TWO RELATED MINIMUM PROBLEMS WITH FREE BOUNDARIES GOVERNED BY THE p-LAPLACE OPERATOR

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Abstract. In this note we will discuss some aspects of two related minimum problems giving rise to free boundaries. The unifying themes are the Bernoulli boundary condition, and the p-Laplacian as the governing equation.

1. Introduction

We consider two free boundary problems and their corresponding minimization problems. First we need to start with some definitions and notations, frequently used in this note.

For $1 < p < \infty$, the p-Laplacian $\Delta_p u$ is defined as

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

For the properties of Δ_p we refer to [5] and the references therein. We say u is a weak solution of the equation

$$\Delta_p u = f,$$

if $u \in W^{1,p}(\mathbb{R}^N)$ and

$$-\int |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = \int f\varphi, \tag{1.1}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, where

$$W^{1,p}(\mathbb{R}^N) := \{\varphi :\ \varphi,\ |\nabla\varphi| \in L^p(\mathbb{R}^N)\}.$$

Let now $\mathcal{H}^{N-1}\lfloor\partial\Omega$ denote the $(N-1)$-dimensional Hausdorff measure restricted to $\partial\Omega$. our aim is to study the following problems.

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Problem 1. For a given measure $\mu \geq 0$, with compact support in $\mathbb{R}^N$, and a continuous function $g \geq c > 0$, find u such that

$$
\begin{cases} \Delta_p u & = -\mu + g\mathcal{H}^{N-1}\lfloor\partial\Omega, \\ \text{supp } \mu & \subset \Omega, \end{cases} \tag{1.2}
$$

where $\Omega := \{u > 0\}$, and the p-Laplacian is to hold in the sense of distributions, i.e.,

$$
-\int |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = -\int \mu\varphi + \int_{\partial\Omega} \varphi g d\mathcal{H}^{N-1}, \tag{1.3}
$$

for all $\varphi \in C_o^\infty(\mathbb{R}^N)$.

Problem 2. For a given bounded domain $D \subset \mathbb{R}^N$, continuous functions $g \geq c > 0$ and $h > 0$ find u such that

$$
\begin{cases} \Delta_p u = g\mathcal{H}^{N-1}\lfloor\partial\Omega & \text{in } \mathbb{R}^N \setminus D \\ \quad D \subset \Omega & \\ \quad u = h & \text{on } D, \end{cases} \tag{1.4}
$$

where $\Omega := \{u > 0\}$.

For C^1 boundaries (boundaries with a unique normal vector) the following implication is immediate:

$$
\Delta_p u = g\mathcal{H}^{N-1}\lfloor\partial\Omega \quad \text{(locally)} \qquad \Longrightarrow \qquad \lim_{x\to\partial\Omega} |\nabla u(x)| = g(x), \quad x \in \Omega.
$$

The latter, in turn, implies

$$
\frac{\partial u}{\partial \nu} = -g \qquad \text{on } \partial\Omega,
$$

where ν denotes the unit outward normal vector.

These problems for $p = 2$ have been intensively studied, in the past 20-years. Existence, regularity, and certain geometric configurations (inherited from the data), are among those problems that have been in focus. We refer the reader to the paper of Alt and Caffarelli for existence and regularity of solutions for Problem 2, and to Gustafsson and Shahgholian for Problem 1; see also [8], [9].

Using the method of sub- and supersolutions, A. Beurling [2] proved the existence of solutions in the plane, for the case $p = 2$ in Problem 2. This method was adapted later by A. Acker to prove the existence in $\mathbb{R}^N$ for Problem 2, and with the restriction $p = 2$ and convex D. Recently the second author and A. Henrot [6] extended Beurlings method to $\mathbb{R}^N$, $1 < p < \infty$, and convex D. The problem is still open for general p and general domains D in $\mathbb{R}^N$, $N \geq 2$.

Problems 1, and 2 are naturally connected with the following minimization problems.

Problem 1′. Find $u \in \mathbb{J}$ such that

$$
J(u) \leq J(v) \qquad \forall\, v \in \mathbb{J},
$$

where

$$J(u) = J_{g,\mu} := \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p} - \mu u + \left(\frac{p-1}{p}\right) g^p \chi_{\{u>0\}} \right) dx$$

and

$$\mathbb{J} := \{v \geq 0;\ v \in W^{1,p}(\mathbb{R}^N)\}.$$

Problem 2′. Find $u \in \mathbb{I}$ such that

$$I(u) \leq I(v) \qquad \forall\ v \in \mathbb{I},$$

where

$$I(u) = I_g(u) := \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p} + \left(\frac{p-1}{p}\right) g^p \chi_{\{u>0\}} \right) dx$$

and

$$\mathbb{I} = \mathbb{I}_{D,h} := \{v \geq 0;\ v = h \text{ on } D,\ v \in W^{1,p}(\mathbb{R}^N)\},$$

with $h \in W^{1,p}(\mathbb{R}^N)$ and $D \subset \mathbb{R}^N$ given.

Definition. (Local minimizer) A function $u \in \mathbb{J}$ (respectively $\mathbb{I}$) is a local minimum of J (respectively I) if, for some $\epsilon > 0$, $J(v) \geq J(u)$ (respectively $I(v) \geq I(u)$) for every $v \in \mathbb{J}$ (respectively $\mathbb{I}$) with

$$\int \left(|\nabla(v-u)|^p + |\chi_{\{v>0\}} - \chi_{\{u>0\}}| \right) dx < \epsilon.$$

For regular boundaries one can show that any (local) minimizer of Problems 1′, and 2′ are solutions of the corresponding free boundary problems. Indeed, (see [3]) if $F(x,z,w) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, is a smooth function with certain structure conditions that guarantee the existence of minimizer(s) to the functional

$$\int_{\mathbb{R}^N} F(x, u, \nabla u)\ dx,$$

over a certain function space, then any (local) minimizer u of this functional satisfies

$$\begin{cases} \nabla \cdot F_w - F_z = 0 & \text{in } \{u > 0\} \\ \nabla u \cdot F_w - F = 0 & \text{on } \partial\{u > 0\}, \end{cases} \tag{1.5}$$

where $F_w = \nabla_w F$ and $F_z = \partial F / \partial z$. Now with

$$F(x,z,w) = \frac{|w|^p}{p} - \mu z + \frac{p-1}{p} \chi_{\{z>0\}},$$

we obtain the free boundary in Problem 1, while with

$$F(x,z,w) = \frac{|w|^p}{p} + \frac{p-1}{p} \chi_{\{z>0\}},$$

the second free boundary problem is achieved.

The method in proving the first equation in (1.5) is quite standard and classical (see Theorem 2.1). The fulfillment of the second equation, however, is one of the major mathematical problems in the theory of variational inequalities. In their pioneering paper, Alt and Caffarelli [1], proved that for $p = 2$, the second equation in (1.5) is satisfied in the strong sense on the so-called flat free boundaries, and that these boundary points are essentially the whole boundary, minus a set of $\mathcal{H}^{N-1}$-measure zero. In this paper we will not be concerned with the second equation in (1.5), and we only prove that (1.5) is satisfied in a very weak sense (Theorem 2.2 below).

Example 1. For general (non-negative) measures μ, the functional J in Problem 1, is not necessarily bounded from below. To illustrate this, let μ be the Dirac measure at the origin, $g = 1$, and define

$$v_m(x) = \min(\max(|x|^{(p-n)/(p-1)} - 1, 0), m) \quad \in \quad \mathbb{J}.$$

Now simple calculations show

$$J(v_m) \quad \to \quad -\infty.$$

Hence

$$\min_{\mathbb{J}} J_{\mu,g} = -\infty.$$

Example 2. In one space dimension the functional is in general not bounded even if we let μ to be a bounded function. Indeed, let

$$u_R = \max(R - |x|, 0), \qquad \mu = 2\chi_{[-1,1]} \qquad g = 1.$$

Then

$$J(u_R) = \int_R^R \left(\frac{|u'|^p}{p} - \mu u + \left(\frac{p-1}{p}\right) \chi_{[-R,R]} \right) dx = 2 - 2R$$

which tends to $-\infty$ with $R \to \infty$.

Obviously we can approximate the function $\mu = 2\chi_{[-1,1]}$ with a C^∞ function and still obtain $-\infty$ as the minimum value for the functional.

Example 2, shows that we must restrict ourselves to the case $N \geq 2$. The first example also gives us some trouble while working with general measures. In general the functional J is not bounded from below if μ is not in the dual space $(W^{1,p})^*$ A way of circumventing this problem for $p = 2$ was introduced in [4]. The method in [4] makes strong use of the (sub) meanvalue property for (sub) harmonic functions, which is not available for $p \neq 2$. Before closing this section we give a simple outline of the this method. We need first the following definition.

Definition For a given nonnegative measure μ, we define μ_ϵ to be the function

$$\mu_\epsilon(x) = h_\epsilon * \mu(x) = \int h_\epsilon(x - y)\ d\mu(y),$$

where $h_\epsilon(x) = \epsilon^{-N} h(x/\epsilon)$ and $h \in C_o^\infty(B(0,1))$ is a radially symmetric, non-increasing function of $|x|$, with $\int h = 1$.

Proof of the case $p = 2$. (An outline; for the full proof see [4]). First one proves that for bounded functions μ there are minimizers of the functional J. Then one considers the functional J_{g,μ_ϵ}, and proves the existence of a minimizer. For μ enough concentrated one shows that supp $\mu_\epsilon \subset \Omega_\epsilon$, where $\Omega_\epsilon = \{u_\epsilon > 0\}$, and u_ϵ is a minimizer of J_{g,μ_ϵ}. The boundary condition $|\nabla u| = 1$ (in some weak sense) now follows as in [1]. Next one shows that

$$u_\epsilon = U^{\mu_\epsilon} - U^{\sigma_\epsilon},$$

where ($\sigma_\epsilon = \mathcal{H}^{N-1}\lfloor\partial\Omega_\epsilon$), and U^{μ_ϵ}, U^σ_ϵ denote the normalized Newtonian potential of the measures μ_ϵ and σ_ϵ, respectively. Now for $\epsilon < \epsilon_0$ define

$$v = U^\mu - U^{\sigma_\epsilon}.$$

Then, by the mean-value property for harmonic functions, and that h_ϵ is a radial mollifier one gets

$$U^\mu = U^{\mu_\epsilon}, \qquad \text{in } \mathbb{R}^N \setminus \text{ supp } \mu_\epsilon,$$

and consequently

$$v = U^{\mu_\epsilon} - U^{\sigma_\epsilon} = u_\epsilon \qquad \text{in } \mathbb{R}^N \setminus \text{ supp } \mu_\epsilon. \tag{1.6}$$

This in turn implies that the function v and u_ϵ share the same boundary conditions, i.e. $v = 0$, and $|\nabla v| = 1$ on $\partial\Omega_\epsilon$. Finally (1.6) gives that $\Delta v = -\mu$ in $\{v > 0\} = \Omega_\epsilon$, and this completes the proof for the case $p = 2$.

In this paper we will show (Theorem 2.6) that there are solutions to problems 1,2, with the restriction $d\mu = f(x)dx$, $f \in L^\infty(\mathbb{R}^N)$, $f \geq 0$ and it has compact support.

2. Existence

Theorem 2.1. *If $u, v \in W^{1,p}(\mathbb{R}^N)$ are local minima of $J_{g,\mu}$ and I_g, over $\mathbb{J}$, $\mathbb{I}_{D,h}$ respectively, then*

$$\Delta_p u + \mu \geq 0 \qquad in \quad \mathbb{R}^N, \tag{2.1}$$

$$\Delta_p u + \mu = 0 \qquad in \quad \Omega_u, \tag{2.2}$$

$$\Delta_p v \geq 0 \qquad in \quad \mathbb{R}^N \setminus D, \tag{2.3}$$

$$\Delta_p v = 0 \qquad in \quad \Omega_v \setminus D, \tag{2.4}$$

where $\Omega_u = \{u > 0\}$ and $\Omega_v = \{v > 0\}$, and the equations are satisfied in the weak sense (1.1).

Proof. Take $0 \leq \phi \in C_o^\infty(\mathbb{R}^N)$ and define, for $\epsilon > 0$, $v_\epsilon = (u - \epsilon\phi)_+$. Then $v_\epsilon \in \mathbb{J}$, $0 \leq v_\epsilon \leq u$. Set $D_\epsilon = \{u \leq \epsilon\phi\} = \{v_\epsilon = 0\}$. Clearly $|D_\epsilon \cap \Omega| \to 0$ as $\epsilon \to 0$. Since $v_\epsilon - u = -u$ in D_ϵ, $v_\epsilon - u = -\epsilon\phi$ outside D_ϵ it follows that $v_\epsilon \to u$ in $W^{1,p}(\mathbb{R}^N)$ and that

$$\int g^p|\chi_{\{v_\epsilon>0\}} - \chi_{\{u>0\}}|dx = \int_{\Omega\cap D_\epsilon} g^p dx \quad \to 0.$$

Since u is a local minimum we conclude that $J(u) \leq J(v_\epsilon)$ for $\epsilon > 0$ small enough.

Next using the mean value theorem (for the function $|\xi|^p$ in $\mathbb{R}^N$) and the fact that $u \le \epsilon\phi$ in D_ϵ we obtain

$$
\begin{aligned}
0 &\le J(v_\epsilon) - J(u) \\
&= \frac{1}{p}\int (|\nabla v_\epsilon|^p - |\nabla u|^p) - \int \mu(v_\epsilon - u) + \left(\frac{p-1}{p}\right)\int g^p(\chi_{\{v_\epsilon>0\}} - \chi_{\{u>0\}}) \\
&\le \frac{1}{p}\int (|\nabla(u-\epsilon\phi)|^p - |\nabla u|^p) + \epsilon\int_{D_\epsilon^c}\mu\phi + \int_{D_\epsilon}\mu u - \left(\frac{p-1}{p}\right)\int_{\Omega\cap D_\epsilon} g^p \\
&\le \frac{1}{p}\int (|\nabla(u-\epsilon\phi)|^p - |\nabla u|^p) + \epsilon\int_{D_\epsilon^c}\mu\phi + \epsilon\int_{D_\epsilon}\mu\phi \\
&= \epsilon\left(\int |\nabla u + \xi|^{p-2}(\nabla u + \xi)\cdot(-\nabla\phi) + \int_{\mathbb{R}^N}\mu\phi\right),
\end{aligned}
$$

where $\xi \in [0, -\epsilon\nabla\phi]$ (the line segment from the origin to $-\epsilon\nabla\phi(x)$ in $\mathbb{R}^N$ with x fixed. Dividing both sides by ϵ and letting $\epsilon \to 0$ we obtain,

$$0 \le -\int |\nabla u|^{p-2}\nabla u\cdot\nabla\phi + \int \mu\phi$$

for all $0 \le \phi \in C_o^\infty(\mathbb{R}^N)$, and hence that $\Delta_p u + \mu \ge 0$ in $\mathbb{R}^N$, which proves (2.1).

If supp $\phi \subset \Omega$ we can take $v = u \pm \epsilon\phi \in \mathbb{J}$ for $\epsilon > 0$ small enough (and $0 \le \phi \in C_o^\infty(\mathbb{R}^N)$) and this readily gives that $\Delta_p u + \mu = 0$ in Ω. This proves (2.2). The proof of (2.3)–(2.4) is similar. $\square$

Theorem 2.2. *Let u be a local minimum and assume $g^p \in W^{1,1}(\mathbb{R}^N)$. Then*

$$\lim_{\epsilon\searrow 0}\int_{\partial\{u>\epsilon\}} (|\nabla u|^p - g^p)\eta\cdot\nu d\mathcal{H}^{N-1} = 0$$

for every $\eta \in C_o^\infty(\mathbb{R}^N \setminus$ supp $\mu, \mathbb{R}^N)$. (ν denotes the outward normal vector of $\partial\{u > \epsilon\}$.)

Proof. ([1; 2.5]) For $\eta \in C_o^\infty(\mathbb{R}^N \setminus \text{supp}\ \mu, \mathbb{R}^N)$ and $\epsilon > 0$ small, define u_ϵ by $u_\epsilon(\tau_\epsilon(x)) = u(x)$, where $\tau_\epsilon(x) = x + \epsilon\eta(x)$. Since ϵ is small $\det(D\tau_\epsilon) \ne 0$, and thus τ_ϵ^{-1} exists. Hence u_ϵ is well defined.

Then

$$0 \le J(u_\epsilon) - J(u) = \int_{\{u>0\}}\left(\frac{|(D\tau_\epsilon)^{-1}\nabla u|^p}{p} - \mu\circ\tau_\epsilon u + \left(\frac{p-1}{p}\right)g^p\circ\tau_\epsilon\right)\det D\tau_\epsilon - \int_{\{u>0\}}\left(\frac{|\nabla u|^p}{p} - \mu u + \left(\frac{p-1}{p}\right)g^p\right),$$

where $\circ$ denotes decomposition. Next using

$$(D\tau_\epsilon)^{-1} = (E + \epsilon D\eta)^{-1} = E - \epsilon D\eta + o(\epsilon),$$

where E is the unit matrix in $\mathbb{R}^N$, we'll have

$$|(D\tau_\epsilon)^{-1}\nabla u|^p = |\nabla u|^p - \epsilon p|\nabla u|^{p-2}\nabla u\cdot(D\eta\nabla u) + o(\epsilon).$$

Also,

$$\det(D\tau_\epsilon) = 1 + \epsilon\nabla \cdot \eta + o(\epsilon),$$
$$g^p \circ \tau_\epsilon = g^p + \epsilon\nabla g^p \cdot \eta + o(\epsilon).$$

Inserting these in the above, and making use of the fact that $\eta \equiv 0$ inside support of μ, we obtain

$$0 \le J(u_\epsilon) - J(u) =$$
$$\epsilon\int_{\{u>0\}} -|\nabla u|^{p-2}\nabla u \cdot (D\eta\nabla u) + \frac{1}{p}|\nabla u|^p\nabla\cdot\eta + \left(\frac{p-1}{p}\right)(g^p\nabla\cdot\eta + \nabla g^p\cdot\eta) + o(\epsilon) =$$
$$\epsilon\int_{\{u>0\}} \nabla\cdot\left[\left(\frac{|\nabla u|^p}{p} + \left(\frac{p-1}{p}\right)g^p\right)\eta - (\eta\cdot\nabla u)|\nabla u|^{p-2}\nabla u\right] + o(\epsilon).$$

Dividing both sides by ϵ and letting $\epsilon \to 0$, and observing that we can in similar way work with $x - \epsilon\eta$, it follows that the linear term in ϵ vanishes. Hence, using the fact that $\nabla u/|\nabla u| = \nu$ on $\partial\{u > \epsilon\}$, we'll have

$$\begin{aligned}
0 &= \int_{\{u>0\}} \nabla\cdot\left[\left(\frac{|\nabla u|^p}{p} + \left(\frac{p-1}{p}\right)g^p\right)\eta - (\eta\cdot\nabla u)|\nabla u|^{p-2}\nabla u\right] \\
&= \lim_{\epsilon\to 0}\int_{\partial\{u>\epsilon\}} \left(\frac{|\nabla u|^p}{p} + \left(\frac{p-1}{p}\right)g^p - |\nabla u|^p\right)\eta\cdot\nu \\
&= -\left(\frac{p-1}{p}\right)\lim_{\epsilon\to 0}\int_{\partial\{u>\epsilon\}} (|\nabla u|^p - g^p)\eta\cdot\nu\ .
\end{aligned}$$

This proves the theorem. □

Our first lemma concerns a comparison type result.

The following lemma turns out to be very useful. Cf. [4] for $p = 2$.

Lemma 2.3. *For $k = 1, 2$, let $J_k(u) = J_{\mu_k, g_k}(u)$, where $g_1 \ge g_2$ and $\mu_1 \le \mu_2$ and are bounded functions. For $u_1, u_2 \in \mathbb{J}$, define $v_1 = \min(u_1, u_2)$ and $v_2 = \max(u_1, u_2)$. Then $v_1, v_2 \in \mathbb{J}$ and*

$$J_1(v_1) + J_2(v_2) \le J_1(u_1) + J_2(u_2).$$

In particular, if u_1 minimizes J_1 then $J_2(v_2) \le J_2(u_2)$ and if u_2 minimizes J_2 then $J_1(v_1) \le J_1(u_1)$. If u_k minimizes J_k for $k = 1, 2$ then v_1 minimizes J_1 and v_2 minimizes J_2.

Proof. In general, if $\Phi(t)$ is a nondecreasing function of $t \in \mathbb{R}$ and $\varphi_1 \le \varphi_2$ then, as is easily seen,

$$\int (\varphi_1\Phi(u_1) + \varphi_2\Phi(u_2)) \le \int (\varphi_1\Phi(v_1) + \varphi_2\Phi(v_2)).$$

Applying this with $\varphi_j = \mu_j$, $\Phi(t) = t$ we find that

$$\int(\mu_1 u_1 + \mu_2 u_2) \le \int(\mu_1 v_1 + \mu_2 v_2),$$

and choosing $\varphi_j = -g_j^p$, $\Phi(t) = 0$ for $t \le 0$, $\Phi(t) = 1$ for $t > 0$ we find

$$\int(g_1^p\chi_{\{u_1>0\}} + g_2^p\chi_{\{u_2>0\}}) \ge \int(g_1^p\chi_{\{v_1>0\}} + g_2^p\chi_{\{v_2>0\}}).$$

The proof now follows by observing $\int(|\nabla u_1|^p + |\nabla u_2|^p) = \int(|\nabla v_1|^p + |\nabla v_2|^p)$. □

Similarly one can prove the following lemma.

Lemma 2.4. *Let $I_k(u) = I_{g_k}(u)$, $\mathbb{I}_k = \mathbb{I}_{D,h_k}$, $h_1 \le h_2$, $g_1 \ge g_2$ and $k = 1, 2$. For $u_1, u_2 \in \mathbb{I}$ define $v_1 = \min(u_1, u_2)$ and $v_2 = \max(u_1, u_2)$. Then $v_k \in \mathbb{I}_k$ and*

$$I_1(v_1) + I_2(v_2) \le I_1(u_1) + I_2(u_2).$$

In particular, if u_1 minimizes I_1 then $I_2(v_2) \le I_2(u_2)$ and if u_2 minimizes I_2 then $I_1(v_1) \le I_1(u_1)$. If u_k minimizes I_k for $k = 1, 2$ then v_1 minimizes I_1 and v_2 minimizes I_2.

In order to get comparison solutions we shall first prove the existence of solutions (minimizers) in a special case.

Lemma 2.5. *Let $\mu = M\chi_{B(0,R)}$, $g = c$, and $h = b$, where M, c, b, and R are nonnegative constants. Then $J_{\mu,g}$ and I_g, respectively, have at least one minimum (minimizer) u in $\mathbb{J}$, and $\mathbb{I}_{B,h}$ respectively, where B is any ball with center at the origin. Any minimizing u is radially symmetric, radially nonincreasing and vanishes outside a compact set.*

Proof. Since $I \ge 0$, we need only to show $J \ge$ const. $> -\infty$. Then the rest follows as in standard theory. For u in $\mathbb{J}$ we let $u^* \in \mathbb{J}$ denote the usual radially symmetric decreasing rearrangement of u. (see [7]). Then

$$\int |\nabla u^*|^p \le \int |\nabla u|^p, \quad \int \mu u^* \ge \int \mu u, \quad \int g^p \chi_{\{u^*>0\}} = \int g^p \chi_{\{u>0\}};$$

where the first inequality is classical (see [m, Theorem 4.1]) and the second inequality uses the fact that μ is nonincreasing as function of $r = |x|$. It follows that $J(u^*) \le J(u)$ and hence that we only need to look for minima in $\mathbb{J}^* = \{u \in \mathbb{J} : u^* = u\}$. It should be observed also that $J(u^*) < J(u)$ unless $u^* = u$.

We first prove that J is bounded from below on $\mathbb{J}^*$ (and hence on $\mathbb{J}$). For u in $\mathbb{J}^*$ there is a unique ρ in $[0, \infty]$ depending on u such that $u(x) > 0$ for $|x| < \rho$ and $u(x) = 0$ for $|x| \ge \rho$. Set $\Omega = \{u > 0\} = B(0, \rho)$. Regarding μ, g and u as functions of $r = |x|$ we have

$$\frac{1}{\omega_N} J(u) = \int_0^\rho \frac{|u'(r)|^p}{p} r^{N-1} dr - \int_0^\rho \mu u r^{N-1} dr + \left(\frac{p-1}{p}\right) \frac{c^p}{N} \rho^N.$$

Since $J(u) = +\infty$ if $\rho = +\infty$ we need only to consider u with $\rho < \infty$. Set

$$\lambda = \left(\int_0^\rho \frac{|u'(r)|^p}{p} r^{N-1} dr\right)^{1/p} = \frac{1}{\sqrt[p]{\omega_N}} \|\nabla u\|,$$

$$\phi(r) = \int_0^r \mu(s) s^{N-1} ds = \frac{M}{N} \min(r, R)^N.$$

Since $\phi(r) \ge 0$ and $u' \le 0$ we get

$$\int_0^\rho \mu u r^{N-1} dr = -\int_0^\rho \phi u' dr \le \left(\int_0^\rho |u'|^p r^{N-1}\right)^{1/p} \left(\int_0^\rho \left(\frac{\phi}{r^{(N-1)/p}}\right)^q\right)^{1/q}$$
$$= p^{1/p} \lambda A_\rho,$$

where

$$A_\rho = \left(\int_0^\rho \left(\frac{\phi}{r^{(N-1)/p}}\right)^q\right)^{1/q}, \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

and A_ρ is well defined and finite due to the definition of ϕ. Now

$$A_\rho \le C\left(1+\int_1^\rho r^{(1-N)\frac{q}{p}}\right)^{1/q} \le \text{ const. } \rho^{(1-N)\frac{1}{p}+\frac{1}{q}} = C\rho^{1-\frac{N}{p}},$$

where C is a generic constant (changing its value from a one appearance to another) and independent of ρ. It thus follows, using $N \ge 2$ and $\rho \ge 1$, (for $\rho \le 1$ the functional is obviously bounded from below)

$$\frac{1}{\omega_N}J(u) \ge \lambda^p - C_1\lambda\rho^{(1-\frac{2}{p})} + C_2\rho^2, \tag{2.5}$$

for some positive constants C_1, C_2. It thus remains to prove that the right hand side of (2.5) is bounded from below. We consider two cases
Case 1: Suppose $1 < p \le 2$. Then $\rho^{1-2/p} \le 1$, and we thus have

$$\frac{1}{\omega_N}J(u) \ge \lambda^p - C_1\lambda + C_2\rho^2 \ge -\left(\frac{C_1}{p}\right)^{p/(p-1)} + C_2\rho^2 \tag{2.6}$$

Case 2: Suppose $2 < p < \infty$. Then $\rho^{1-2/p} \le \rho$, and we thus have

$$\frac{1}{\omega_N}J(u) \ge \lambda^p - C_1\lambda\rho + C_2\rho^2 \ge -\left(\frac{C_1^2}{4C_2}\right)^{2/(p-2)}. \tag{2.7}$$

Let $\{u_n\} \subset \mathbb{J}^*$ be a minimizing sequence, ρ_n the radius of support of u_n and $\lambda_n^p = \int_0^{\rho_n} |u_n'|^p r^{N-1}dr = \frac{1}{\sqrt[p]{\omega_N}}\|\nabla u_n\|$. Then it follows from (2.5)–(2.7) that

$$\rho_n \le \text{const.} < \infty, \tag{2.8}$$

$$\lambda_n \le \text{const.} < \infty. \tag{2.9}$$

Now from (2.8) and (2.9) the existence of a minimum for J follows by standard arguments. In fact, (2.8) shows that we may work in $\mathbb{J} \cap W_o^{1,p}(B)$ for some fixed ball B and (2.9) then shows that the minimizing sequence $\{u_n\}$ is precompact in the w-$W_o^{1,p}(B)$ (i.e. $W_o^{1,p}(B)$ provided with the weak topology). As J is easily checked to be lower semi-continuous in w-$W_o^{1,p}(B)$ the existence of a minimum follows . □

Theorem 2.6. *Let $g \ge c > 0$ be as before and take $d\mu = f(x)dx$, with $0 \le f \le M$ and $supp(f) \subset B(0,R)$. Then there is a function u, which has compact support, and satisfies*

$$\Delta_p u = -\mu, \qquad \text{in } \Omega$$
$$|\nabla u| = g, \qquad \text{on } \partial\Omega \setminus (supp(\mu)),$$

where $\Omega = \{u > 0\}$, the p-Laplacian is satisfied in the sense of (1.1), and the boundary condition is satisfied in the sense of Theorem 2.2. Moreover, if μ is enough concentrated then $supp\ \mu \subset \Omega$.

Proof. According to Theorems 2.1 and 2.2 we need only to prove the existence of a minimizer to the functional $J_{g,\mu}$. Since the function g is bounded below and f bounded above we may use symmetric barriers as in Lemma 2.5.

Let $\tilde{f} = M\chi_{B(0,R)} \geq f$, $\tilde{g} = c \leq g$, and set $\tilde{J} = J_{\tilde{f},\tilde{g}}$, $J = J_{f,g}$.

By Lemma 2.5 there is a largest minimizer $\tilde{u}$ in $\mathbb{J}$ of $\tilde{J}$. Clearly

$$J(u) \geq \tilde{J}(u) \qquad \text{for all } u \in \mathbb{J} \tag{2.10}$$

and also, by Lemma 1.1,

$$J(\min(u,\tilde{u})) \leq J(u). \tag{2.11}$$

Thus $J(u)$ decreases if u is replaced by $\min(u,\tilde{u})$. Choose an open ball B such that $\operatorname{supp} \tilde{u} \subset B$. (2.10) together with Lemma 2.5 shows that J is bounded from below and (2.11) shows that if $\{u_n\}$ is a minimizing sequence then so is $\{\min(u_n,\tilde{u})\}$. Thus there exists a minimizing sequence $\{u_n\}$ with $\operatorname{supp} u_n \subset B$. By Poincaré's lemma then $\|u_n\| \leq C\|\nabla u_n\|$, so that

$$J(u_n) \geq \frac{\|\nabla u_n\|^p}{p} - \|f\chi_B\|\|u_n\| \geq \frac{\|\nabla u_n\|^p}{p} - C\|\nabla u_n\| \geq -\text{constant}.$$

Thus J is bounded from below, $\|\nabla u_n\| \leq C$ and the existence of a minimizer follows as in Lemma 2.5.

If now $u \in \mathbb{J}$ denotes any minimizer of J then Lemma 2.3 shows that $\max(u,\tilde{u}) \leq \tilde{u}$, since $\tilde{u}$ is the largest minimizer of $\tilde{J}$, hence that $u \leq \tilde{u}$. This shows that u has compact support in a fixed compact set. Now to complete the proof one needs to make sure that the support of the measure μ is inside Ω provided μ is concentraed enough. This also can be done easily by the comparison Lemma above. One just compares the solution with one smaller, by taking a new measure $\mu' = a\chi_{\{>0\}}B \leq \mu$ for large a and some open ball B inside the support of μ. The proof is completed. $\square$

In a similar fashion one can state and prove the existence of a solution to problem 2. Here one needs to use Lemma 2.4 and make some small and obvious changes in the proof.

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