1. Problem set 2

Problem 1. Let $H^n_\nu(r)$ and $S^n_\nu(r)$ be the pseudohyperbolic space and pseudosphere respectively, cf. Definition 23, p. 110 of O'Neill's book. For a general semi-Riemannian manifold (M, g), let

$$G = \mathrm{Ric} - \frac{1}{2}Sg,$$

where Ric is the Ricci tensor and S is the scalar curvature of (M, g). G is called the *Einstein tensor*. Compute the Einstein tensor of $H^4_1(r)$ and of $S^4_1(r)$.

It should be remarked that $S^4_1(r)$ is *de Sitter space*, a Lorentz manifold of interest in general relativity. The universal covering space of $H^4_1(r)$ is called *anti de Sitter* space.

Problem 2. According to Lemma 25, p. 110 of O'Neill's book, $S^4_1(r)$ is diffeomorphic to $\mathbb{R}\times S^3$ and $H^4_1(r)$ is diffeomorphic to $S^1\times\mathbb{R}^3$. What are the metrics on $\mathbb{R} \times S^3$ and $S^1 \times \mathbb{R}^3$ induced by these diffeomorphisms?

Let $(\bar{M}, \bar{g})$ be a Lorentz manifold which is time oriented, cf. p. 144-145 of O'Neill's book. In other words, there is a timelike vector field T on $\bar{M}$ and we say that a causal vector $v \in T_p\bar{M}$ is future oriented if $\bar{g}(v, T_p) < 0$. Let M be a spacelike hypersurface of $(\bar{M}, \bar{g})$, with induced metric g, let $i : M \to \bar{M}$ be the embedding and let N be a future directed unit timelike vector field such that for every $p \in M$, $\bar{g}(N_p, i_*v) = 0$ for every $v \in T_pM$. Then the second fundamental form of M is the covariant 2-tensor field k on M defined by

$$k(v, w) = \bar{g}(\bar{D}_{i_*v}N, i_*w),$$

for $v, w \in T_pM$, where $\bar{D}$ is the Levi-Civita connection of $(\bar{M}, \bar{g})$. In what follows, we shall not distinguish between v and i_*v.

Problem 3. Prove that $II(v, w) = k(v, w)N_p$ for $v, w \in T_pM$, where N_p is the future directed unit normal to M at p.

Problem 4. If $\bar{G}$ is the Einstein tensor of $(\bar{M}, \bar{g})$ and N is as above, prove that

$$\bar{G}(N_p, N_p) = \frac{1}{2}[S - k_{ij}k^{ij} + (\mathrm{tr}_g k)^2](p), \tag{1}$$

$$\bar{G}(N_p, v) = [\mathrm{div} k - d(\mathrm{tr}_g k)](v), \tag{2}$$

where $\bar{G}$ is the Einstein tensor of $(\bar{M}, \bar{g})$, $p \in M$, $v \in T_pM$, D is the Levi-Civita connection and S the scalar curvature of (M, g). For the definition of $\mathrm{div} k$, cf. (2), p. 86 of O'Neill's book.

The object $\mathrm{tr}_g k$ is referred to as the mean curvature of the hypersurface. The equations (1)-(2) are of central importance in general relativity; they constitute the starting point for formulating Einstein's equations as an initial value problem. Einstein's equations state that $\bar{G} = T$, where T is the stress-energy tensor of the matter. What the stress energy tensor is depends on the specific matter model. Combining Einstein's equations with (1)-(2), one obtains the so-called *constraint equations*. Classical matter is expected to satisfy the *weak energy condition* which states that $T(v, v) \geq 0$ for timelike vectors. If the hypersurface is *maximal*, i.e. if $\mathrm{tr}_g k = 0$, then (1) implies $S \geq 0$ if the weak energy condition is satisfied, i.e. that the scalar curvature is non-negative. It should be noted that this is often constitutes a strong restriction on the possible topologies of the hypersurface.

Problem 5. Let

$$I^+(0) = \{x \in \mathbb{R}^4 : x^0 > 0, -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 < 0\}.$$

Let $\mathbb{R}_+ = (0, \infty)$. Define $\phi : \mathbb{R}_+ \times \mathbb{R}^3 \to I^+(0)$ by

$$\phi(t, x) = (t(1 + |x|^2)^{1/2}, tx),$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$. Show that ϕ is a diffeomorphism. If one pulls back the Minkowski metric on $I^+(0)$ by ϕ, what is the resulting metric, say g_+, on $M_+ = \mathbb{R}_+ \times \mathbb{R}^3$?

The Lorentz manifold (M_+, g_+) is clearly a vacuum solution to Einstein's equations, i.e. the corresponding Einstein tensor is zero. It is of interest in cosmology.

Problem 6. With M_+ and g_+ as in the previous problem, compute the left and right hand sides of (1) and (2) in the case of the spacelike hypersurfaces $\{t\} \times \mathbb{R}^3$ in (M_+, g_+).