UNIQUENESS FOR AN OVERDETERMINED BOUNDARY VALUE PROBLEM FOR THE P-LAPLACIAN

FARID BAHRAMI AND HENRIK SHAHGHOLIAN

ABSTRACT. For $p > 1$ set $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and let μ be a measure with compact support. Suppose, for $j = 1, 2$, there are functions $u_j \in W^{1,p}$ and (bounded) domains Ω_j, both containing the support of μ with the property that $\Delta_p u_j = \chi_{\Omega_j} - \mu$ in $\mathbf{R}^N$ (weakly) and $u_j = 0$ in the complement of Ω_j. If in addition $\Omega_1 \cap \Omega_2$ is convex, then $\Omega_1 \equiv \Omega_2$ and $u_1 \equiv u_2$.

1. INTRODUCTION

The inverse "exterior" domain problem in classical potential theory is concerned with finding the shape of a domain, given its Newtonian potential at the exterior points. One of the main features of this problem is to give geometric conditions so that there is at most one domain having prescribed exterior potential.

A method for approaching this problem is to assume the contrary, i.e. to assume that there are, at least, two solutions Ω_j $(j = 1, 2)$, whose Newtonian potentials

$$U^{\Omega_j}(x) = c_N \int_{\Omega_j} \Gamma(x-y)dy \qquad (j = 1, 2)$$

coincide outside their union. Here

$$\Gamma(x) = \begin{cases} |x|^{2-N} & N \geq 3 \\ \log|x| & N = 2, \end{cases}$$

and c_N is a normalization factor so that $\Delta U^{\Omega_j} = \chi_{\Omega_j}$ in $\mathbf{R}^N$. Next, one defines

$$U = \begin{cases} U^{\Omega_1} & \text{in} \quad \Omega_1^c \\ U^{\Omega_2} & \text{in} \quad \Omega_2^c \\ \text{arbitrary} & \text{in} \quad \Omega_1 \cap \Omega_2. \end{cases}$$

Since $U^{\Omega_1} = U^{\Omega_2}$ in $(\Omega_1 \cup \Omega_2)^c$, U is well defined. One may also assume U is C^1 in a small neighborhood of any point $x \in (\partial\Omega_1 \cap \Omega_2) \cup (\partial\Omega_2 \cap \Omega_1)$. This depends on the fact that the Newtonian potential, with bounded density, is $C^1(\mathbf{R}^N)$.

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Now setting

$$u_j = U^{\Omega_j} - U, \qquad \text{in } \mathbf{R}^N, \qquad (j = 1, 2)$$

we'll have, for $j = 1, 2$,

$$\begin{aligned} &\Delta u_j = \chi_{\Omega_j} - \mu \qquad \text{in } \mathbf{R}^N, \\ &u_j = 0 \qquad \text{in } \mathbf{R}^N \setminus \Omega_j, \end{aligned} \tag{1}$$

where $\mu = \Delta U$, with

$$\text{supp}\mu \subset (\Omega_1 \cap \Omega_2) \cup (\partial\Omega_1 \cap \partial\Omega_2). \tag{2}$$

We also remark that $\Omega_1 \cap \Omega_2 \neq \emptyset$, since otherwise U^{Ω_j} can be continued, harmonically, into Ω_j and violate Liouville's theorem, unless $\Omega_j = \emptyset$, which is excluded.

The above described problem, through equation (1), now becomes a "free boundary problem", which has been studied by several people (see [sa], [sh], [gu], [g-s], [k-m]). For the original problem of uniqueness in the class of domains with the same exterior potentials we refer to the book of Isakov [i] and the references therein, (see also [z]).

A famous result of P.S. Novikov [n] states that uniqueness holds in the class of starshaped domains with the same exterior potentials. It was also conjectured that if Ω_1 is convex, Ω_2 is solid (no holes) and they have the same exterior potentials in the complement of their union, then they must be identical. This is still an open problem. The second author, however, proved the following partial result [shah]: If two domains, whose intersection is convex, have the same exterior potentials in the complement of their union, then they must be identical.

A generalization of the above concept to nonlinear potential theory seems not so easy, as there are no integral representations for nonlinear operators in general. However, the reformulation of the problem of potentials to the language of free boundaries (see equation (1)) seems to be a possible way of extending the concept in mind to nonlinear operators. Now the problem becomes to prove uniqueness, under certain geometric conditions, for the solutions to

$$\begin{cases} A(u) = \chi_\Omega - \mu & \text{in } \mathbf{R}^N, \\ u = 0 & \text{in } \mathbf{R}^N \setminus \Omega, \\ \text{supp}\mu \subset \Omega, \end{cases} \tag{3}$$

where μ is a given measure and A is a general operator.

In this note we will be concerned with the p-Laplace operator Δ_p, where $1 < p < \infty$ and

$$\Delta_p u = \text{div } (|\nabla u|^{p-2}\nabla u). \tag{4}$$

By a solution to (3) when $A = \Delta_p$ we mean a function $u \in W^{1,p}(\Omega)$ such that

$$\int_{\mathbf{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi dx = \int_{\mathbf{R}^N} \phi\chi_\Omega(dx - d\mu),$$

whenever $\phi \in C_o^\infty(\mathbf{R}^N)$. Both existence and regularity of the solutions to equation (3), when A is the ordinary Laplacian ($p = 2$ in (4)), has been intensively studied (see [sa], [gu], [g-s], [k-s]). Most of those results may easily be generalized to hold for the p-Laplacian.

We also remark that outside the support of the measure μ, any solution u to equation (3) (with $A = \Delta_p$) is $C^{1,\alpha}$, which is the maximum regularity that one should expect [l]. Therefore we will always assume that our solutions are $C^{1,\alpha}$ outside the support of the measure and that they are in $W^{1,p}$ in the entire Euclidean space. The only point where we use the C^1 regularity is in connection with a Hopf's type maximum principle.

The main result of this paper states that if u_1 and u_2 are solutions to equation (3) with supports Ω_1 and Ω_2 respectively and A being the p-Laplacian, then $u_1 = u_2$, provided $\Omega_1 \cap \Omega_2$ is convex. We prove this by contradiction using the comparison and Hopf's maximum principle. For the readers convenience we state the comparison principle (cf. ([h-k-m], 6.5, 7.6) and [l]).

Comparison Principle. Let G be a bounded subset of $\mathbf{R}^N$ and u, v be two functions in $W^{1,p}(G)$ satisfying $\Delta_p u \geq \Delta_p v$ (weakly) in G and $\limsup_{y\to x} u \leq \liminf_{y\to x} v$, where $x \in \partial G$. Then $u \leq v$ in G.

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2. Main Result

THEOREM 1. Let μ be a measure with compact support and suppose $u_j \in W^{1,p}(\mathbf{R}^N)$ ($j = 1, 2$) are solutions to

$$\begin{cases} \Delta_p u_j = \chi_{\Omega_j} - \mu & \text{in } \mathbf{R}^N, \\ u_j = 0 & \text{in } \mathbf{R}^N \setminus \Omega_j, \\ \mathrm{supp}\mu \subset \Omega_j, \end{cases} \tag{5}$$

where Ω_j is compact. Suppose also $\Omega_1 \cap \Omega_2$ is convex. Then $\Omega_1 \equiv \Omega_2$, and $u_1 \equiv u_2$.

In order to prove this theorem we need some lemmas. First we prove a simple form of Hopf type maximum principle (which is also called the Hopf's boundary point lemma). However, the reader should be aware that the usual Hopf's boundary point lemma is also valid for non-degenerate elliptic operators (see [t]).

LEMMA 2. Let B_r be a ball of radius r. Let $u \in C^1(\overline{B}_r) \cap W^{1,p}(B)$ satisfy $u \leq 0$ and $\Delta_p u \geq \alpha$, for some $\alpha > 0$. Suppose also there is a point $z \in \partial B_r$ such that $u(z) = 0$. Then $\frac{\partial u}{\partial n}(z) > 0$, where n is the unit outward normal on ∂B_r.

Proof: Consider the function $h(x) = c(|x|^q - r^q)$, where q is the conjugate of p, i.e. $1/p + 1/q = 1$, and $c > 0$ is chosen such that $\Delta_p h = \alpha$. Thus in B_r $\Delta_p u \geq \Delta_p h$, while on ∂B_r, $h \geq u$. Hence by comparison principle $h \geq u$ in B_r. Since $h(z) = u(z) = 0$, we obtain

$$\frac{\partial u}{\partial n}(z) \geq \frac{\partial h}{\partial n}(z) = cqr^{q-1} > 0.\Box$$

The above lemma holds true for $\alpha = 0$. However, in the proof one should take the fundamental solution of the p-Laplacian as an auxiliary function.

LEMMA 3. Let u be a solution of (5) and Ω the corresponding domain. Then for each boundary point $z \in \partial\Omega$ there is a sequence $\{z^j\} \subset \Omega$, such that $z^j \to z$ and $u(z^j) > 0$.

Proof: Let $z \in \partial\Omega$ be fixed and suppose on the contrary that there is a small ball B centered at z such that $u \leq 0$ on B. According to (5), u is a subsolution in B and therefore, by the strong maximum principle [hkm; 6.5], it can not attain its maximum in the interior of B. Since $u(z) = 0$, we obtain a contradiction, and the proof is completed. $\Box$

LEMMA 4. Under the assumptions of Theorem 1,

$$\sup_{\Omega_1 \setminus \Omega_2} u_1 \geq \frac{1}{q} d_1^q,$$

where $d_1 = \sup_{z \in \Omega_1}$ dist $(z, \Omega_1 \cap \Omega_2)$. The same conclusion holds for u_2, mutatis mutandis.

Proof: Let $z \in \Omega_1 \setminus \Omega_2$ and $u(z) > 0$. Let $y \in \partial(\Omega_1 \cap \Omega_2)$ be the nearest point to z. By translation and rotation we may assume y is the origin and z is on the positive x_1 axis. Now let us denote by D the domain $\{x \in \Omega_1; x_1 > 0\}$. Set $v(x) = 1/q|x_1 - z_1|^q$, and observe that $\Delta_p v = 1$ and $v(x) > 0 > u_1(x) - u_1(z)/2$ on $\partial D \setminus \{x_1 = 0\}$. If also $v(x) \geq u_1(x) - u_1(z)/2$ on $\{x_1 = 0\}$, then by the comparison principle $v \geq u_1 - u_1(z)/2$ in D. This contradicts the simple fact that $v(z) = 0 < u_1(z)/2$. Therefore we conclude that

$$v \leq \sup_{\{x_1=0\}} u_1 - u_1(z)/2 \leq \sup_{\Omega_1 \setminus \Omega_2} u_1 - u_1(z)/2. \tag{6}$$

By Lemma 3, for each boundary point $z \in \partial\Omega_1 \setminus \overline{\Omega}_2$ one may consider a sequence $z^j \to z$ and such that $u(z^j) > 0$. Hence (6) holds also for all boundary points z off $\overline{\Omega}_2$. Taking the supremum for v over z yields the desired result. $\Box$

LEMMA 5. Under the hypotheses of Theorem 1 and Lemma 4, if $\Omega_1 \neq \Omega_2$ and $d_1 \leq d_2$, then there exists $y^1 \in \Omega_1 \setminus \Omega_2$ such that $-u_1(y^1) > d_1^q/q$, and $|\nabla u_1(y^1)| = 0$. Similar conclusions hold if $d_2 \leq d_1$, mutatis mutandis.

Proof: Let $a = -\inf_{\Omega_1\setminus\Omega_2} u_1 \geq 0$ and $y^1 \in \overline{\Omega}_1 \setminus \Omega_2$ be such that $u_1(y^1) = -a$. Then as measures

$$\Delta_p(u_1 + a) \leq \Delta_p u_2 \quad \text{in } \Omega_2, \quad \text{while} \qquad u_2 \leq u_1 + a \quad \text{on } \partial\Omega_2.$$

Therefore, by the comparison principle $u_2 \leq u_1 + a$ in Ω_2. In $\Omega_1 \setminus \Omega_2$, $u_2 = 0$ and $u_1 + a \geq 0$. We thus conclude

$$u_1 + a - u_2 \geq 0 \qquad \text{in } \Omega_1 \cup \Omega_2. \tag{7}$$

In particular, $u_2 \leq a$ in $\Omega_2 \setminus \Omega_1$ and $a > 0$. Hence $\sup_{\Omega_2\setminus\Omega_1} u_2 \leq a$.

We now claim

$$\sup_{\Omega_2\setminus\Omega_1} u_2 < a.$$

Suppose on the contrary that there is a $z \in \Omega_2\setminus\Omega_1$ such that $u_2(z) = a$. Now, by (7), $u_1 + a - u_2 \geq 0$ in Ω_2, and therefore it has a minimum value at z, and consequently a vanishing gradient. Since $z \in \Omega_2\setminus\Omega_1$, u_1 has also a vanishing gradient at z. Therefore we conclude that $|\nabla u_2|(z) = 0$. By the convexity of $\Omega_1 \cap \Omega_2$ we may take a ball B in $\Omega_2 \setminus \Omega_1$ with z on its boundary and apply Lemma 2 to obtain a contradiction. This proves our claim. Summing up we have

$$a = -u_1(y^1) > \sup_{\Omega_2\setminus\Omega_1} u_2 \geq \frac{1}{q} d_2^q \geq \frac{1}{q} d_1^q.$$

This proves the first part of the lemma. To prove the second statement observe by (7) that $u_1 + a - u_2$ is nonnegative in Ω_1. Also, by the definition of a and since $y^1 \notin \Omega_2$ we may conclude that $u_1 + a - u_2$ has a minimum value at y^1 and thus $\nabla u_1(y^1) = \nabla u_2(y^1) = 0$. □

Remark:

We remark that if $\inf(d_1, d_2) = 0$, then $\Omega_1 \equiv \Omega_2$, and consequently $u_1 \equiv u_2$. This depends on the fact that Ω_1 and Ω_2 have the same volume. The latter may be shown as follows. Since

$$|\Omega_1| - |\Omega_2| = \int (\Delta_p u_1 - \Delta_p u_2),$$

we may integrate by parts, and use the fact that $|\nabla u_j| = 0$ $(j = 1, 2)$ outside a large ball, to conclude that $|\Omega_1| = |\Omega_2|$.

Proof of Theorem 1: By the above remark if $\inf(d_1, d_2) = 0$, then the theorem follows. Therefore suppose $0 < d_1 \leq d_2$. By Lemma 5, there is a point $y^1 \in \Omega_1 \setminus \Omega_2$ such that

$$a = -\inf_{\Omega_1\setminus\Omega_2} u_1 = -u_1(y^1) > \frac{1}{q} d_1^q, \qquad |\nabla u_1(y^1)| = 0. \tag{8}$$

Let H be the hyperplane passing through y^1 and containing $\Omega_1 \cap \Omega_2$ on one side of it. By translation and rotation, we may assume that $y^1 = 0$, $H = \{x_1 = 0\}$ and $\Omega_1 \cap \Omega_2$ lies in $\{x_1 < 0\}$. Now, choose $z \in \partial\Omega_1 \setminus \Omega_2$ with largest distance to H. Then $z_1 \leq d_1$ and by (8), $z_1^q/q < a$. Let $\epsilon > 0$ be such that $(z_1 + \epsilon)^q/q \leq a$, and define $w(x) = u_1(x) + a$, $v(x) = ((x_1 + \epsilon)^q - \epsilon^q)/q$. Then, since $\Delta_p w = \Delta_p v = 1$ in $D := \{x \in \Omega_1 : x_1 > 0\}$ and $w \geq v$ on ∂D, we'll have (by comparison principle) $w \geq v$ in D. Since also $w(0) = v(0)$, it follows that

$$\frac{\partial w}{\partial x_1}(0) \geq \frac{\partial v}{\partial x_1}(0);$$

i.e.

$$\frac{\partial u_1}{\partial x_1}(0) \geq \frac{\partial v}{\partial x_1}(0) = \epsilon^{q-1},$$

which contradicts (8). This proves the theorem in the case $d_1 \leq d_2$. If $d_2 \leq d_1$ we interchange Ω_1, Ω_2 and repeat the same argument. The proof is now completed. □

Remark:

It follows from the proof of Lemma 5 that one may weaken the convexity assumption on $\Omega_1 \cap \Omega_2$ to an exterior ball condition if one assumes one of the solutions u_1, u_2 is positive.

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Math. Dept. University of Tehran, P.O.Box 13145-1873 Tehran, Iran

Dept. of Math., The Royal Inst. of Technology, 100 44 Stockholm Sweden
E-mail address: henriks@math.kth.se