Mathematical Cosmology - the shape and dynamics of the universe

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1. Standard picture of cosmology

Standard assumptions

In the standard picture of cosmology, one assumes the universe to be spatially **homogeneous** and **isotropic**. Roughly speaking, this means that at one "instant in time", an observer considers the universe to look the same regardless of at which spatial point he is located (homogeneity) and in which direction he is looking (isotropy). With a standard choice of matter model, one then gets the following conclusions. A **big bang**, with arbitrarily strong gravitational fields, took place in the finite past, and in the future the universe will expand indefinitely or recollapse. Mathematically, one can prove that there are only three geometries consistent with the assumptions, the only freedom left being a scale factor. This is illustrated for the three cases in Figure 1.

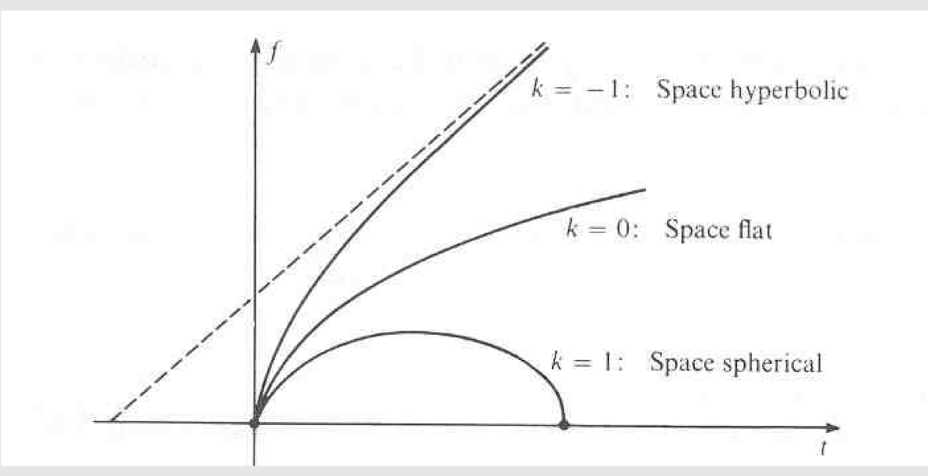

Figure 1: *The three standard possibilities for the scale factor of the universe. The picture is taken from O'Neill [1].*

Motivation for and stability of the standard picture?

The restrictions imposed are obviously strong, but not completely uncalled for. A combination of experimental observations and philosophical preconceptions yield strong support for the standard model. However, the question arises to what extent a similar picture is true in more general situations. Is the existence of a singularity simply a consequence of the symmetry assumptions? (The word singularity is here used as a synonym for Big Bang: in the standard examples this means that the entire universe shrinks to a point and that the gravitational fields become arbitrarily large.) If one perturbs the standard initial data slightly, does one get a similar picture? Furthermore, even though the standard model is consistent with experiment, it is of interest to analyze to what extent the experiments imply that our universe is close to the standard one. Finally, one would like to know to what extent one can deduce isotropy and homogeneity as a consequence of the equations.

In mathematical cosmology, the main interest is not to give a reasonable model of the universe, but to try to see if more general solutions to the Einstein equations behave in a way similar to the standard cosmologies.

2. Concepts of singularity (Big Bang)

Singularities with bounded gravitational fields

The question concerning the existence of singularities in general is, at least partially, answered by the singularity theorems of Hawking and Penrose. These theorems state that cosmological spacetimes under quite general conditions have a singularity. However, the existence of singularities is here equated with causal geodesic incompleteness. What this means is that if you trace the history of a freely falling observer into the past, the corresponding curve will end after a finite time. This concept of a singularity does not imply that the gravitational fields become arbitrarily large. Thus it is not clear that the standard picture of arbitrarily strong gravitational fields hold in a more general situation.

Cosmology as an initial value problem

In order to try to analyze whether the standard picture holds in a more general situation, one considers Einstein's equations as an initial value problem. One specifies the "state of the universe" at one point in time and then tries to determine the rest by equations describing the evolution. One cannot solve the resulting equations explicitly. Instead one has to be satisfied with proving that solutions exist and then to try to analyze the qualitative features of them. After having carried out this analysis, one can hopefully prove that there is a singularity and that the gravitational fields become arbitrarily strong there. In practice, one usually gets stuck on trying to prove existence.

3. Shape of the universe?

Figure 2:*Over the edge?*

Shape of the earth? When looking at the earth on a small scale, it looks pretty much like a plane. It is therefore not completely unnatural to think that the earth globally has the shape of a plane. As a consequence, one expects it to be either finite, and thus have a boundary, or to be infinite in extent. As we know, the earth can certainly be finite in extent and still be without boundary. In fact the sphere is not the only possibility. Figure 3 illustrates that beyond the sphere, there are infinitely many other possibilities.

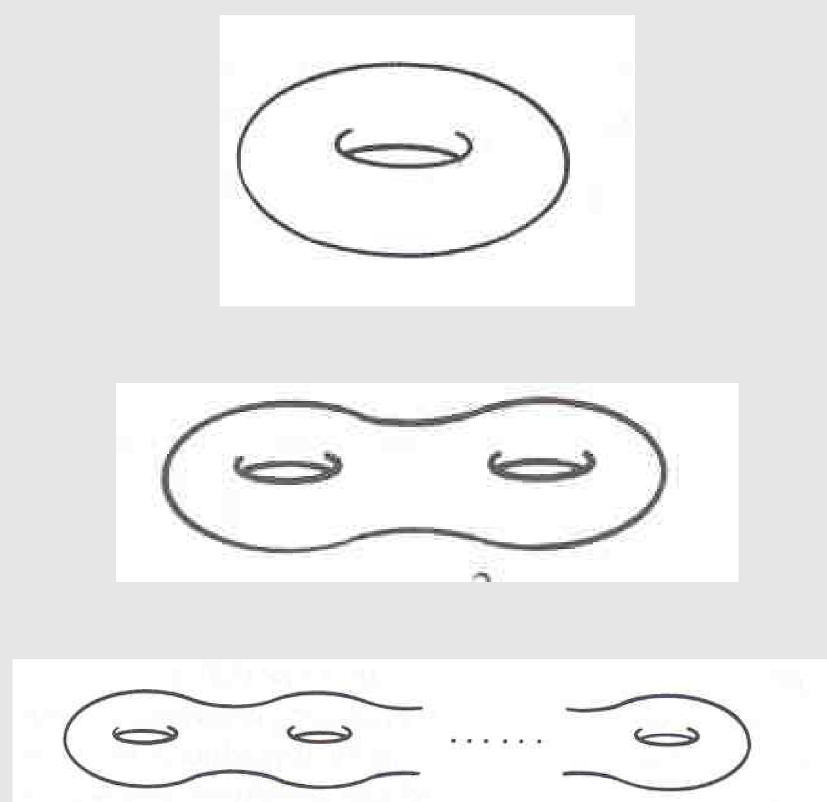

Figure 3: *Bounded two dimensional surfaces that locally look like a plane. The pictures are taken from Stillwell [2].*

Shape of the universe? Similarly to the earth, the universe, at "one point in time", looks pretty much like a three dimensional analogy of a plane locally, and it is tempting to again argue that it either has to be finite and have a boundary or to be infinite. This need however not be the case. An assumption often made in mathematical cosmology is that the universe is finite in extent and without boundary.

Mathematical concepts of shape - manifolds

In order to be able to address the question of the shape of the universe, at least from a mathematical point of view, one needs to have a precise definition of what one means by "the shape of the universe" at "one point in time". In mathematics the concept **manifold** has been defined and it serves this purpose (for the technically minded I am here interested in compact, orientable manifolds without boundary). This concept makes sense for any dimension. When asking the question what the possible shapes of the earth are, one mathematically asks the question what the possible two dimensional manifolds are. Interestingly enough, all the possibilities have been classified in dimension two. They are the sphere and the possibilities illustrated in Figure 3. In dimension three, the analogous question has not yet been answered, but recent developments indicate that the question might not be that far from receiving an answer.

Shape of the universe - 3 dimensional manifolds. To imagine 3-dimensional manifolds is of course somewhat more difficult than to imagine surfaces. One naturally thinks of surfaces as being a part of something bigger, but this is not so natural when thinking of a 3-dimensional manifold. When one wants to think not only of shape but also of geometry, it becomes even more difficult. The best way is to try to imagine how it would be to live inside it. Two examples of this are given in Figure 4. The repeated patterns should not be understood as many objects but as multiple images of the same object. The pictures should of course not be taken too seriously, the visualizations presented here are merely intended to illustrate some aspects of the mathematical properties of some particular 3-dimensional manifolds.

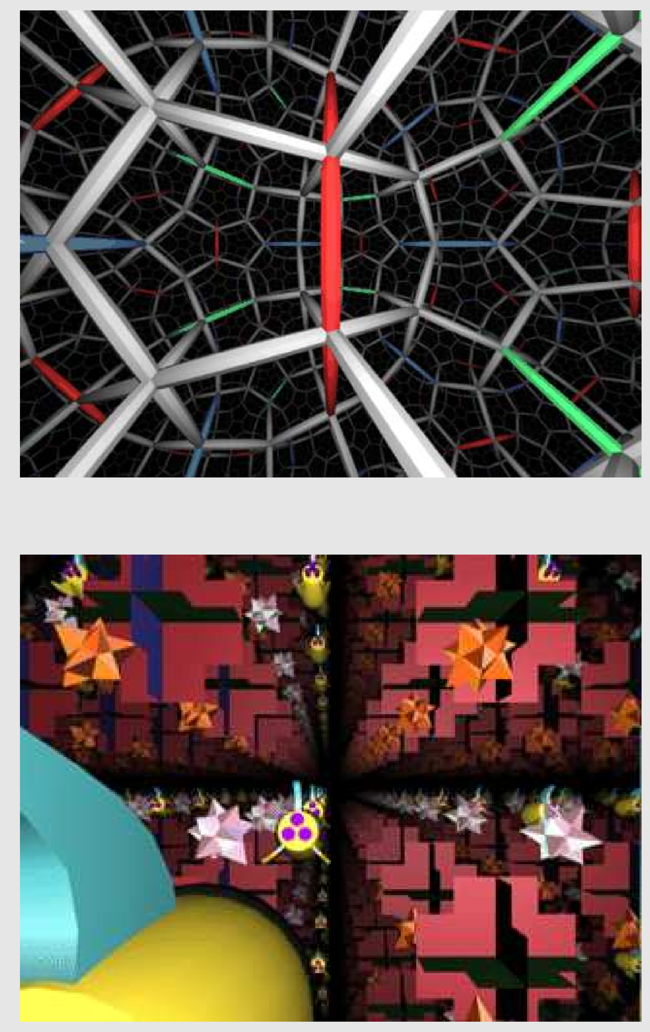

Figure 4: *Life inside hyperbolic space and a 3-torus respectively. The reader interested in more information on 3-manifolds is referred to the work of Thurston [3].*

4. Global shape and asymptotics

Symmetry and global shape

One interesting question is if there is any relation between the asymptotic behaviour of the universe as it expands and its global shape. As has been mentioned, the standard assumptions one makes are spatial homogeneity and isotropy. These are global conditions and lead to drastic restrictions of the possible shapes of the universe. Even when one drops the isotropy condition and limits oneself to consider universes which are only locally spatially homogeneous, one still restricts the possibilities severely. In other words, the study of these classes of solutions unavoidably makes it impossible to address certain questions.

Isotropization for "most" observers?

One way of trying to answer the question of whether the universe tends to become more isotropic or not is to consider spatially homogeneous universes that are not isotropic and see if they isotropize. One can then see that they in general do not. However, recent conjectures indicate that this might be due to the fact that the global restrictions on the shape implied by the condition of spatial homogeneity are too severe. According to these conjectures, in a general manifold, the parts which have anisotropies will asymptotically have a volume which is negligible in comparison with the isotropic pieces.

5. Technical comments

The concrete mathematical problems one ends up with consist of analyzing the asymptotics of solutions to certain evolution equations. The equations have similarities with the wave equation, but they are non-linear. Furthermore, the spatial domain is a 3-manifold and not some open domain of 3-dimensional Euclidean space. It is not possible to solve the equations explicitly, so one has to be satisfied with proving existence and then try to analyze the asymptotics. Proving existence for a sufficiently large time interval is often a major obstacle. In fact, trying to analyze when solutions to non-linear wave equations exist for all future times is a problem which occupies a large number of mathematicians.

6. References

This poster is concerned with mathematical cosmology, but for those interested in the physics point of view, the book *The new cosmos* by Unsöld A. and Baschek B. might be of interest. It covers astronomy and astrophysics more generally, but it should be mentioned that it is not completely popular.

[1] O'Neill B, *Semi Riemannian Geometry*, Academic Press.

[2] Stillwell J, *Classical Topology and Combinatorial Group Theory*, Springer.

[3] Thurston W, *Three-Dimensional Geometry and Topology*, Princeton University Press.

All these books are mathematics texts.