REGULARITY OF A FREE BOUNDARY PROBLEM

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Abstract. Let u be the Newtonian potential of a real analytic distribution in an open set Ω. In this paper we assume u is analytically continuable from the complement of Ω into some neighborhood of a point $x_0 \in \partial\Omega$, and we study conditions under which the analytic continuation implies that $\partial\Omega$ is a real analytic hypersurface in some neighborhood of x_0.

0. Introduction

This paper is concerned with the regularity of free boundaries arising from the inverse problem of the Newtonian potential theory.

Let Ω be a bounded open set in $\mathbb{R}^n$, χ_Ω denote the characteristic function of Ω, and f be a function defined in $\mathbb{R}^n$. Define v to be the Newtonian potential of $f\chi_\Omega$, that is,

$$v(x) = \int_\Omega E_n(x-y)f(y)dy, \tag{0.1}$$

where

$$E_n(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{(2-n)\omega_n}|x|^{2-n} & n \geqslant 3, \end{cases}$$

and ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Obviously v is harmonic off $\overline{\Omega}$. Moreover, if $\partial\Omega$ is a real analytic hypersurface and f is a real analytic function in a neighborhood of $x_0 \in \partial\Omega$, then, as a result of the Cauchy-Kowalevski theorem, there is a harmonic function h in $B_r(x_0)$ such that $v(x) = h(x)$ and $\nabla v(x) = \nabla h(x)$ for $x \in B_r(x_0) \setminus \Omega$ (see e.g. [Sha2]). In this case we say that v has a harmonic (analytic) continuation h, from $B_r(x_0) \setminus \Omega$ into $B_r(x_0) \cap \Omega$. In the present paper we consider the inverse problem. Namely, suppose f is real analytic in some neighborhood of x_0 ($f \in C^{m+\lambda}(B_r(x_0))$) and v, given by (0.1), has a harmonic continuation h into a neighborhood of $x_0 \in \partial\Omega$. Under which

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conditions can we conclude that $\partial\Omega \cap B_\rho(x_0)$ $(0 < \rho \leqslant r)$ is a real analytic surface $(\partial\Omega \cap B_\rho(x_0) \in C^{m+1+\lambda})$?

The differential equation corresponding to the harmonic continuation h is

$$\begin{cases} \Delta v(x) = f(x) & x \in B \cap \Omega, \\ v(x) = h(x),\ \nabla v(x) = \nabla h(x) & x \in B \setminus \Omega, \end{cases} \tag{0.2}$$

where h is harmonic in B and $B = B_r(x_0)$. Equation (0.2) also appears in the local analysis of a number of problems concerning potentials fields and steady-state transport processes: the inverse problem of potential theory [Is2], [Z]; quadrature domains [Sa2], [Sha2]; the capacitor problem [Sc1]; moving boundary problems of Hele-Shaw type [EE], [EV], [Gu], [Ho], [Ma]. Putting $u = v - h$, (0.2) becomes

$$\begin{cases} \Delta u(x) = f(x) & x \in B \cap \Omega, \\ u(x) = |\nabla u(x)| = 0 & x \in B \setminus \Omega. \end{cases} \tag{0.3}$$

According to a theorem of Isakov [Is1], and independently of Kinderlehrer and Nirenberg [KN] (see also [Fr; Sec. 2, Theorem 1.1]), if $\partial\Omega \cap B \in C^1$ and f is analytic in B $(f \in C^{m+\lambda}(B))$, then $\partial\Omega$ is an analytic $(C^{m+1+\lambda})$ surface in some neighborhood of x_0. Thus, our main problem is to show that $\partial\Omega \cap B_\rho(x_0) \in C^1$ in order to conclude that (0.3) implies $\partial\Omega \cap B_\rho(x_0)$ is real analytic.

In the obstacle problem, the energy functional is minimized over a convex set contained in a certain Sobolev space which consists of functions grater or equal than the obstacle Ψ (see e.g. [Fr], [KS]). Let u_0 be the unique minimizer and $\Omega := \{u_0 > \Psi\}$ be the noncoincidence set. The Euler equation for $u := u_0 - \Psi$ in B is

$$\begin{cases} \Delta u(x) = f(x) & x \in B \cap \Omega, \\ u(x) = |\nabla u(x)| = 0 & x \in B \setminus \Omega, \\ u(x) \geqslant 0 & x \in B, \end{cases} \tag{0.4}$$

which is a special case of (0.3).

In two outstanding and fundamental papers, [Ca1] and [Ca2], Caffarelli proved that if u satisfies (0.4) and $\Omega^c := \mathbb{R}^n \setminus \Omega$ is not too small in some neighborhood of $x_0 \in \partial\Omega$, then $\partial\Omega \in C^1$ in a neighborhood of that point. Our aim is to obtain the corresponding result for equation (0.3). In so doing, we encounter mainly two obstacles. First, for u satisfying (0.4) it is known that

$$u \in W^{2,\infty}_{loc}(B), \tag{0.5}$$

i.e., u has bounded second-order derivatives (see [BK], [CK] and [F]). While for solutions to (0.3) this is unknown. Secondly, Caffarelli's estimate for the pure second-order derivatives, [Ca1; Theorem 1], implies

$$\varliminf_{x \to y} D_{ii} u(x) \geqslant 0, \qquad y \in \partial\Omega \tag{0.6}$$

holds for all solutions to (0.4). But (0.6) fails in general (e.g. $u = 2x_1^2 - x_2^2$).

Our method to show (0.5), for u satisfying (0.3), is based on the quadratic growth of u near the boundary. We show that a solutions to (0.3) with $f \in L^\infty$ has quadratic growth, provided that $\{u < 0\}$ has small upper Lebsegue density on $\partial\Omega$. In the particular case of equation (0.4), we obtain a short and simple proof of (0.5) which also supplies an estimate with universal constants. The idea to utilize the growth of a solution in order to obtain maximal regularity appears in the work of Alt and Caffarelli, but for a different free boundary problem [AC], where they showed that a linear growth implies Lipschitz continuity of the solution.

To overcome the failure of Caffarelli's estimate (0.6) we impose a certain geometric condition on the set $\{u < 0\}$. Using this condition, we modify Caffarelli's estimate in order to obtain (0.6) for equation (0.3). We also show that our geometric condition holds true, if the complement of Ω is not too small near the boundary point x_0, and if all solutions to the global equation

$$\begin{cases} \Delta v(x) = 1 & x \in \Omega, \\ v(x) = |\nabla v(x)| = 0 & x \in \Omega^c, \\ |v(x)| \leqslant C(1 + |x|)^2 \end{cases} \tag{0.7}$$

are nonnegative. The latter holds true in $\mathbb{R}^2$ by Sakai's classifications of solutions to the global equation (0.7) (see [Sa1], [Sa3]). In $\mathbb{R}^n$, $n \geqslant 3$ there are only partial results concerning the nonnegativity of solutions to (0.7), ([KM], [Sh]), but for the general case it is still an open problem [Sh].

Although in this paper we are dealing with the Laplace operator, our techniques also apply to more general second-order elliptic operators with suitable regularity of the coefficients.

The organization of the paper is as follows: Section 1 deals with the quadratic growth of subharmonic functions. In Section 2 we show the limitation of the second-order derivatives as a consequence of the quadratic growth. Section 3 is devoted to the extension of Caffarelli's regularity result. The connection between the geometrical condition on $\{u < 0\}$ and the nonnegativity of solutions to equation (0.7) is discussed in Section 4.

We use the most standard notations. For a set $S \subset \mathbb{R}^n$, ∂S is the boundary, $\bar{S}$ is the closure, $\mathrm{int}(S)$ is the interior, S^c is the complement and $|S|$ is the Lebesgue measure of S. For a sequence of sets $\{S_j\}_{j=1}^\infty$, $\overline{\lim}_j S_j$ is the set of all limit points of sequences $\{x_j\}_{j=1}^\infty$ with $x_j \in S_j$. $B_r(x_0) := \{x : |x - x_0| < r\}$, $B_r = B_r(0)$ and $B = B_1$. $\nabla u = (D_1 u, ..., D_n u)$, $\Delta u = D_{11} u + ... + D_{nn} u$. Various of absolute constants are denoted by $C, C_0, C_1, ...$ which might change value from one appearance to another.

1. Subharmonic Functions with Quadratic Growth

In this section we study the local behavior of a function u, with $\Delta u \in L^\infty$, near a point where the function and its gradient vanish.

Let $B := B_1(0)$ be the unit ball and let u satisfies the Poisson equation $\Delta u = f \in L^\infty(B)$ in the distributional sense. It is well known that $D_i u$ $(i = 1, ..., n)$ belongs to the Zygmund class, and hence

$$|\nabla u(x) - \nabla u(y)| \leqslant C_0 \|f\|_\infty |x-y| \log \frac{1}{|x-y|}, \qquad |x-y| \leqslant e^{-2}$$

(cf. [Zy; Theorem 4.4]). Therefore, if $u(0) = |\nabla u(0)| = 0$, then

$$|u(x)| \leqslant C_0 \|f\|_\infty |x|^2 \log \frac{1}{|x|}, \qquad |x| \leqslant e^{-2}. \tag{1.1}$$

The main result of this section asserts that the logarithmic term in (1.1) is dispensable provided that

$$L(\rho) = L(\rho, u) := \frac{|B_\rho \cap \{u < 0\}|}{|B_\rho|} \tag{1.2}$$

is small for small ρ.

Theorem 1.1. *Let $f \in L^\infty(B)$, $f \geqslant 0$ and set $\eta_n = 2^{(2n+6)}$. Then any solution u, to $\Delta u = f$ in B with the properties $u(0) = 0$ and*

$$\overline{\lim_{\rho \to 0}}\, L(\rho, u) < \frac{1}{\eta_n}, \tag{1.3}$$

satisfies

$$|u(x)| \leqslant C|x|^2, \qquad x \in B, \tag{1.4}$$

and

$$|\nabla u(x)| \leqslant C|x|, \qquad x \in B_{1/2}. \tag{1.5}$$

In the case $u \geqslant 0$, the constant C depends only on the supremum norm of f and the dimension.

Remarks.

(i) Note that $\overline{\lim}_{\rho\to 0} L(\rho, u) < 2^{-1}$ implies $\nabla u(0) = 0$. Indeed, $u(x) = \mathbf{a}\cdot x + o(|x|)$, where $\mathbf{a} = \nabla u(0)$. Thus, if $\mathbf{a} \neq 0$, then $\lim_{\rho\to 0} L(\rho, u) = \lim_{\rho \to 0} L(\rho, \mathbf{a}\cdot x) = 2^{-1}$.

(ii) The estimate (1.4) fails if $\overline{\lim}_{\rho\to 0} L(\rho)$ is not small, even though it is less than a half. Consider the following example. Let K be a spherical cone $\{x : x_n > |x| \cos(\alpha/2)\}$, $\alpha \in (0, \pi/2)$. The function

$$u(x) = c_\alpha (n x_n^2 - |x|^2) \log|x| \ + \ |x|^2 \phi(\frac{x}{|x|})$$

satisfies the Poisson equation $\Delta u = \chi_K$, where $c_\alpha > 0$ and ϕ is a C^1- function homogeneous of degree zero (see [KM; Sec. 3]). Clearly (1.4) does not hold for u, while $u(0) = |\nabla u(0)| = 0$ and

$$\lim_{\rho\to 0} L(\rho) = \beta_n > \frac{1}{\eta_n},$$

where

$$\beta_n = \frac{\int_0^{\arccos(1/\sqrt{n})} \sin^{n-2}(\varphi)d\varphi}{\int_0^{\pi/2} \sin^{n-2}(\varphi)d\varphi},$$

$(\beta_3 \approx 0.423, \quad \beta_4 \approx 0.391, \quad \beta_5 \approx 0.374)$.

For the proof of Theorem 1.1 we need some lemmas. We also assume, without loss of generality, $\|f\|_\infty \leqslant 1$. The Laplace operator is always considered in the distributional sense.

Lemma 1.2. *(H.S. Shapiro) Suppose* $\Delta u = f$ *in* B *and* $u(0) = 0$, *then*

$$\int_{B_\rho} u dx \leqslant C_1 b_n \rho^{n+2},$$

where $C_1 = 1/2(n+2)$ *and* b_n *is the volume of the unit ball.*

Proof. Let $G_r(y,x)$ be the Greens function for the ball B_r, and first assume u is C^2 in B. Then

$$0 = u(0) = \int_{\partial B_r} u(x)\frac{\partial G_r}{\partial n}(0,x)dS - \int_{B_r} f(x)G_r(0,x)dx.$$

Now

$$G_r(0,x) = \begin{cases} \frac{1}{nb_n}\left(\log \frac{r}{|x|}\right) & n = 2, \\ \frac{1}{n(n-2)b_n}\left(|x|^{2-n} - r^{2-n}\right) & n \geqslant 3, \end{cases}$$

and for $|x| = r$

$$\frac{\partial G_r}{\partial n}(0,x) = \frac{1}{nb_n} r^{1-n}.$$

Thus

$$\int_{\partial B_r} u(x)dS \leqslant nb_n r^{n-1} \int_{B_r} G_r(0,x)dx = \frac{b_n}{2} r^{n+1},$$

which implies

$$\int_{B_\rho} u(x)dx = \int_0^\rho \left(\int_{\partial B_r} u(x)dS\right) dr \leqslant \frac{b_n}{2}\int_0^\rho r^{n+1}dr = \frac{b_n}{2(n+2)}\rho^{n+2}.$$

In the general case, we complete the proof by means of a regularization. ■

Let us define $u^+ = \max(u,0)$ and $u^- = \min(u,0)$.

Lemma 1.3. *Let* u *and* f *be as in Theorem 1.1, then for* $0 < \rho < 2^{-1}$,

$$\sup_{B_\rho} |u| \leqslant 2^{n+1} \sup_{B_{2\rho}} u + C_1\rho^2.$$

Proof. For fixed $y \in B$, let $\psi_y(x) = \frac{|x-y|^2}{2n}$, then by Kato inequality [K],

$$\Delta\left(|u| + \psi_y\right) \geqslant f(\text{sgn } u) + 1 \geqslant 0.$$

Hence $|u|+\psi_y(x)$ is subharmonic, so for $|y|\leqslant\rho$,

$$\begin{aligned} b_n\rho^n|u(y)| &\leqslant \int_{B_\rho(y)}(|u|+\psi_y)dx \leqslant \int_{B_{2\rho}(0)}(u^+ + u^-)dx + \frac{b_n}{2}\int_0^\rho t^{n+1}dt \\ &\leqslant 2\int_{B_{2\rho}(0)} u^+dx + \frac{b_n}{2(n+2)}\rho^{n+2} \\ &\leqslant 2^{n+1}b_n\rho^n \sup_{B_{2\rho}(0)} u + b_nC_1\rho^{n+2}. \end{aligned} \tag{1.6}$$

Here, in the third step we have used the inequality

$$\int_{B_{2\rho}} u^- \leqslant \int_{B_{2\rho}} u^+,$$

which holds for subharmonic u with $u(0)=0$. Now, dividing both sides of (1.6) by $b_n\rho^n$ we'll have the desired result. ■

Lemma 1.4. *Let u and f be as Theorem 1.1 and $0<\rho<1$. If*

$$\sup_{B_\rho} u \leqslant M\rho^2, \tag{1.7}$$

then

$$\sup_{B_{(\rho/4)}} u \leqslant \eta_n\left(\frac{\rho}{4}\right)^2\left(1+L(\frac{\rho}{2})M\right), \tag{1.8}$$

where $L(\rho)=L(\rho,u)$ is defined in (1.2).

Proof. Let $|y|=\rho/4$. Then, by Lemmas 1.2-1.3, by (1.7), and since u is subharmonic, we have

$$\begin{aligned} b_n\left(\frac{\rho}{4}\right)^n u(y) &\leqslant \int_{B_{(\rho/4)}(y)} udx \leqslant \int_{B_{(\rho/2)}(0)}|u|dx \\ &= \int_{B_{(\rho/2)}(0)} udx + 2\int_{B_{(\rho/2)}(0)} u^-dx \\ &\leqslant C_1b_n\left(\frac{\rho}{2}\right)^{n+2} + 2b_n\left(\frac{\rho}{2}\right)^n L(\frac{\rho}{2}) \sup_{B_{(\rho/2)}(0)}|u| \\ &\leqslant C_1b_n\left(\frac{\rho}{2}\right)^{n+2} + 2b_n\left(\frac{\rho}{2}\right)^n L(\frac{\rho}{2})\left(2^{n+1}\sup_{B_\rho(0)} u + C_1\left(\frac{\rho}{2}\right)^2\right) \\ &\leqslant 3C_1b_n\left(\frac{\rho}{2}\right)^{n+2} + \ 2^{n+2}b_n\frac{\rho^{n+2}}{2^n}L(\frac{\rho}{2})M \\ &\leqslant 2^{(n+4)}b_n\left(\frac{\rho}{2}\right)^{n+2}\left(1+L(\frac{\rho}{2})M\right). \end{aligned}$$

Dividing the above inequality by $b_n\rho^n4^{-n}$, we obtain (1.8). ■

Proof of Theorem 1.1. By (1.3) there is $0 < \rho_0 \leqslant e^{-2}$ such that

$$L(\frac{\rho}{2}, u) \leqslant \theta < \frac{1}{\eta_n} \qquad \text{for } \rho \leqslant \rho_0. \tag{1.9}$$

For $\rho_0 < |x| \leqslant 1$,

$$|u(x)| \leqslant \left(\sup_B |u|\right) \frac{|x|^2}{\rho_0^2}.$$

By (1.1), (1.7) holds for ρ_0 with $M = C_0 \log \frac{1}{\rho_0}$. Applying Lemma 1.4 and (1.8) inductively, we deduce

$$\sup_{B_{(\rho_0/4^k)}} u \leqslant \left(\frac{\rho_0}{4^k}\right)^2 \eta_n M \sum_{j=0}^{k} (\theta\eta_n)^j \leqslant \left(\frac{\rho_0}{4^k}\right)^2 \eta_n M \sum_{j=0}^{\infty} (\theta\eta_n)^j, \quad k = 0, 1, 2... \tag{1.10}$$

Let now, $\rho_0/4^{k+1} \leqslant \rho < \rho_0/4^k$ and set $C_2 = \frac{\eta_n M}{1-\eta_n M}$. Then, since u is subharmonic, we have by (1.10)

$$\sup_{B_\rho} u \leqslant \sup_{B_{(\rho_0/4^k)}} u \leqslant C_2 \left(\frac{\rho_0}{4^k}\right)^2 \leqslant C_2 4^2 \rho^2.$$

Finally, set $C_3 = \max\left(\frac{(\sup_B |u|)}{\rho_0^2},\ 4^2 C_2\right)$. Then for $0 < \rho < \frac{1}{2}$, Lemma 1.3 gives

$$\sup_{B_\rho} |u| \leqslant 2^{n+1} 4 C_3 \rho^2 + C_1 \rho^2.$$

Thus, we have obtained the estimate (1.4) with $C = 2^{n+3} C_3 + C_1$.

In order to show (1.5) we use a gradient estimate combined with a regularization (cf. [GT; Theorem 3.9]). This gives that for any $\rho < 1/2$,

$$\rho \sup_{B_\rho} |\nabla u| \leqslant C' \left(\sup_{B_{2\rho}} |u| + \rho^2 \|f\|_\infty \right).$$

Thus, (1.5) follows from (1.4) and the above estimate. ■

Remark. Theorem 1.1 can be slightly generalized for $f \in L^\infty(B)$ which it not necessarily nonnegative. Letting $v(x) = u(x) + \|f\|_\infty |x|^2/(2n)$, we'll have $\Delta v \geqslant 0$ and $\{x : v(x) < 0\} \subset \{x : u(x) < 0\}$. Now Theorem 1.1 applies.

2. Limitation of the Second-Order Derivatives

We consider now a solution u to the free boundary problem

$$\begin{cases} \Delta u(x) = f(x) & x \in B \cap \Omega, \\ u(x) = |\nabla u(x)| = 0 & x \in B \setminus \Omega. \end{cases} \tag{2.1}$$

We have $u \in W^{2,p}(B) \cap C^{1,\lambda}(B),\ 1 < p < \infty,\ 0 < \lambda < 1$ whenever $f \in L^\infty(B)$. We will prove that if u has a quadratic growth near all $x \in \partial\Omega \cap B$ and f is Hölder continuous, then $u \in W^{2,\infty}_{loc}(B)$, i.e., the second order-derivatives of u are bounded in B. Note that if in addition to (2.1), $u \geqslant 0$ in B, or equivalently u arises out of the obstacle problem, then the quadratic growth of u for all $x \in \partial\Omega \cap B$ is assured by the previous section. In that case, the limitation of the second-order derivatives was proved by many authors, including, Freshe [F], Gerhat [G], Brezis and Kinderlehrer [BK], and Caffarelli and Kinderlehrer [CK] (see also [KS]). For $u \geqslant 0$ our approach provides an estimation with a universal constant, which was obtained also in [CK]. In both [G] and [BK] the nonlinear case was also considered.

Theorem 2.1. *Let u be a solution of the free boundary problem (2.1) with $f \in C^\lambda(B),\ 0 < \lambda < 1$. If*

$$\sup_{B_\rho(x)} |u| \leqslant C\rho^2, \qquad \forall x \in \partial\Omega \cap B \quad (B_\rho(x) \subset B) \tag{2.2}$$

then for any $0 < r < 1$ there is a positive constant $M(r)$ depending on C in (2.2), the Hölder's norm of f and the dimension, such that

$$|D_{ij}u(x)| \leqslant M(r) \qquad \forall x \in \bar{B}_r. \tag{2.3}$$

Using Theorems 1.1 and 2.1 we have the following corollary.

Corollary 2.2. *Let u be a solution of (2.1) in B, where $f \geqslant 0$ and $f \in C^\lambda(B)$. Assume*

$$\overline{\lim_{\rho\to 0}} \sup_{x\in\partial\Omega\cap B} \frac{|B_\rho(x) \cap \{u < 0\}|}{|B_\rho(x)|} < \frac{1}{2^{2n+6}},$$

then (2.3) holds. Moreover, if $u \geqslant 0$, then C in (2.2) depends only on the Hölder's norm of f and the dimension, and hence the same conclusion holds for $M(r)$.

Proof of Theorem 2.1. Let $h \in \mathbb{R}^n$ and define

$$u_h(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{|h|^2}.$$

Then for $x \in \bar{B}_r \setminus \Omega$ and $|h| \leqslant 1 - r$, $|u_h(x)| \leqslant 2C$ by (2.2). Hence,

$$|D_{ij}u(x)| \leqslant 2C \quad x \in \bar{B}_r \setminus \Omega. \tag{2.4}$$

Let $d(x) = \operatorname{dist}(x, \Omega^c)$, $\|f\|_\lambda = \sup_{x\in B}|f(x)| + \sup_{x,y\in B}\frac{|f(x)-f(y)|}{|x-y|^\lambda}$ and $M = \sup_B |u|$. If $x \in \bar{B}_r \cap \Omega$ with $d(x) \geqslant 1-r$, then

$$\begin{aligned}(2.5)\qquad &\left(\frac{1-r}{2}\right)^2 \sup_{B_{(d(x)/2)}(x)} |D_{ij}u(x)| \leqslant \left(\frac{d(x)}{2}\right)^2 \sup_{B_{(d(x)/2)}(x)} |D_{ij}u(x)| \\ &\leqslant C\left(\sup_{B_{d(x)}(x)} |u| + \left(\frac{d(x)}{2}\right)^2 d(x)^\lambda \|f\|_\lambda\right) \leqslant C\left(M + \|f\|_\lambda\right),\end{aligned}$$

where the second inequality is standard estimate for the second order derivatives (See e.g. [GT; Theorem 4.6]).

For $x \in \bar{B}_r \cap \Omega$ with $d(x) < 1-r$, we set $\rho = d(x)/3$ and $d(x) = |x - x_0|$ for some $x_0 \in \partial\Omega$. Using (2.2) and applying the same estimate [GT; Theorem 4.6], we obtain

$$\begin{aligned}(2.6)\qquad &\rho^2 \sup_{B_\rho(x)} |D_{ij}u(x)| \leqslant C\left(\sup_{B_{2\rho}(x)} |u| + \rho^2(2\rho)^\lambda \|f\|_\lambda\right) \\ &\leqslant C\left(\sup_{B_{5\rho}(x_0)} |u| + \rho^2(2\rho)^\lambda\|f\|_\lambda\right) \leqslant C\left((5\rho)^2 + \rho^2(2\rho)^\lambda\|f\|_\lambda\right).\end{aligned}$$

Combining (2.4), (2.5) and (2.6) we obtain (2.3), and the proof is completed. ■

Remark. The results of sections 1 and 2 can be generalized to second-order uniformly elliptic operators, with suitable regularity of the coefficients. Of course the constant η_n depends on the modulus of the ellipticity of the operator. The explicit computations in Lemma 1.2, by means of the Green's function, can be replaced by known estimates of the Green's function for the corresponding operator (cf. [LSW], [S]).

3. Regularity of The Free Boundary

In the celebrated papers [Ca1] and [Ca2], Caffarelli proved that a nonnegative solution to the free boundary problem (0.4) has a free boundary which is C^1 near $x_0 \in \partial\Omega$, provided that the complement of Ω is not too "small" near x_0 (see Definition 3.8). In this section, we follow Caffarelli's method in [Ca2] and extend his result to a larger class of solutions. The reader can find a more detailed description of Caffarelli's works [Ca1] and [Ca2] in [Fr; Chapter 2].

In the previous sections we gave conditions which ensure the boundedness of the second-order derivatives of solutions to (0.3). Therefore, in this section we will assume that any solution has this latter property, and we will study the regularity of free boundaries in the following class.

Definition 3.1. A function u is said to be of class $P_r = P_r(M, z)$ if,

(1) $\sup_{B_r} |D_{ij}u| \leqslant M, \quad \forall i,j = 1...n, \qquad (B_r = B_r(z))$
(2) $\Delta u = 1$ in $\Omega(u) := B_r \setminus \Lambda(u)$,
(3) $z \in \Gamma(u) := \partial\Omega \cap B_r$,

where $\Lambda(u) = \{x \in B_r : u(x) = |\nabla u(x)| = 0\}$.

We also define

$$\Omega^+(u) = \{x \in B_r : u(x) > 0\}, \qquad \Omega^-(u) = \{x \in B_r : u(x) < 0\}.$$

The classes

$$P_r^+ = \{u \in P_r : \Omega^-(u) = \emptyset\} \text{ and } P_r^* = \{u \in P_r^+ : u \text{ is convex}\}$$

coincide with Caffarelli's definitions [Ca2]. The scaled function u_s is defined as

$$u_s(x) = \frac{u(sx)}{s^2}.$$

It is obvious that $u_s \in P_{r/s}$ if $u \in P_r$ and

$$\Lambda(u_s) = \Lambda_s(u), \qquad \Omega(u_s) = \Omega_s(u), \qquad \Gamma(u_s) = \Gamma_s(u),$$

where $S_s = \{x : \ sx \in S\}$ for any set S.

The maximal radius of a set S, $MR(S)$, is the radius of the largest ball contained in S. We measure the "thinness" of S near a point x by the quantity

$$l(r, x, S) := MR\left(S \cap B_r(x)\right).$$

Obviously, $l(r, x, S)$ is an upper semicontinuous function of x, and if S has zero upper Lebesgue density at x, then $\overline{\lim}_{r\to 0}\, l(r, x, S)/r = 0$.

Let $u \in P_1(M, 0)$, $\ 0 < \varepsilon \leqslant 1$, and set

$$l(r) = \sup_{y \in \bar{B}_\varepsilon \cap \Gamma(u)} l(r, y, \Omega^-(u)). \tag{3.1}$$

A real valued function $\omega(r)$ is a modulus of continuity, if it is a nonnegative, right continuous and nondecreasing function of $0 \leqslant r < 1$ with $\omega(0^+) = 0$.

Definition 3.2. We say that the function $u \in P_1(M, 0, \omega, \varepsilon)$ if $u \in P_1(M, 0)$ and if

$$\frac{l(r))}{r} \leqslant \omega(r), \qquad 0 \leqslant r < 1 - \varepsilon,$$

where $\omega(r)$ is a modulus of continuity and $l(r)$ is given by (3.1).

The geometrical condition $\overline{\lim}_{r\to 0}\,(l(r)/r) = 0$ can be replaced by a growth condition u^-. Indeed, for $y \in \Gamma(u)$ set

$$m(r, y) = -\inf_{B_r(y)} u,$$

and

$$m(r) = \sup_{y \in \bar{B}_\varepsilon \cap \Gamma(u)} m(r, y).$$

Then we have the following lemma.

Lemma 3.3. *Let $u \in P_1(M,0)$, then $u \in P_1(M,0,\omega,\varepsilon)$ for some modulus of continuity ω, if and only if $m(r) = o(r^2)$ as $r \to 0$.*

Remark. In remaining sections convergence of subsequences will be used frequently. Therefore, we do not distinguish between sequences and subsequences.

Proof of Lemma 3.3. We may assume $m(r)$ is not identically zero. For any positive r we let $x_r \in \bar{B}_r(y)$ be the point where $-u(x_r) = m(r,y)$, and $z \in \bar{B}_r(y)$ be a point nearest to x_r such that $u(z) = 0$. Hence,

$$
\begin{aligned}
m(r,y) &= -u(x_r) = u(z) - u(x_r) \\
&= \int_0^1 \nabla u(tz + (1-t)x_r) \cdot (z - x_r)dt \leqslant \sqrt{n} M l(r) r,
\end{aligned} \tag{3.2}
$$

for all $y \in \Gamma(u) \cap \bar{B}_\varepsilon$. Dividing by r^2 both sides of (3.2), we get the necessary condition.

For the sufficiency, assume $u \notin P_1(M,0,\omega,\varepsilon)$ for any modulus of continuity ω, then $\overline{\lim}_{r\to 0}\,(l(r)/r) = c > 0$. Hence, there exists a sequence $\{r_j\}, r_j \searrow 0$ such that

$$
l(r_j) \geqslant c r_j \qquad c > 0. \tag{3.3}
$$

Since $l(r,y,\Omega^-(u))$ is a upper semicontinuous in y, there is for all j, $y_j \in \bar{B}_\varepsilon \cap \Gamma(u)$ with the property

$$
l(r_j) = l(r_j, y_j, \Omega^-(u)).
$$

It is also clear that for all j there is a ball $B_{l(r_j)}(x_j)$ such that

$$
B_{l(r_j)}(x_j) \subset \Omega^-(u) \cap B_{r_j}(y_j). \tag{3.4}
$$

¿From inequality (3.3) we have

$$
c\, r_j \leqslant \; |x_j - y_j| \; \leqslant \; (1-c) r_j. \tag{3.5}
$$

We now set $u_j(x) = r_j^{-2} u(r_j x + y_j)$ and $B_j = B_{(l(r_j)/r_j)}((x_j - y_j)/r_j)$. By compactness, (3.3) and (3.5), the sequence $\{B_j\}$ converges to a ball $B_{l_0} := B_{l_0}(x_0 - y_0)$, with a positive radius l_0 and the center $x_0 - y_0 \neq 0$, $y_0 \in \Gamma(u)$. Compactness also gives that $u_j \to u_0$ uniformly on compact sets. Furthermore, by (3.4), $B_j \subset \Omega^-(u_j) \cap B_1$ and hence $u_0 \leqslant 0$ in $B_{l_0} \cap B_1$. Since $\Delta u_j = 1$ in B_j, $\Delta u_0 = 1$ in B_{l_0}. Therefore u_0 is not identically zero in $B_{l_0} \cap B_1$, and hence

$$
\frac{m(r_j)}{r_j^2} \geqslant \frac{m(r_j,y_j)}{r_j^2} = -\frac{1}{r_j^2} \inf_{B_{r_j}(y_j)} u = -\inf_{B_1} u_j \not\to 0, \qquad \text{as } r_j \searrow 0.
$$

This completes the proof of the lemma. ■

Lemma 3.4. *Suppose $u_j \in P_{r_j}(M,0)$, $u_j \to u_0$ and $\nabla u_j \to \nabla u_0$ uniformly on compact sets of B_{r_0}, as $r_j \to r_0$. Then $u_0 \in P_{r_0}(M,0)$. In particular, if $u_j = u_{\rho_j}$ for a sequence $\{\rho_j\}$ with $\rho_j \searrow 0$, and $u \in P_1(M,0,\omega,\varepsilon)$ for some modulus of continuity ω, then $u_j \to u_0 \in P^+_\infty(M,0)$.*

Proof. The only difficulty in proving the first statement is to show that $0 \in \Gamma(u_0)$. Since $0 \in \Lambda(u_0)$, it suffices to show that $0 \in \overline{\Omega(u_0)}$. In order to do that, we first note that any $u \in P_r(M,0)$ is subharmonic. Hence, for all $y \in \Gamma(u)$ and small (positive) ε, we have $\Omega^+(u) \cap B_\varepsilon(y) \neq \emptyset$. Therefore, $0 \in \Gamma(u_j)$ implies $0 \in \overline{\Omega^+(u_j)}$ and by using Lemma 1 of [Ca2] one obtains that

$$\overline{\lim_{j\to\infty}}\ \overline{\Omega^+(u_j)} \subset \overline{\Omega^+(u_0)} \subset \overline{\Omega(u_0)},$$

(see [Ca2; Corollary 1]) which implies that $0 \in \overline{\Omega(u_0)}$.

To show the second statement , let $R > 0$, then we have

$$-\inf_{B_R} u_j = -\frac{1}{\rho_j^2} \inf_{B_{(R\rho_j)}} u = -R^2 \frac{1}{(R\rho_j)^2} \inf_{B_{(R\rho_j)}} u \leqslant R^2\ \frac{m(R\rho_j)}{(R\rho_j)^2}.$$

Since $u_j \to u_0$ uniformly on compact sets, Lemma 3.3 and the latter inequality imply that u_0 is nonnegative. ■

Our next aim is to modify Caffarelli's estimate of pure second-order derivatives. This estimate has a key role both in [Ca1] and [Ca2].

Lemma 3.5. *There is a modulus of continuity ω_1, depending on ω and ε, such that for any $u \in P_1(M,0,\omega,\varepsilon)$, holds*

$$D_{ii}u(x) \geqslant -\omega_1(|x-y|), \qquad y \in \Gamma(u) \cap \bar{B}_{\varepsilon/4}.$$

In particular,

$$\varliminf_{x\to y\in\Gamma(u)} D_{ii}u(x) \geqslant 0, \qquad |y| \leqslant \varepsilon/4.$$

Lemma 3.6. *Let $u \in P_1(M,0,\omega,\varepsilon)$, $x \in \Omega(u) \cap \bar{B}_{\varepsilon/2}$ and dist$(x,\Gamma(u)) \leqslant \rho$. Then, for any given $0 < \delta < 1$, there is $z_\delta \in B_{\rho(1-\delta/2)}(x)$ such that*

$$D_{ii}u(z_\delta) \geqslant -C_1\sqrt{\delta} - C_2\omega(2\rho).$$

Proof. Let $x \in \Omega(u)$, $\mathbf{e}_i$ be any direction, and $0 < s < 1 - |x|$. Then $u(x+s\mathbf{e}_i) \geqslant -m(\rho+s,y)$, where y is the nearest point to x on $\Gamma(u)$. Since $|x| \leqslant \varepsilon/2$, $|y| \leqslant \varepsilon$. Therefore, by (3.2) and Definition (3.2), $m(r,y) \leqslant \sqrt{n}Ml(r)r \leqslant \sqrt{n}M\omega(r)r^2$. So,

$$\begin{aligned}-\sqrt{n}M(\rho+s)^2\omega(\rho+s) &\leqslant -m(\rho+s,y)\\ &\leqslant u(x+s\mathbf{e}_i) = u(x) + sD_iu(x) + \int_0^s\int_0^\tau D_{ii}u(x+t\mathbf{e}_i)dtd\tau.\end{aligned}$$

Since $u \in P_1(M,0)$, $|D_i u(x)| \leqslant \sqrt{n} M\rho$ and $|u(x)| \leqslant nM\rho^2$. Thus we have obtained

$$\int_0^s \int_0^\tau D_{ii} u(x + t\mathbf{e}_i) dt d\tau \geqslant -C_1\rho(\rho+s) - C_2(\rho+s)^2\omega(\rho+s). \tag{3.6}$$

Following [Ca1, Lemma 1], we let $x_0 = (1-\delta)(y-x)+x$, and we let $I = [x_0, \hat{x}]$ be the segment started at x_0 with length $\sqrt{\delta}\rho/2$ and in the direction $\mathbf{e}_i$, or $-\mathbf{e}_i$ so that it is contained in $B_{\rho(1-\delta/2)}(x)$. Applying (3.6) above at the point x_0 and the segment I, we have

$$\frac{\delta\rho^2}{8} \sup_I D_{ii} u \geqslant -C_1 \left(\frac{3}{2}\right) \rho^2 \delta^{3/2} - C_2 \left(\frac{3}{2}\right)^2 \rho^2 \delta \omega(2\rho)$$

which gives the conclusion of the lemma. ■

Proof of Lemma 3.5. Let $y \in \Gamma(u) \cap \bar{B}_{\varepsilon/4}$ and set

$$M_k(y) = -\inf_{B_{2^{-k}}(y)} D_{ii} u, \qquad M_k = \sup\{M_k(y) : y \in \Gamma(u) \cap \bar{B}_{\varepsilon/4}\} \qquad k = 1, 2, \ldots$$

Then $M_{k+1} \leqslant M_k$. We conduct the proof of the lemma by showing that

$$\lim_k M_k = 0. \tag{3.7}$$

If (3.7) fails, then for some $\varepsilon_0 > 0$

$$M_k \geqslant \varepsilon_0 \qquad \text{for all } k. \tag{3.8}$$

Now take k_0 so that $2^{-(k_0+1)} \leqslant \varepsilon/4$, and let $k \geqslant k_0$. Applying Lemma 3.6 with $\rho = 2^{-(k+1)}$, and Harnack inequality, we get

$$D_{ii} u(x) + M_k \geqslant C\delta^{n-1} \left(M_k - C_1\sqrt{\delta} - C_2\omega(2^{-k}) \right), \qquad x \in B_{2^{-(k+1)}}(y).$$

Hence,

$$-M_{k+1}(y) + M_k \geqslant C\delta^{n-1} \left(M_k - C_1\sqrt{\delta} - C_2\omega(2^{-k}) \right), \qquad y \in \Gamma(u) \cap \bar{B}_{\varepsilon/4}.$$

Taking the supremum, we obtain

$$-M_{k+1} + M_k \geqslant C\delta^{n-1} \left(M_k - C_1\sqrt{\delta} - C_2\omega(2^{-k}) \right).$$

Let k be sufficiently large so that $C_2\omega(2^{-k}) \leqslant \varepsilon_0/4$, we get by the above inequality,

$$-M_{k+1} + M_k \geqslant C\delta^{n-1} \left(\frac{3}{4} M_k - C_1\sqrt{\delta} \right), \qquad k \geqslant k_1.$$

For each $k \geqslant k_1$, we can now choose an appropriate δ such that we obtain

$$-M_{k+1} \geqslant -M_k + CM_k^{(2n-1)}, \qquad k \geqslant k_1. \tag{3.9}$$

Inequality (3.9) implies $M_k \leqslant C(1/k)^\beta$, $\beta = (2(n-1))^{-1}$ which contradicts (3.8). ■

Corollary 3.7. *Let* $\{u_j\} \subset P_1(M, 0, \omega, \varepsilon)$ *and* $\{r_j\}$ *be a sequence with* $r_j \searrow 0$. *Set* $(u_j)_{r_j}(x) = r_j^{-2} u_j(r_j x + y)$ *for* $y \in \Gamma(u) \cap \bar{B}_{\varepsilon/4}$. *Then* $(u_j)_{r_j} \to u_y$ *and* $\nabla (u_j)_{r_j} \to \nabla u_y$ *uniformly on compact sets. Moreover* $u_y \in P^*_\infty(M, 0)$.

Proof. By compactness and Lemma 3.4, it is clear that $u_y \in P^+_\infty(M, 0)$. To show the convexity, let for any $R > 0$,

$$v_j(x) = r_j^{-2} u_j(r_j x + y) + \omega_1(r_j R)\frac{|x - y|^2}{2n}.$$

Then $D_{ii} v_j(x) \geqslant 0$ in $B_R(y)$ and $v_j \to u_y$ uniformly in $B_R(y)$. Hence u_y is convex in $B_R(y)$. ■

The main result of this section is extension of Caffarelli's regularity theorem [Ca2; Theorem 1] to the class $P_1(M, 0, \omega, \varepsilon)$ (instead of $P^+(M, 0)$). Before stating the result, let us recall the definition of the minimal diameter.

Definition 3.8. The minimum diameter of a set S, denoted $\mathrm{MD}(S)$, is the infimum of distances between pairs of parallel planes such that S is contained in the stripe determined by the planes. We also define the "thickness" of the set S with respect to a point x by

$$\delta(r, x, S) = \frac{\mathrm{MD}(S \cap B_r(x))}{r}.$$

Theorem 3.9. *For any* $u \in P_1(M, 0, \omega, \varepsilon_0)$, *where* ω *is a modulus of continuity and* $0 < \varepsilon_0 < 1$. *There exists a positive nondecreasing function* $\sigma(r)$ *with* $\sigma(0^+) = 0$, *such that if* $\delta(r, 0, \Lambda(u)) > \sigma(r)$ *for some positive* $r \leqslant \varepsilon/4$, *then* $\Gamma(u) \cap B_{\tilde{r}}$ *is given by the graph of a* C^1*-function for some positive* $\tilde{r}$.

Proof. We replace $\Omega(u)$ in Section 6-7 in [Ca2] by $\Omega^+(u)$, and we use Lemmas 3.4, 3.5 and Corollary 3.7 in order to verify all the steps in those sections. To convince the reader, we show that u satisfies the following "flatness" condition [Ca2; Lemma 6]:

Given $\varepsilon > 0, \delta > 0$, there exists $\rho_0 = \rho_0(\varepsilon, \delta)$ such that: if $u \in P_1(M, 0, \omega, \varepsilon_0)$ and $\delta(\rho, 0, \Lambda(u)) \geqslant \varepsilon$ for some $0 < \rho < \rho_0$, then in an appropriate system of coordinates

$$\begin{cases} \Lambda(u) \supset B_{\rho\lambda/2} \cap \{x;\ -x_n \geqslant 2\rho\lambda\delta\} \\ \Omega^+(u) \supset B_{\rho\lambda/2} \cap \{x;\quad x_n \geqslant 2\rho\lambda\delta\} \end{cases}. \tag{3.10}$$

Proof of the flatness condition. If the statement fails, then there are sequences $\{u_j\} \subset P_1(M, 0, \omega, \varepsilon_0)$ and $\{\rho_j\}$ with $\rho_j \searrow 0$ such that: $\delta(\rho_j, 0, \Lambda(u_j)) \geqslant \varepsilon$ and at least one of (3.10) doesn't hold in any system of coordinates. Let $(u_j)_{\rho_j}(x) = \rho_j^{-2} u_j(\rho_j x)$. Then $(u_j)_{\rho_j} \to u_0$ and $\nabla (u_j)_{\rho_j} \to \nabla u_0$ uniformly on compact sets, and by Corollary 3.7, u_0 is convex ($u_0 \in P^*_\infty(M, 0)$). Since

$$\Lambda(u_0) \supset \overline{\lim_j}\, \Lambda\left((u_j)_{\rho_j}\right) \ \text{ and } \ \overline{\Omega(u_0)} \supset \overline{\lim_j}\, \overline{\Omega^+\left((u_j)_{\rho_j}\right)},$$

and since

$$\delta\left(1,0,\Lambda(u_0)\right) \geqslant \overline{\lim_j}\, \delta\left(1,0,\Lambda\left((u_j)_{\rho_j}\right)\right) \geqslant \varepsilon,$$

we get a contradiction to Lemma 5 in [Ca2] which asserts the following:

Given $\varepsilon > 0, \delta > 0$, there exists $\lambda = \lambda(\varepsilon,\delta)$ such that: if $u \in P_1^*(M,0)$ and $\delta(1,0,\Lambda(u)) \geqslant \varepsilon$, then in an appropriate system of coordinates

$$\begin{cases} \Lambda(u) \supset B_\lambda \cap \{x; -x_n \geqslant |x|\cos(\frac{\pi}{2}-\delta)\} \\ \Omega(u) \supset B_\lambda \cap \{x; \quad x_n \geqslant |x|\cos(\frac{\pi}{2}-\delta)\}. \end{cases}$$

$\lambda(\varepsilon,\delta)$ is the λ of the relation (3.10). ∎

The general case. For $f \in C^\lambda(B)$, $f(x) \geqslant c > 0$ we consider the class P_r in Definition 3.1 with (2) replaced by $\Delta u = f$ in $\Omega(u)$.

The conclusion of Theorem 3.9 remains true for the above class, but Lemma 3.5 and Lemma 7 in [Ca2] should be modified (see Lemma A1 in [Ca1], or Theorem 3.8 in [Fr] for the modification).

4. The global equation

In this section we will discuss conditions so that a solution u, to equation (0.3) ($0 \in \partial\Omega$), belongs to $P_1(M,0,\omega,\varepsilon_0)$ for some modulus of continuity ω. We will show that a sufficient condition for $u \in P_1(M,0,\omega,\varepsilon_0)$ is that $\Lambda(u)$ is not too small and all solutions to the global equation

$$\begin{cases} \Delta u(x) = 1 & x \in \Omega, \\ u(x) = |\nabla u(x)| = 0 & x \in \mathbb{R}^n \setminus \Omega =: \Omega^c, \\ u(x) = o(|x|^3) & |x| \to \infty. \end{cases} \tag{4.1}$$

are nonnegative. That is indeed the case in the two dimensional space (see Theorem 4.2 below). We recall that if $u \in P_1(M,0)$ and u_0 is the blow-up of u, then u_0 satisfies equation (4.1).

Definition 4.1. Let u satisfies (4.1). If Ω^c has a non-empty interior, then Ω is called a *null quadrature domain.*

That terminology is motivated by the below identity, which holds for any null quadrature domain Ω;

$$\int_\Omega h dx = 0$$

for all harmonic and integrable h in Ω (see e.g. [KM; Sec. 2], [Sa1]). Null quadrature domains appear naturally in several problems of the Mathematical-Physics [BF], [EE], [EV], [Ho], and [Le]. For example, in the case Ω^c is bounded, then Ω is a null quadrature domain if and only if the the Newtonian potential of χ_Ω is equal to a quadratic polynomial inside Ω^c [FS]. The latter means that in appropriate system of coordinates, the shell $t\Omega^c \setminus \Omega^c, (t > 1)$ induces no gravity force in the cavity and by Dive's theorem [Di], Ω^c is an ellipsoid (see [KM; Sec. 4] for details). In the plane Sakai gave a complete description of null quadrature domains [Sa1], [Sa3].

Theorem 4.2 (Sakai). *The function u is a solution to (4.1) with $\bar{\Omega} \neq \mathbb{R}^2$ if and only if Ω is one of the following:*

1. *The exterior of an ellipse.*
2. *The non-convex domain bounded by a parabola.*
3. *A half-plane.*
4. *The complement of a stripe.*

Moreover, any solution to (4.1) with $\bar{\Omega} \neq \mathbb{R}^2$ is nonnegative.

Sakai's result supply the motivation to the following conjectures:

Conjecture 4.3 ([KM], [Sh]). *The function u is a solution to (4.1) with $\bar{\Omega} \neq \mathbb{R}^n$ if and only if Ω is one of the following:*

1. *The exterior of an ellipsoid.*
2. *The non-convex domain bounded by an elliptical paraboloid.*
3. *A cylinder over* 1. *or* 2.
4. *A half-space.*
5. *The complement of a stripe.*

Conjecture 4.4 ([Sh]). *Let u satisfies (4.1) with $\bar{\Omega} \neq \mathbb{R}^n$, then $u \geqslant 0$.*

Conjecture 4.3 was verified by Friedman and Sakai in the case Ω^c is bounded [FS], and by Karp and Margulis for Ω^c which are contained in a cylinder of co-dimension 2, [KM]. Shahgholian showed that Conjecture 4.4 holds true for all Ω which appear in the list of Conjecture 4.3, [Sh]. A verification of Conjecture 4.4 which is independent of the classification of null quadrature domains is not known for the authors (even in the two dimensional case). If the growth condition at infinity in (4.1) is drooped, then there do exist nonnegative solutions to

$$\begin{cases} \Delta u(x) = 1 & x \in \Omega, \\ u(x) = |\nabla u(x)| = 0 & x \in \Omega^c, \end{cases}$$

with $\bar{\Omega} \neq \mathbb{R}^n$ (see [Ri], [Sha1]).

Let $\delta(r, y, \Lambda(u))$ denote the normalized minimal diameter at the point $y \in \Gamma(u)$ and set

$$\delta(r) = \inf_{y \in \Gamma(u) \cap \bar{B}_{\varepsilon_0}} \delta(r, y, \Lambda(u)).$$

Let $u \in P_1(M, 0)$, we claim

$$u \in P_1(M, 0, \omega, \varepsilon_0) \qquad \text{for some modulus of contiuity } \omega, \tag{4.2}$$

provided that

$$\varliminf_{r \to 0} \delta(r) > 0, \tag{4.3}$$

and in $\mathbb{R}^n$, $n \geqslant 3$, we also assume Conjecture 4.4 holds true.

Proof. If (4.2) fails, then by Lemma 3.3, there is a sequence $\{r_j\}_{j=1}^{\infty}$, $r_j \searrow 0$ such that $m(r_j)/r_j^2 \geqslant c > 0$, where $m(r)$ is defined below Definition 3.2. Since u is continuous, for each j there is $y_j \in \Gamma(u) \cap \bar{B}_{\varepsilon_0}$ such that $m(r_j) = m(r_j, y_j) := \inf_{B_{r_j}(y_j)} u$. Hence, we have

$$\frac{m(r_j, y_j)}{r_j^2} \geqslant c > 0. \tag{4.4}$$

Now, let $u_j(x) = r_j^{-2} u(r_j x + y_j)$, then by compactness, $u_j \to u_0$ and $\nabla u_j \to \nabla u_0$ on compact sets of $\mathbb{R}^n$. Obviously, u_0 satisfies (4.1). Further, let $\Lambda_\infty = \overline{\lim}_j \Lambda(u_j)$, e.i., Λ_∞ is the limit points of all sequences $\{x_j\}$, $x_j \in \Lambda(u_j)$. Then, $\Lambda_\infty \subset \Lambda(u_0)$, and for any $\varepsilon > 0$ there is J_0 such that

$$\Lambda(u_j) \cap \bar{B}_1 \subset (\Lambda_\infty)_\varepsilon \cap \bar{B}_1 \qquad \text{for } j \geqslant J_0,$$

where $(\Lambda_\infty)_\varepsilon = \{x : \operatorname{dist}(x, \Lambda_\infty) < \varepsilon\}$. Therefore,

$$\begin{aligned} &\mathrm{MD}\,(\Lambda(u_0) \cap B_1) = \delta\,(1, 0, \Lambda(u_0)) \geqslant \delta\,(1, 0, \Lambda_\infty) \geqslant \overline{\lim_j}\, \delta\,(1, 0, \Lambda(u_j)) \\ &= \overline{\lim_j}\, \frac{MD\left(\Lambda(u) \cap B_{r_j}(y_j)\right)}{r_j} = \overline{\lim_j}\, \delta(r_j, y_j, \Lambda(u)) \geqslant \underline{\lim_j}\, \delta(r_j). \end{aligned} \tag{4.5}$$

So, by (4.3) and (4.5), we have $\mathrm{MD}\,(\Lambda(u_0) \cap B_1) > 0$. Further, we observe that for any v satisfying (4.1) with $\operatorname{int}(\Lambda(v)) = \emptyset$, yields $\Delta v = 1$ a.e., and by Liouville's theorem v is a quadratic polynomial. Hence $\Lambda(v)$ is a linear space of dimension $\leqslant n - 1$. Since u_0 satisfies (4.1), it follows from the above observation that $\mathrm{MD}\,(\Lambda(u_0) \cap B_1) > 0$ implies $\operatorname{int}(\Lambda(u_0)) \neq \emptyset$. Therefore $u_0 \geqslant 0$ (for $n = 2$ by Theorem 4.2, and for $n \geqslant 3$ provided that Conjecture 4.4 is true). But by (4.4),

$$-\inf_{B_1} u_j = -\inf_{B_{r_j}(y_j)} u = \frac{m(r_j, y_j)}{r_j^2} \geqslant c > 0$$

which contradicts the nonnegativity of u_0. ■

Remark. It would be interesting to prove (4.2) with (4.3) replaced by a condition which depends only on $0 \in \Gamma(u)$. For example, $|\Lambda(u) \cap B_1| > 0$ is a very naturally condition, since $\Gamma(u) \cap B_1$ has Lebsegue measure zero by Lemma 1.2 in [CR] (or in a greater generality Lemma 2.11 in [KM]), and hence we may assume $\Lambda(u) = \overline{\operatorname{int}\,(\Lambda(u))}$. But the blow-up techniques seems to be inefficient for this purpose. Indeed, by the example of Schaeffer [Sc2], there is a solution u to (0.3) with $f \in C^\infty(B)$, $f(x) \geqslant c > 0$ and $|\Lambda(u) \cap B_1| > 0$. Moreover, $\Lambda(u) \cap B_1$ has infinitely many components. For that particular solution, we can choose a sequence $\{r_j\}_{j=1}^\infty$, $r_j \searrow 0$, such that $\Lambda_\infty = \{0\}$, here $\Lambda_\infty = \overline{\lim}_j \Lambda(u_j)$ and $u_j(x) = r_j^{-2} u(r_j x)$. Thus, even in the two dimensional space, $\overline{\lim}_j \Lambda(u_j)$ can be too small in order to conclude that the blow-up u_0 is nonnegative.

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