to the properties of the vectorfields Λ_i, $\Lambda^\alpha u$ is also a solution to the homogeneous wave equation. Consequently

$$\frac{1}{2}\int_{\mathbb{R}^n} \{[\partial_t(\Lambda^\alpha u)]^2 + |\nabla(\Lambda^\alpha u)|^2\} dx \tag{195}$$

are conserved quantities. On the other hand, due to the Klainerman Sobolev inequalities, we have

$$(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}|\partial u(t,x)| \leq C \sum_{|\alpha|\leq(n+2)/2} \|\Lambda^\alpha \partial u(t,\cdot)\|_2.$$

We would like to relate the right hand side of this inequality to (195). Since

$$[\Lambda_i, \partial_\nu] = \sum_{\mu,\nu,\lambda} a_{i\nu\lambda}\partial_\lambda,$$

where $a_{i\nu\lambda}$ are constants, we obtain

$$\sum_{|\alpha|\leq(n+2)/2} \|\Lambda^\alpha \partial u(t,\cdot)\|_2 \leq C \sum_{|\alpha|\leq(n+2)/2} \|\partial(\Lambda^\alpha u)(t,\cdot)\|_2.$$

However, since (195) is a conserved quantity, the right hand side is bounded by a constant depending on the initial data, assuming that the initial data have compact support for instance. Note that the Λ_i sometimes depend on t and x, but this dependence is of no greater importance if the initial data have compact support. Adding up these estimates, we obtain

$$(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}|\partial u(t,x)| \le C.$$

Thus we obtain decay for ∂u. In odd dimensions, this inequality can be used to obtain decay for u of the form $(1+t^2)^{-(n-1)/4}$, but not in even dimensions. The main point of this argument is however that we obtain a decay estimate without knowing the fundamental solution. If one is interested in equations of the form

$$\Box u = F(u, \partial u),$$

then it is possible to use the fundamental solution to good effect, cf. Proposition 18. However, in the case of the Einstein equations, $\Box$ should be replaced by $g^{\alpha\beta}\partial_\alpha\partial_\beta$, where $g^{\alpha\beta}$ is not known. Trying to get a decay estimate by first finding the fundamental solution might consequently be complicated. For that reason, the Klainerman Sobolev inequalities are a very attractive tool.

An observation that naturally leads to the introduction of the null condition is that the decay rate for different derivatives is different. Let us assume that $t, |x| > 0$. One can compute that

$$\partial_t + \partial_r = \frac{1}{t+|x|}\left(L_0 + \frac{x^j}{|x|}\Omega_{0j}\right),$$

where we sum over j and

$$\partial_r = \frac{x^j}{|x|}\partial_j.$$

As a consequence, if u solves the homogeneous wave equation,

$$\begin{aligned} |(\partial_t u + \partial_r u)(t,x)| &\le \frac{1}{t+|x|}\left(|(L_0 u)(t,x)| + \sum_{j=1}^{n}|(\Omega_{0j}u)(t,x)|\right) \\ &\le C(1+t)^{-1}(t+|x|)^{-1}, \end{aligned}$$

where the last step is due to the fact that $L_0 u$ and $\Omega_{0j}u$ are solutions to the homogeneous wave equation, which decay as $(1+t)^{-1}$. By similar arguments, cf. Sogge, one can prove that all derivatives that are tangent to the forward light cone have better decay properties. However, for derivatives that are transversal to the light cone one does not get any improvement. Consider (194) with $F(u, \partial u) = u_t^2$. Let us express the non-linearity in derivatives that are tangential and derivatives that are transversal to the forward light cone. We have

$$\begin{aligned} u_t^2 &= \left[\frac{1}{2}(\partial_t u - \partial_r u) + \frac{1}{2}(\partial_t u + \partial_r u)\right]^2 \\ &= \frac{1}{4}[(\partial_t u - \partial_r u)^2 + 2(\partial_t u - \partial_r u)(\partial_t u + \partial_r u) + (\partial_t u + \partial_r u)^2]. \end{aligned}$$

The problem with this expression is that there is a term which is the square of a derivative transversal to the forward light cone. This is something that cannot be dealt with in 3+1-dimensions. The idea behind the null condition is to prevent such terms from appearing. Let us simply quote a result from Sogge's book. Consider (194) where u, F, f, g take values in $\mathbb{R}^N$.

DEFINITION 34. We say that F satisfies the *null condition* if the following condition holds:

$$F = F_0 + O(|u|^3 + |\partial u|^3),$$

where F_0 only depends on ∂u and for every $I = 1, ..., N$,

$$F_0^I = \sum_{J,M=1}^{N} \sum_{\mu,\nu=0}^{3} f_{\mu\nu}^{IJM} \partial_\mu u^J \partial_\nu u^M$$

where $f_{\mu\nu}^{IJM}$ are constants such that for all I, J, M,

$$f_{\mu\nu}^{IJM} \xi^\mu \xi^\nu = 0$$

whenever ξ is a null vector with respect to the Minkowski metric, i.e.

$$\eta_{\mu\nu} \xi^\mu \xi^\nu = 0.$$

This condition prevents the occurrence of the square of derivatives transversal to the forward light cone. Consequently, one obtains better decay estimates for the non-linearity than for a general F satisfying

$$F = O(|\partial u|^2 + |u|^3).$$

The main result is the following.

THEOREM 33. *Let $n = 3$ and assume F satisfies the null condition. Then, if we fix $f, g \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^N)$, (194) always has a global solution for ϵ small enough.*

For a proof, we refer the reader to Sogge's book. One can in fact generalize the result to situations where $\Box$ is replaced by

$$g^{\mu\nu} \partial_\mu \partial_\nu,$$

and g depends on u and ∂u. In order to obtain such a result, one does however need to impose some sort of null condition on $g_{\mu\nu}$.

To what extent are these observations of any relevance to the study of Einstein's equations? Say that we consider Einstein's equations in wave coordinates gauge (193). Do these equations satisfy the null condition? Unfortunately, the answer is no. This led to Christodoulou and Klainerman using quite a different approach to prove the global non-linear stability of Minkowski space, see *The Global Non-linear Stability of the Minkowski Space* (1993), Princeton University Press, by Christodoulou, D. and Klainerman, S. Unfortunately, the argument is a book of more than 500 pages. One reason the argument is so long is that the authors cannot use the conformal Killing fields Λ_i in their arguments, but rather have to construct approximate conformal Killing fields adapted to the particular solution. Recently, the work of Hans Lindblad and Igor Rodnianski has showed that it does make sense to look at Einstein's equations in wave coordinates gauge and that it is possible to prove the global non-linear stability of Minkowski space using these equations as well. One attractive feature of the argument is that it is enough to use the conformal Killing vectors of Minkowski space, Λ_μ, rather than construct vector-fields adapted to the particular solution. This fact, among other things, reduces the size of the argument (in terms of pages) roughly by a factor of ten. However, that is not to say that all the conclusions that are obtained in the book can be obtained by the arguments of Lindblad and Rodnianski. Another advantage of the new argument is that it seems to be easier to generalize to cases with matter. In fact, the case of a scalar field seems to come for free. This distinguishes the argument from that of Christodoulou and Klainerman, since their argument depends in a crucial way upon properties of the vacuum equations that do not hold for a scalar field. So how do these things add up? Einstein's equations in wave coordinates gauge do not satisfy the null condition, and yet we obtain global existence for small initial data. In their work, Lindblad and Rodnianski introduce what they refer to as the *weak null condition*. The rough idea is to construct a new system by ignoring all terms of degree higher than two and all terms satisfying the null condition. If the new system allows global existence for small data, the system is said to satisfy the weak null condition.