ON THE POROUSITY OF FREE BOUNDARIES IN DEGENERATE VARIATIONAL INEQUALITIES

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Abstract. In this note we consider certain degenerate variational problems with constraints. The exact growth of the solution near the free boundary is established. A consequence of this is that the free boundary is locally porous and thus has 0 Lebesgue measure.

1. Introduction

In this paper we establish the optimal growth near the free boundary of the solutions to the obstacle problem for the so-called inhomogeneous p-Laplace equation, $1 < p < \infty$,

$$\Delta_p v = f(x) \tag{1.1}$$

in a bounded domain D of $\mathbb{R}^N$. Here $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ is the p-Laplace operator. Recall that a function v is a weak solution to (1.1) in D if $v \in W^{1,p}(D)$ and

$$\int_D (\nabla v|^{p-2}\nabla v \cdot \nabla \eta + f\eta)dx = 0$$

whenever $\eta \in C_0^\infty(D)$. The associated obstacle problem is to minimize

$$J(v) = \int_D \left(\frac{|\nabla v|^p}{p} + f(x)v\right) dx \tag{1.2}$$

over the set

$$\mathcal{K} = \{v \in W^{1,p}(D) : \ v - \theta \in W_o^{1,p}(D), \ v \geq \psi\}.$$

The function ψ is called the *obstacle*; θ determines the boundary values. The minimizer u to (1.2) satisfies in the weak sense

$$\Delta_p u = f\chi_\Omega \quad \text{in } D, \qquad u \geq \psi \quad \text{a.e. in } D, \tag{1.3}$$

where

$$\Omega = \{x \in D : u(x) > \psi(x)\}.$$

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We study the free boundary

$$\Gamma = \partial\Omega \cap D$$

under the following assumptions on the data. First, the function f satisfies

$$0 < \lambda_0 \le f(x) \le \Lambda_0 \quad \text{in } D. \tag{1.4}$$

Second, for the obstacle either $\psi \equiv 0$ or in the weak sense

$$\Delta_p \psi = g, \qquad 0 \le -g \le \Lambda_1 \quad \text{in } D. \tag{1.5}$$

Besides we assume that ψ and θ are in $W^{1,p}(D) \cap L^\infty(D)$. Along with (1.3), (1.4) and (1.5) this provides local $C^{1,\alpha}$ regularity for some $\alpha \in (0,1)$ of functions u and ψ; see [L].

Theorem A. *Let u be the minimizer to (1.2) with $\psi \equiv 0$. Then in any subdomain $D' \subset\subset D$*

$$u(x) \le C(d(x))^{p/(p-1)}$$

where $d(x) = dist(x, \Gamma)$, and $C = C(u, D', D)$ is some constant. Moreover $\Gamma \cap D'$ is porous.

Recall here that a set E in $\mathbb{R}^N$ is called *porous* if there are positive constants δ and r_0 such that for each $x \in E$ and $0 < r < r_0$ there is a point y such that $B_{\delta r}(y) \subset B_r(x) \setminus E$. The porous sets have Hausdorff dimension $< N$ and as a consequence Lebesgue measure 0; see [S]. The last statement in Theorem A says that Γ is *locally porous.*

The case (1.5) brings difficulties connected with the singular set

$$F = \{x \in D : \nabla\psi(x) = 0\}.$$

However, we are able to show the following.

Theorem B. *$\Gamma \setminus \partial F$ is locally porous, provided ψ satisfies (1.5).*

2. Main Lemma

We use notation $B_r(z)$ for the open ball with center z and radius r; $B_r = B_r(0)$.

Define $\mathcal{G} = \mathcal{G}(p)$ to be the class of all functions u in B_1 satisfying:

$$\|\Delta_p u\|_\infty \le 1; \tag{2.1}$$

$$0 \le u \le 1; \tag{2.2}$$

$$u(0) = 0. \tag{2.3}$$

Our main lemma is as follows.

Lemma 2.1. *There is a positive constant K such that for any $u \in \mathcal{G}$, there holds*

$$|u(x)| \le K|x|^{p/(p-1)} \qquad \forall x \in B_1.$$

Remark. Lemma 2.1, in an obvious way, can be stated as follows:
Suppose there is a function u in B_1 satisfying

$$\|\Delta_p u\|_\infty \le \Lambda_0, \qquad 0 \le u \le M, \qquad u(0) = 0.$$

Then

$$|u(x)| \le KK'|x|^{p/(p-1)} \qquad \forall x \in B_1,$$

where

$$K' = \max\left(M, \Lambda_0^{1/(p-1)}\right),$$

and K as in Lemma 2.1.

Set

$$S(r,u,z) = \sup_{x \in B_r(z)} |u(x)|, \qquad S(r,u) = S(r,u,0).$$

For a given bounded function u, define the set $\mathbb{M}(u)$ to be the maximal subset of all natural numbers $\mathbb{N}$ such that the following doubling condition holds

$$2^{p/(p-1)} S(2^{-j-1},u) \ge S(2^{-j},u) \qquad \forall\ j \in \mathbb{M}(u). \tag{2.4}$$

Lemma 2.2. *There exsits a constant $K = K(p,N)$ such that*

$$S(2^{-j-1},u) \le K\left(2^{-j}\right)^{p/(p-1)},$$

for all $u \in \mathcal{G}$, and $j \in \mathbb{M}(u)$.

Proof. We argue by contradiction. Thus we assume that for any $k \in \mathbb{N}$, there are $u_k \in \mathcal{G}$ and $j_k \in \mathbb{M}(u_k)$ such that

$$S(2^{-j_k-1},u_k) \ge k\left(2^{-j_k}\right)^{p/(p-1)}. \tag{2.5}$$

Define now

$$\tilde{u}_k(x) := \frac{u_k(2^{-j_k}x)}{S(2^{-j_k-1},u_k)} \qquad \text{in } B_1. \tag{2.6}$$

Then

$$0 \le \tilde{u}_k \le 2^{p/(p-1)} \qquad (\text{by } (2.1)), \tag{2.7}$$

$$\sup_{B_{(1/2)}} |\tilde{u}_k| = 1 \qquad (\text{by } (2.6)), \tag{2.8}$$

$$\tilde{u}_k(0) = 0 \qquad (\text{by } (2.3)), \tag{2.9}$$

and by (2.1) and (2.5)

$$\|\Delta_p \tilde{u}_k(x)\|_\infty \leq k^{1-p}.$$

Now all the above and $C^{1,\alpha}$ estimates (for some $\alpha > 0$) of solutions (see e.g. [L]) gives a compactness for the set $\{\tilde{u}_k\}$ in $C^{1,\alpha}$-norm, locally in $\mathbb{R}^N$. Hence for a subsequence $\tilde{u}_k \to u$ in $C^{1,\alpha}$ norm. Moreover, the limit function $u \not\equiv 0$, by (2.8) and satisfies

$$\Delta_p u = 0, \qquad u(0) = 0, \qquad u \geq 0,$$

in B_1. Obviously this contradicts the minimum principle, and the conclusion in the lemma follows. □

Proof of Lemma 2.1. We claim that

$$S(2^{-j}, u) \leq K(2^{-j+1})^{p/(p-1)} \tag{2.10}$$

for all $j \in \mathbb{N}$, with the constant K in Lemma 2.2. With no loss of generality we may assume that $K \geq 1$. Thus (2.10) holds for $j = 0$. Next, let (2.10) hold for some $j \in \mathbb{N}$. Then it holds also for $j+1$. Indeed, if $j \in \mathbb{M}(u)$ then this follows from Lemma 1.2. Otherwise, (2.4) fails and we obtain

$$S(2^{-j-1}) \leq 2^{-p/(p-1)} S(2^{-j}) \leq 2^{-p/(p-1)} K(2^{-j+1})^{p/(p-1)} = K(2^{-j})^{p/(p-1)}.$$

This proves the claim.

To complete the proof, let $2^{-j-1} \leq r \leq 2^{-j}$. Then by (2.10)

$$S(r, u) \leq S(2^{-j}, u) \leq K\left(2^{-j-1}\right)^{p/(p-1)} \leq K r^{p/(p-1)}.$$

The lemma is proved. □

3. Proof of Theorem A

Lemma 3.1. *Let $\Delta_p u = f\chi_\Omega$, with $f \geq \lambda_0$, and $\Omega = \{u > 0\}$. Then, for all $z \in \overline{\Omega}$,*

$$S(r, u, z) \geq C_0 r^{p/(p-1)} + u(z),$$

where

$$C_0 = \frac{p-1}{p}\left(\frac{\lambda_0}{N}\right)^{1/(p-1)}.$$

The original idea of this lemma comes from [C]; see also [BS].

Proof. First suppose $z \in \Omega$, and set

$$w_\epsilon(x) = u(x) - u(z)(1-\epsilon), \qquad v(x) = C_0|x-z|^{p/(p-1)}.$$

Then $\Delta_p v = \lambda_0$ and therefore $\Delta_p w_\epsilon = \Delta_p u \geq \Delta_p v$ in $\Omega \cap B_r(z)$, and $w_\epsilon \leq v$ on $\partial\Omega \cap B_r(z)$. If also $w_\epsilon \leq v$ on $\partial B_r(z) \cap \Omega$, then we may apply the comparison principle (see [HKM]) to obtain $w_\epsilon \leq v$ in $B_r(z) \cap \Omega$, which contradicts the fact that $w_\epsilon(z) = \epsilon u(z) > 0 = v(z)$. Hence

$$\sup_{\partial B_r(z)} w_\epsilon \geq \sup_{\partial B_r(z)} v = C_0 r^{p/(p-1)}.$$

Letting $\epsilon \to 0$, we obtain the desired result, for all $z \in \Omega$, and by continuity for all $z \in \overline{\Omega}$. The proof is completed. $\square$

Proof of Theorem A. Without loss of generality we may assume that the domain D' in Theorem A is the ball B_1, and moreover that $B_3 \subset D$.

Let $x \in B_1 \cap \Omega$. Consider the following two cases.
(i) $d(x) \geq 1$;
(ii) $d(x) < 1$.

In case (i) we may write that $u(x) \leq S(1,u) \leq S(1,u)(d(x))^{p/(p-1)}$.

In case (ii) we take $z = z_x \in \partial\Omega$ with $d(x) = |z - x|$. Then $B_1(z_x) \subset B_3 \subset D$. Define $\tilde{u}(y) = u(z_x + y)$ for $y \in B_1$. Then

$$\|\Delta_p \tilde{u}\|_\infty \leq \|f\|_\infty, \qquad \tilde{u}(0) = 0, \qquad 0 \leq \tilde{u} \leq \max_{\partial B_1} \tilde{u}_0.$$

Hence by the remark following Lemma 2.1, we will have

$$u(x) = \tilde{u}(x - z) \leq KK'|x - z|^{p/(p-1)} = C_1(d(x))^{p/(p-1)}.$$

Taking $C = \max\{C_1, S(1,u)\}$ we obtain the estimate in Theorem A.

Show now that $\Gamma \cap B_1$ is porous. Let $z \in \Gamma \cap B_1$. Then for $0 < r < 1$ by Lemma 3.1 there is $x = x_z \in \partial B_r(z)$, such that

$$u(x) \geq C_0 r^{p/(p-1)}.$$

Now by the first part of this theorem,

$$C_0 r^{p/(p-1)} \leq u(x) \leq C(d(x))^{p/(p-1)},$$

implying

$$d(x) \geq \delta r \qquad (\delta = \min\{1, (C_0/C)^{p/(p-1)})\},$$

i.e.,

$$B_{\delta r}(x) \subset \Omega.$$

Since $x \in \partial B_r(z)$, there is a ball $B_{(\delta/2)r}(y) \subset B_r(z) \cap B_{\delta r}(x) \subset B_r(z) \setminus \Gamma$. This shows that $\Gamma \cap B$ is porous. The theorem is proved. $\square$

4. Proof of Theorem B

Consider first the interior F° of the singular set

$$F = \{\nabla\psi = 0\}.$$

Evidently, in every connected component U_α of F° there is a constant c_α such that $\psi \equiv c_\alpha$ in U_α. Then the difference $u - c_\alpha$ is solution to the obstacle problem in U_α with 0 obstacle. Applying Theorem A we see that $\Gamma \cap U_\alpha$ is locally porous. Since this is true for every component U_α of F° we conclude that

$$\Gamma \cap F^\circ \text{ is locally porous.} \tag{4.1}$$

Next, consider the difference $v = u - \psi$ in some ball $B \subset\subset D \setminus F$. Then we assert that v satisfies in the weak sense

$$\operatorname{div}(\theta(x)\nabla v) = h(x) \quad \text{in } B, \qquad v \geq 0 \quad \text{a.e. in } B \tag{4.2}$$

where

$$h(x) = f\chi_\Omega - g(x)$$

and θ is a symmetric $N \times N$ matrix function in $C^\alpha(B)$, satisfying the ellipticity condition

$$\lambda_B|\xi|^2 \leq \theta(x)\xi \cdot \xi \leq \Lambda_B|\xi|^2 \tag{4.3}$$

for each $x \in B$ and $0 \neq \xi \in \mathbb{R}^N$. Indeed, if we denote $\Theta(\xi) = |\xi|^{p-2}\xi$ then subtracting (1.5) from (1.3) we obtain

$$\int_B [(\Theta(\nabla u) - \Theta(\nabla\psi)) \cdot \nabla\eta + h\eta]dx = 0$$

for any test function $\eta \in C_0^\infty(B)$. Then (4.2) follows with

$$\theta(x) = \int_0^1 \Theta'(\nabla u_t)dt,$$

where $u_t = tu + (1-t)\psi$. Besides one can see that (4.3) holds since $|\nabla\psi|$ is bounded away from 0 on $\overline{B}$. ¿From (1.4) and (1.5) it follows that in B

$$\lambda_0\chi_\Omega \leq \operatorname{div}(\theta(x)\nabla v) \leq \Lambda_0 + \Lambda_1, \quad v \geq 0 \quad \text{a.e}, \quad \Omega = \{v > 0\}. \tag{4.4}$$

Now we make observation that tools used in the proof of Theorem A work also for functions satisfying (4.4), with $p = 2$. So, we may conclude that $\Gamma \cap B$ is porous. Since $B \subset\subset D \setminus F$ is arbitrary,

$$\Gamma \setminus F \text{ is locally porous.} \tag{4.5}$$

Theorem B now folows from (4.1) and (4.5). □

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