ON QUADRATURE DOMAINS AND THE SCHWARZ POTENTIAL

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Abstract. Let T be a distribution in R^n whose support is compact. A domain Ω in R^n is said to be a quadrature domain with respect to T if supp $T \subset \Omega$ and $\int_\Omega h =< h, T >$, for all functions h which are harmonic and integrable over Ω. We denote this by $\Omega \in Q(T, HL^1)$. This is equivalent to saying that there is a solution u (called modified Schwarz potential) to the following overdetermined Cauchy problem:

$$\Delta u = 1 - T \text{ in } \Omega, \qquad \text{and} \qquad u = |\nabla u| = 0 \text{ on } \partial\Omega.$$

In this paper we investigate different properties of quadrature domains. We show that if T is a fixed distribution, then bounded domains in $Q(T, HL^1)$ are uniformly bounded. Later we prove that if the modified Schwarz potential of a domain Ω is a polynomial, then Ω is a half-space. We also study the uniqueness problem. Namely, we show that if D and Ω are in $Q(T, HL^1)$, v and u are the MSPs of D and Ω respectively and $\sup_{\partial D} u \geq 0$, then $\partial\Omega$ meets $\{x \in \Omega : v(x) < 0\}$. Consequently if $v \geq 0$, then $|\Omega \mathrm{M} D| = 0$.

0. INTRODUCTION

Let Ω be a domain in R^n and T a distribution compactly supported in Ω and recall the definition of a quadrature domain (QD) in the abstract. When $T = 0$ we also consider the subharmonic test functions and we call these domains *null quadrature domains* (NQDs). To be more precise, we say Ω is a NQD for the test class $SL^1(\Omega)$ (the space of functions which are subharmonic and integrable over Ω) and write $\Omega \in Q(0, SL^1)$ if

$$\int_\Omega g(x)\ dx \geq 0 \qquad \forall g \in SL^1(\Omega).$$

The main purpose of this paper is to study some properties of QDs. The paper is organized as follows.

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In section 2, using an approximation theorem due to L. Karp, we show that if Ω is a domain in R^n whose complement contains arbitrarily large balls then the subset of $HL^1(\Omega)$ which consists of rapidly decreasing functions ($O(|x|^{-m})$ for arbitrary m) is dense in $HL^1(\Omega)$.

In section 3 we show that if $\Omega_i \in Q(T_i, HL^1)$, for $i = 1, 2, ..., \chi_{\Omega_i} \to \chi_\Omega$ (pointwise) and $T = \lim T_i$ exists and is compactly supported in Ω, then $\Omega \in Q(T, HL^1)$. This is very useful for constructing new QDs from old. Here we also show that if $\Omega \in Q(T, HL^1)$ then there is a solution to the following Cauchy problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \text{ near } \partial\Omega, \\ u = \frac{1}{2}|x|^2 & \text{on } \partial\Omega, \\ \nabla u = x & \text{on } \partial\Omega. \end{cases}$$

The solution of this will be called the *Schwarz potential* (SP) of Ω or $\partial\Omega$, and

$$v = \frac{1}{2n}|x|^2 - \frac{1}{n}u,$$

will be called the *modified Schwarz potential* (MSP) of Ω or $\partial\Omega$. Using this we give a new and simple proof (cf. [3] and [9]) of the following. Let T be a fixed distribution. Then there is a ball $B(= B_T)$ such that if $\Omega \in Q(T, HL^1)$, then $\Omega \subset B$.

In section 4 we mainly treat null quadrature domains (NQDs). We show that the half-space, the exterior of a cylinder and the non-convex component of the complement of a paraboloid are in $Q(0, SL^1)$. We also prove that if the MSP of $\partial\Omega$ is analytic and positive in Ω (with no growth restrictions), then $\Omega \in Q(0, SL^1)$. Our last result in this section reveals that if Ω is a domain in R^n whose MSP is a polynomial then $\partial\Omega$ is necessarily a hyperplane. This was also observed by the authors in [7] by means of different techniques.

In section 5 we treat the uniqueness problem. The main question, here, which still remains open, is "Do there exist two different solid domains in $Q(T, HL^1)$?", where by a solid domain we mean a domain without cavities. Our main result here states that if D and Ω are in $Q(T, HL^1)$, v and u are the MSPs of D and Ω respectively and $\sup_{\partial D} u \geq 0$, then $\partial\Omega$ meets $\{x \in \Omega : v(x) < 0\}$. Consequently if $v \geq 0$, then $|\Omega \mathrm{M} D| = 0$.

1. NOTATIONS

The basic notations adopted in this note are as follows.

By Ω we shall always mean an open set in $\mathrm{R}^n (n \geq 3)$ such that
i) $\overline{\Omega} \neq \mathrm{R}^n$,
ii) $\Omega = int(\overline{\Omega})$.
$\Omega_R = \Omega \cap B(0,R)$ where $B(0,R)$ is the n-dimensional open ball with radius R centered at the origin. $\Omega \triangle D$ means the symmetric difference of Ω and D; $|\Omega|$ will denote the n-dimensional Lebesgue measure of Ω. An open set $\Omega \subset \mathrm{R}^n$ is said to be an *(L)-domain* if there exists a sequence $y^k \in \mathrm{R}^n \setminus \Omega$ such that

$$\lim_{k\to\infty} \int_\Omega \frac{dx}{|x-y^k|^{n+1}} = 0.$$

This condition is certainly satisfied if $\mathrm{R}^n \setminus \Omega$ contains arbitrarily large balls. Γ, K_m and K are defined as follows. $\Gamma(x-y) = c_n|x-y|^{2-n}$ where $c_n = \frac{1}{(2-n)A_n}$ and A_n is the area of the $(n-1)$-dimensional unit sphere.

$$K_m^a(x,y) = \Gamma(x-y) - \sum_{|\alpha|<m} \frac{(y-a)^\alpha}{\alpha!}\Gamma^{(\alpha)}(x-a) \quad \left(\Gamma^\alpha(z) = \frac{\partial^\alpha}{\partial z^\alpha}\Gamma(z)\right),$$

$K^a(x,y) = K_3^a(x,y), \quad K(x,y) = K^0(x,y), \quad K_m(x,y) = K_m^0(x,y).$
$\partial_j K(x,y) = \dfrac{\partial}{\partial x_j} K(x,y).$

REMARK. *The kernel $K_m(x,y)$ satisfies:*

$$K_m(x,y) = O\left(\frac{|y|^m}{|x|^{m+n-2}}\right),$$

for $|x| > |y|$ and y fixed.

If $\Sigma \subset \mathrm{R}^n$ is any set and f, g are continuously differentiable functions in a neighbourhood of Σ we write $f \equiv g$ on Σ for $g = f$ and $\nabla g = \nabla f$ on Σ.

We define the *generalized Newtonian potential* of Ω with density g (where $g \in L^\infty(\mathrm{R}^n)$ and supp $g \not\equiv \mathrm{R}^n$) and with respect to $a \notin$ supp g by

$$U^{g,a} = \int K^a(x,y)\ g(x)\ dx\ .$$

If $a = 0$ we write $U^g = U^{g,0}$. When $g = \chi_\Omega$ (the characteristic function of Ω) we set $U^\Omega = U^g$. Observe also that U^g solves the problem $\Delta u = g$ in R^n, $g \in L^\infty(\mathrm{R}^n)$ with $u \in C^1(\mathrm{R}^n)$.

2. A NONSTANDARD APPROXIMATION THEOREM

In this section we recall an approximation theorem (due to L. Karp[5]) which makes it possible to handle unbounded QDs. In the sequel, for convenience, we will assume that $0 \notin \overline{\Omega}$.

THEOREM 2.1. *(L. Karp) Let H be the subset of $HL^1(\Omega)$ consisting of all finite linear combinations of $K(x,y)$, $\partial_j K(x,y)$ for $y \in \mathrm{R}^n \setminus \Omega$ and $j = 1, 2, ..., n$. Then H is dense in $HL^1(\Omega)$.*

REMARK. *If in the above theorem we replace H by S (i.e. the subset of $SL^1(\Omega)$ consisting of all finite linear combinations of $\pm K(x,y)$, $\pm\partial_j K(x,y)$ for $y \in \mathrm{R}^n \setminus \Omega$, $j = 1, 2, ..., n$ and $K(x,y)$ for $y \in \Omega$ with positive coefficients), then S is dense in $SL^1(\Omega)$.*

Using the above theorem we can prove the following.

COROLLARY 2.2. *Let Ω be an (L)-domain and define H_m to be the set of all finite linear combinations of $K_m(x,y)$, $\partial_j K_m(x,y)$ for $y \in \mathrm{R}^n \setminus \Omega$, $j = 1, 2, ..., n$ and $m > 3$. Then H_m is dense in $HL^1(\Omega)$.*

PROOF: Let F be a bounded linear functional on $L^1(\Omega)$ which is zero on H_m. Then by the Hahn-Banach theorem and theorem 2.1 it suffices to show that F is also zero on H_3 .

Let $g = g^F$ be the bounded measurable function corresponding to F and extend it to R^n by setting it equal to zero in $\mathrm{R}^n \setminus \Omega$.

Putting

$$N^g(y) = \int K_m(x,y)\ g(x)\ dx,$$

we'll have

$$N^g(y) = U^g(y) - \sum_\alpha b_\alpha \frac{y^\alpha}{\alpha!} \qquad (3 \le |\alpha| \le m-1),$$

where

$$b_\alpha = \int \Gamma^{(\alpha)}(x)\ g(x)\ dx\ .$$

Since $N^g = 0$ in $\mathrm{R}^n \setminus \Omega$ it is sufficient (in view of theorem 2.1) to show $b_\alpha = 0 \quad \forall\alpha$.
Let now $|\alpha| = m-1$ then $\partial^\alpha U^g(y) = b_\alpha$ for $y \in \mathrm{R}^n \setminus \overline{\Omega}$ and moreover

$$|\partial^\alpha U^g(y)| = |\int g(x)\ \partial^\alpha K(x,y)\ dx| \le C \int_\Omega |x-y|^{2-n-m}\ dx$$
$$\le C d_m(y) \int_\Omega |x-y|^{-n-1}\ dx,$$

where C is a constant and $d_m(y) = [dist(y, \partial\Omega)]^{3-m}$.
Thus

$$|b_\alpha| \le C d_m(y) \int_\Omega \frac{dx}{|x-y|^{n+1}} \qquad \forall y \in \mathrm{R}^n \setminus \overline{\Omega}.$$

By the assumption there exists a sequence $\{y^k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \int_\Omega \frac{dx}{|x-y^k|^{n+1}} = 0 \ .$$

Hence (by the definition of d_m) $d_m(y^k) \le$ const. for $k = 1, 2, ...$; and this in turn gives

$$|b_\alpha| \le C' \int_\Omega \frac{dx}{|x-y^k|^{n+1}} \qquad \forall k,$$

where C' is a positive constant. Letting $k \to \infty$ we will have $b_\alpha = 0 \ \forall \alpha :$ $|\alpha| = m-1$.

Similarly, we can show $b_\alpha = 0$ when $|\alpha| = m-2$. Proceeding in this way we can show that $b_\alpha = 0 \quad \forall \alpha : |\alpha| \ge 3$. Thus the theorem follows. ∎

Next we state a lemma due to L. Karp [5], this lemma will be used later.

LEMMA 2.3. *(L. Karp) Let $g \in L^\infty(\mathrm{R}^n)$ and $0 \notin supp\ g$, then U^g is in $C^1(\mathrm{R}^n)$ and satisfies*

$$U^g(y) = O(|y|^2 \log|y|),$$
$$\nabla|U^g(y)| = O(|y| \log|y|).$$

For a proof see [5].

REMARK. *In the above corollary we can omit the (L)-domain assumption but replace H_m by S_m (i.e. the subset of $SL^1(\Omega)$ consisting of all finite linear combinations of $\pm K_m(x,y)$, $\pm\partial_j K_m(x,y)$ for $y \in \mathrm{R}^n \setminus \Omega$, $j = 1, 2, ..., n$ and $K(x,y)$ for $y \in \Omega$ with positive coefficients) then S_m is dense in $SL^1(\Omega)$.*

To prove this it suffices, in view of the Hahn-Banach theorem, to show that every bounded linear functional on $L^1(\Omega)$ which is positive on S_m is also positive on $SL^1(\Omega)$. So let N^g be as in the proof of theorem 2.1, where g is a bounded function corresponding to a bounded linear functional which is positive on S_m. Hence $N^g \ge 0$ on R^n. Thus

$$N^g(y) = U^g(y) - \sum_\alpha b_\alpha \frac{y^\alpha}{\alpha!} \ge 0 \quad y \in \mathrm{R}^n \qquad (3 \le |\alpha| \le m-1),$$

and by lemma 2.3

$$\sum_{3\leq|\alpha|\leq m-1} b_\alpha \frac{y^\alpha}{\alpha!} \leq C(|y|^2+1)\log(|y|+2).$$

Thus by Liouville's theorem $b_\alpha = 0$, which is the desired conclusion.

3. SOME PROPERTIES OF QD

THEOREM 3.1. *Let Ω be given and suppose $\Omega_i \in Q(T_i, HL^1)$ $(i = 1, 2, ...)$ is a sequence of QDs with the following properties:*

(1) $\chi_{\Omega_i} \longrightarrow \chi_\Omega$, *pointwise ($\chi_\Omega$ is the characteristic function of Ω);*
(2) $\lim T_i =: T$ *exists in the sense of distributions;*
(3) T *is compactly supported in Ω;*
(4) $\overline{\cup \Omega_i} \not\equiv \mathrm{R}^n$.

Then $\Omega \in Q(T, HL^1)$, provided $\Omega \neq \emptyset$.

PROOF: Assume $0 \notin \overline{\cup \Omega_i}$ and define

$$\hat{T}(y) =< K(.,y), T > .$$

Then by theorem 2.1 it is sufficient to show

$$\int_\Omega K(x,y)\ dx = \hat{T}(y) \qquad \forall y \in \mathrm{R}^n \setminus \Omega\ . \tag{3-1}$$

Since both sides of (3-1) are in $C^1(\mathrm{R}^n \setminus \mathrm{supp}\ T)$ it suffices to prove (3-1) for $y \in \mathrm{R}^n \setminus \overline{\Omega}$. (This depends, partly, on the fact that we merely work with domains which are equal to the interior of their closure.)
Let now $y \in \mathrm{R}^n \setminus \overline{\Omega}$ be fixed and $\epsilon > 0$ be arbitrary. Then there is an i_ϵ such that

$$\begin{aligned} \int_{(\Omega \triangle \Omega_i)} |K(x,y)|\ dx < \epsilon \qquad & \forall i > i_\epsilon, \\ |\hat{T}(y) - \hat{T}_i(y)| < \epsilon \qquad & \forall i > i_\epsilon. \end{aligned} \tag{3-2}$$

For large i_ϵ $y \notin \Omega_i$ as $i \geq i_\epsilon$ and consequently

$$\int_{\Omega_i} K(x,y)\ dx = \hat{T}_i(y) \qquad \text{for} \qquad i \geq i_\epsilon.$$

This together with (3-2) implies

$$\begin{aligned} &|\int_\Omega K(x,y)\ dx - \hat{T}(y)| \\ &\leq \int_{(\Omega \triangle \Omega_i)} |K(x,y)| dx + |\hat{T}(y) - \hat{T}_i(y)| < 2\epsilon. \end{aligned}$$

Since ϵ was arbitrary we arrive at (3-1). ∎

REMARK. *If in the hypothesis of theorem 3.1,* $\Omega_i \in Q(0, SL^1)$ *(i=1,2,...),* $\blacksquare$ *then* $\Omega \in Q(0, SL^1)$. We leave the proof of this case to the reader.

Next we extend a theorem, theorem (4.2) in [7], to unbounded domains.

THEOREM 3.2. *Let* $\Omega \in Q(T, HL^1)$*, then the MSP* u *of* $\partial\Omega$ *with respect to* T *exists and satisfies* $|u| \le C(|x|^2)\log(|x|)$ *for some positive constant* C *and large* $|x|$.

Conversely, let Ω *be an (L)-domain whose MSP* u *exists and satisfies* $|u| \le C(|x|^m)$ *for some constant* C*, and large* $|x|$*. Set* $T = 1 - \Delta u$ *in* Ω *and assume supp* T *is compact in* Ω*. Then* $\Omega \in Q(T, HL^1)$ *and consequently* $m \le 2$*; if in addition* $m \le 3-\epsilon$ $(\epsilon > 0)$ *then the (L)-domain assumption can be relaxed.*

PROOF: Assume $0 \notin \overline{\Omega}$ and let $\Omega \in Q(T, HL^1)$. Set

$$u(y) = \int_\Omega K(x,y)\ dx\ - < K(.,y), T >= U^\Omega(y) - \hat{T}(y),$$

then by the assumption (i.e. $\Omega \in Q(T, HL^1)$) $u = 0$ in $\mathrm{R}^n \setminus \overline{\Omega}$. Since U^Ω and $\hat{T}$ are C^1 in $\mathrm{R}^n \setminus \mathrm{supp}\ T$ we conclude $u \equiv 0$ on $\partial\Omega$. Moreover, u (obvious) satisfies the distribution identity

$$\Delta u = 1 - T \qquad \text{in } \Omega,$$

and the growth hypothesis follows from lemma 2.3.

The converse is an easy consequence of Green's identity and corollary 2.2 (or, theorem 2.1 when $m \le 3-\epsilon$). For details of this for the bounded case we refer to ([7]; theorem (4.2)). $\blacksquare$

Our next result concerns uniform boundedness of QDs for a fixed distribution. This was first observed by Sakai ([9]; page 51) and then by Gustafsson ([3]; proposition 3.5). Our technique is much easier and very flexible in the sense that it can be applied to "generalized" QDs and obstacle type free boundary problems. For this we state a lemma due to L.A. Caffarelli which can be found in ([1]; page 154 lemma 3.1). We state it in terms of MSP.

LEMMA 3.3. *Let* Ω *be a QD with* u *as its MSP. Then for every point* $x^0 \in \partial\Omega$ *the following holds:*

$$\text{(3-3)} \qquad \sup_{x \in (\Omega \cap \partial B_r(x^0))} [u(x)] \ge \frac{r^2}{2n},$$

for any r such that $B_r(x^0)$ does not meet supp $(\Delta u - \chi_\Omega)$.

PROOF: Let $x^0 \in \partial\Omega$ be fixed and take a sequence $\{x^j\}$ in Ω such that $x^j \to x^0$ and $u(x^j) \geq 0$. The function

$$w(x) = u(x) - u(x^j) - \frac{1}{2n}|x - x^j|^2$$

is harmonic in $\Omega \cap B_r(x^j)$, $w(x^j) = 0$ and $w < 0$ on $\partial\Omega \cap B_r(x^j)$ (if it is not empty). Therefore by the maximum principle the maximum value of w in $\Omega \cap B_r(x^j)$ is attained at the boundary of $B_r(x^j)$ and this maximum value is positive, i.e.

$$\sup_{x \in (\Omega \cap \partial B_r(x^j))} w(x) > 0.$$

Letting $x^j \to x^0$ we obtain (3-3).

THEOREM 3.4. *Bounded domains in $Q(T, HL^1)$ are uniformly bounded.*

PROOF: Let $\Omega \in Q(T, HL^1)$ and u its MSP. Fix a ball B containing supp T and assume $\Omega \setminus B$ is not empty. Since u is subharmonic in $\Omega \setminus B$ $u = 0$ on $\partial\Omega$ and $\sup u > 0$ on $\Omega \setminus B$ (by (3-3)) u admits its maximum value on $(\partial B) \cap \Omega$. Therefore, in view of lemma 3.3,

$$\sup_{x \in \partial\Omega} dist(x, B) \leq \left(2n \sup_{\partial B} u\right)^{1/2} \leq \text{const.(independent of } \Omega),$$

since $u = U^\Omega - \hat{T}$ (by theorem 3.2) and $|\Omega| =< 1, T >$. The theorem is proved. ∎

4. NQD

In this section we study NQDs in R^n. We first recall a theorem due to M. Sakai [8].

THEOREM 4.1. *(Sakai). Let Ω be a domain in R^2. Then $\Omega \in Q(0, HL^1)$ if and only if Ω is the exterior of an ellipse, a ball, the non-convex component of the complement of a parabola or a half-plane.*

For a proof of this theorem see [8].
Since the complements of all the domains in theorem 4.1 are limit domains of an increasing sequence of ellipses, we can reformulate the theorem as follows:

THEOREM 4.1$'$. *Let $\Omega \subset \mathrm{R}^2$. Then $\Omega \in Q(0, HL^1)$ if and only if there exists an increasing sequence of ellipses E_i such that:*

$$\Omega = \mathrm{R}^n \setminus \cup E_i.$$

Here we want to extend the "if" part of this theorem to R^n. To do this we need the following theorem, which was first proved in [2]. Also, cf. [7].

THEOREM 4.2. *Let E be an ellipsoid in R^n. Then*

$$\mathrm{R}^n \setminus \overline{E} =: \Omega \in Q(0, HL^1) \ .$$

For the reader's convenience we outline a simple proof.
PROOF: Assume $0 \in int(\overline{E}) = \mathrm{R}^n \setminus \overline{\Omega}$. By theorem 2.1 it suffices to show

$$\int_\Omega K(x,y)\ dx = 0 \qquad\qquad \forall y \in \mathrm{R}^n \setminus \Omega \ .$$

Define E^R to be the ellipsoidal homoeoid with the cavity E i.e. $E^R = RE \setminus \overline{E}$ where $R > 1$.
Let now R be sufficiently large so that for a given $\epsilon > 0$ we have

$$\int_{\Omega\setminus E^R} |K(x,y)| dx < \epsilon, \qquad\qquad \text{for a fixed } \ y \in E.$$

This implies

$$|\int_\Omega K(x,y)\ dx| < \epsilon + |\int_{E^R} K(x,y)\ dx| \ . \tag{4-1}$$

Now we claim

$$\text{(4-2)} \qquad \int_{E^R} K(x,y)\ dx = 0 \qquad \forall y \in E,$$

which together with (4-1), and the fact that ϵ was arbitrary implies

$$\int_{\Omega} K(x,y)\ dx = 0 \qquad \text{in } \ E.$$

Define

$$\hat{E}^R(y) = \int_{E^R} \Gamma(x-y)\ dx\ .$$

Now by a theorem of Newton (for $n = 3$ see [6], page 22) we have

$$\text{(4-3)} \qquad \hat{E}^R(y) = \text{constant in } E.$$

Since

$$K(x,y) = K_3(x,y) = \Gamma(x-y) - \sum_{|\alpha|<3} \Gamma^{(\alpha)}(x)\frac{y^\alpha}{\alpha!},$$

we have

$$\int_{E^R} K(x,y)\ dx = \hat{E}^R(y) - \sum_{|\alpha|<3} \frac{\partial^\alpha}{\partial y^\alpha}\hat{E}^R(0)\frac{y^\alpha}{\alpha!}\ .$$

By (4-3) $\dfrac{\partial^\alpha}{\partial y^\alpha}\hat{E}^R(y) = 0$ for $y \in E$ and $|\alpha| > 0$, and in particular when $y = 0$.
Thus

$$\int_{E^R} K(x,y)\ dx = \hat{E}^R(y) - \hat{E}^R(0) = 0.$$

∎

COROLLARY 4.3. *Let* $\{E_i\}_{i=1}^\infty$ *be an increasing sequence of ellipsoids in* R^n. *Then* $\mathrm{R}^n \setminus \cup E_i \in Q(0, HL^1)$.

PROOF: The proof is an immediate consequence of theorem 3.1 and theorem 4.2. ∎

REMARK. *Since a half-space, the exterior of a cylinder, the non-convex component of the complement of a paraboloid and the exterior of a ball are limits of (decreasing) sequences of the exteriors of ellipsoids,* *those domains are also NQDs. As there are no other examples of NQDs we suggest the following conjecture.*

CONJECTURE.
Let $\Omega \in Q(0, HL^1)$. Then there exists an increasing sequence $\{E_i\}$ of ellipsoids such that

$$\Omega = \mathrm{R}^n \setminus \cup E_i .$$

So far, we have been working with HL^1 but one could ask whether or not $\mathrm{R}^n \setminus \overline{E} \in Q(0, SL^1)$.
In the following we will give an affirmative answer to this question.

THEOREM 4.4. *Let E be an ellipsoid in R^n, then*

$$\mathrm{R}^n \setminus \overline{E} \in Q(0, SL^1) \ .$$

PROOF: Let $E = \left\{ x : \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} \leq 1 \right\}$. In view of theorem 4.2 and the remark following theorem 2.1, it suffices to show that

$$v(y) := \int_{\mathrm{R}^n \setminus E} K(x, y)\ dx \geq 0 \qquad\qquad \text{for } y \in \mathrm{R}^n \setminus E.$$

Set

$$\hat{E}(y) := \int_E \Gamma(x - y)\ dx \ ,$$

then $\hat{E}$ is equal to a quadratic polynomial in the interior of E (see [11]). In fact

$$\hat{E}(y) = b_n \int_0^\infty \frac{ds}{\varphi(s)} (1 - \sum_{j=1}^{n} \frac{y_j^2}{a_j^2 + s}) \qquad y \in E,$$

and

$$\hat{E}(y) = b_n \int_\lambda^\infty \frac{ds}{\varphi(s)} (1 - \sum_{j=1}^{n} \frac{y_j^2}{a_j^2 + s}) \qquad y \in \mathrm{R}^n \setminus E,$$

where λ is the largest of the ellipsoidal coordinates of y, i.e. the positive root of the equation $\sum_{j=1}^{n} \frac{y_j^2}{a_j^2 + \lambda} = 1$, $\varphi^2(s) = (a_1^2 + s) \ldots (a_n^2 + s)$ and

$$b_n = -\frac{a_1 a_2 \ldots a_n}{4} < 0 \ .$$

Let Q be the polynomial which is identical with $\hat{E}$ in the interior of E, and for $y \in \mathrm{R}^n \setminus E$ set

$$u(y) = Q(y) - \hat{E}(y).$$

Then u and v satisfy

$$\text{(4-4)} \qquad \begin{cases} \Delta u = \Delta v = 1 & \text{in } \mathrm{R}^n \setminus E, \\ u \equiv \quad v \equiv 0 & \text{on } \partial E. \end{cases}$$

(Recall that $v = 0$ in $\mathrm{R}^n \setminus E$ by theorem 4.2 and $v \in C^1(\mathrm{R}^n)$). Hence

$$u = v \qquad\qquad \text{in } \mathrm{R}^n \setminus E.$$

Therefore, to prove the theorem it is enough to show that u is non-negative, in the complement of E. By the definition of u and Q we have, for $y \in \mathrm{R}^n \setminus E$,

$$\begin{aligned} u(y) &= Q(y) - \hat{E}(y) \\ &= b_n \int_0^\infty \frac{ds}{\varphi(s)} (1 - \sum_{j=1}^n \frac{y_j^2}{a_j^2 + s}) - b_n \int_\lambda^\infty \frac{ds}{\varphi(s)} (1 - \sum_{j=1}^n \frac{y_j^2}{a_j^2 + s}) \\ &= b_n \int_0^\lambda \frac{ds}{\varphi(s)} (1 - \sum_{j=1}^n \frac{y_j^2}{a_j^2 + s}) \geq 0, \end{aligned}$$

which completes the proof of the theorem. ∎

COROLLARY 4.5. *Let $\{E_i\}_{i=1}^\infty$ be an increasing sequence of ellipsoids in* R^n. *Then*

$$\mathrm{R}^n \setminus \overline{\cup E_i} \qquad \in \qquad Q(0, SL^1).$$

Next we characterize NQDs in terms of the generalized Newtonian potential. Let $0 \notin \overline{\Omega}$, then:

COROLLARY 4.6.

$$\Omega \in Q(0, HL^1) \qquad \Longleftrightarrow \qquad U^\Omega = 0 \qquad \textit{in } \Omega,$$

and

$$\Omega \in Q(0, SL^1) \qquad \Longleftrightarrow \qquad \begin{cases} U^\Omega \geq 0 & \textit{in } \mathrm{R}^n \\ U^\Omega = 0 & \textit{in } \mathrm{R}^n \setminus \Omega. \end{cases}$$

In [8] M. Sakai shows that in R^2 the following is true

$$\text{(4-5)} \qquad Q(0, SL^1) = Q(0, HL^1).$$

As to the case R^n $(n \geq 3)$, it is known (see [14]) that if

(1) $\Omega \in Q(0, HL^1)$,
(2) the MSP u of Ω extends to a neighbourhood of each point in $\partial\Omega$ as a C^2 function,
(3) $\partial^\alpha u \in L^\infty(\Omega)$ for all α, $|\alpha| = 2$,
(4) $\mathrm{R}^n \setminus \Omega$ is "sufficiently thick" at infinity so that the generalized maximum principle holds in Ω for harmonic functions,

then $u \geq 0$. This implies (see next theorem) that $\Omega \in Q(0, SL^1)$.

COROLLARY 4.7. *Let Ω be such that the MSP u of $\partial\Omega$ exists in Ω. Then,*

$$u \geq 0 \qquad \text{and} \qquad \Delta u = 1 \quad \text{in } \Omega \qquad \Longleftrightarrow \qquad \Omega \in Q(0, SL^1).$$

PROOF: Let us assume (without loss of generality) that $0 \notin \overline{\Omega}$. Assume $u \geq 0$ and define w to be the C^1 continuation of u to R^n which is identically zero in the complement of Ω. Then

$$\int_{\Omega_R} K(x,y)\ dx = \int_{\Omega_R} \Delta w\ K(x,y)\ dx = w(y) + I_R(y), \tag{4-6}$$

where $y \in B(0,R)$ and

$$I_R(y) = \int_{\partial B_R \cap \Omega} \left(\frac{\partial w}{\partial n_x}(x)\ K(x,y) - w(x)\ \frac{\partial K}{\partial n_x}(x,y) \right)\ d\sigma_x. \tag{4-7}$$

Set

$$I(y) := \lim_{R\to\infty} I_R(y) = \lim_{R\to\infty} U^{\Omega_R}(y) - w(y) = U^{\Omega}(y) - w(y).$$

Then by lemma (2.3) and the positivity of w we obtain $I(y) \leq C(|y|^2 + 1)\log(|y|+2)$. It is also obvious that $\Delta I = 0$ in R^n, hence by Liouville's theorem I is a second degree polynomial. Consequently,

$$w(y) = O((|y|^2+1)\log(|y|+2))$$

in the interior of Ω. Hence by (4-7) and the remark (concerning the growth of K) in section 1 I_R tends to zero and by (4-6) (as R tends to infinity):

$$\int_{\Omega} K(x,y)\ dx = w(y) \geq 0 \qquad \text{in } \mathrm{R}^n.$$

This together with the remark following theorem 2.1 implies that $\Omega \in Q(0, SL^1)$. The converse is obvious. ∎

Next we conjecture the following.

CONJECTURE.
Let $\Omega \in Q(0, HL^1)$ and let u be the MSP of $\partial\Omega$. Then $u \geq 0$.

If this conjecture is true then $Q(0, HL^1) = Q(0, SL^1)$ in R^n.

THEOREM 4.8. *Let Ω be such that the MSP u of $\partial\Omega$ is a polynomial. Then $\partial\Omega$ is a hyperplane.*

The proof is based on the following lemma.

LEMMA 4.9. *Both Ω and $\mathrm{R}^n \setminus \Omega$ in theorem 4.8 contain arbitrarily large balls.*

PROOF: Let m be the exact degree of u and suppose first that neither of the domains contain arbitrarily large balls. Then there exists $r > 0$ such that $B(x, r)$ meets $\partial\Omega$ for every $x \in \mathrm{R}^n$. Let now $\xi = \xi(x) \in \partial\Omega$ be the nearest point to x, then by the mean value theorem

$$u(x) = u(x) - u(\xi) = O(|x - \xi|.|\nabla u(\eta)|) = O(|\nabla u(\eta)|) = O(|x|^{m-1}),$$

where $\eta = tx + (1 - t)\xi$ for some t: $0 \leq t \leq 1$ and $|x|$ is large. But this contradicts the fact that u is of degree m.

Thus, either Ω or $\mathrm{R}^n \setminus \Omega$ contains arbitrarily large balls. Assume $\mathrm{R}^n \setminus \Omega$ contains arbitrarily large balls i.e. Ω is an (L)-domain. Hence by theorem 3.2 Ω is a NQD. Now by ([12]; theorem 1.6) Ω contains arbitrarily large balls, which completes the proof of the lemma. ∎

PROOF OF THEOREM (4.8): By lemma 4.9 both Ω and $\mathrm{R}^n \setminus \Omega$ contain arbitrarily large balls, therefore by theorem 3.2 both Ω and $\mathrm{R}^n \setminus \Omega$ are NQDs and $u(x) \leq C(|x|^2 + 1) \log(|x| + 2)$. Thus u is a polynomial of degree 2. Since $\Delta u = 1$ we conclude that for some i, $\partial_i u$ is a first degree polynomial, not identically zero, which is zero on $\partial\Omega$. Hence $\partial\Omega$ is a hyperplane. ∎

5. POSITIVITY OF THE MSP AND THE UNIQUENESS PROBLEM

Let $\Omega \in Q(T, HL^1)$, where T is a distribution compactly supported in Ω. Denote by u the MSP of Ω (this exists according to theorem 3.2). It is known that if $u \geq 0$ and Ω is solid (without cavities), then Ω is the unique domain in $Q(T, HL^1)$. For a proof of this we refer to [10]. Also, cf. [3]. It is an open question whether $Q(T, HL^1)$ contains a unique solid domain. Our aim here is to prove uniqueness without assuming the positivity of the MSP. Unfortunately, we have to assume that the domains Ω_j $j = 1, 2$ satisfy the internally tangent ball condition i.e. for any $x\partial\Omega_j$ there is a small open ball $B \subset \Omega_j$ such that $x \in \partial B$. Therefore from now on we assume that all domains in this section (except for those in theorem 5.5) satisfy the internally tangent ball condition. We will also make use of the following notation.
For domains Ω_j $(j = 1, 2)$ we set $\Gamma^j = \partial\Omega_j \cap \Omega_i$ where of course $i \neq j$.

THEOREM 5.1. *Let Ω_j $j = 1, 2$ be bounded domains in $Q(T, HL^1)$. Let u_j denote the MSP of Ω_j $(j = 1, 2)$ and suppose*

$$\sup_{\Gamma^2} u_1 + \inf_{\Gamma^1} u_2 \geq 0.$$

Then

$$|\Omega_1 \mathrm{M} \Omega_2| = 0.$$

PROOF: Since $\Omega_j \in Q(T, HL^1)$ for $j = 1, 2$ we have

$$\int_{\Omega_1} h \, dx = \int_{\Omega_2} h \, dx \qquad \forall h \in HL^1(\Omega_1) \cap HL^1(\Omega_2),$$

and in particular for $h = 1$. Hence $|\Omega_1| = |\Omega_2|$. Now suppose $|\Omega_1 \mathrm{M} \Omega_2| \neq 0$, then $\Omega_1 \setminus \Omega_2$ is not empty. Set $u = u_1 - u_2$ then u is harmonic in $\Omega_1 \cap \Omega_2$ and consequently it attains its maximum on the boundary $\partial(\Omega_1 \cap \Omega_2)$. By the assumption u attains its maximum on Γ^2, say at the point x (observe that $x \notin \Gamma^1$). Then by Hopf's maximum principle [4] the derivative of u at that point in any outward direction is positive, and since u_2 has vanishing gradient on Γ^2 we arrive at

$$\frac{\partial u_1}{\partial \nu}(x) > 0,$$

where ν is any (fixed) outward vector to $\partial\Omega_2$ at x. Now u_1, being a subharmonic function in $\Omega_1 \setminus \Omega_2$, attains its maximum value on $\Omega_1 \setminus \Omega_2$ at $\partial(\Omega_1 \setminus \Omega_2)$, hence at x. Therefore

$$\frac{\partial u_1}{\partial \nu}(x) \leq 0,$$

which is a contradiction. ∎

REMARK. *It follows, from the proof of theorem 5.1, that if* Ω *and* D *are two different domains in* $Q(T, HL^1)$*, then the gradient of the MSP of* Ω *has to vanish at some point in* $\Omega \setminus D$*.* This was also obtained by Sakai [9] and Gustafsson [3].

COROLLARY 5.2. *Let* Ω_j *for* $j = 1, 2$ *be as in theorem 5.1 and suppose* $\sup_{\Gamma^2} u_1 \geq 0$*. Then* Γ^1 *meets* $\{x \in \Omega : u_2(x) < 0\}$*. Consequently if* $u_2 \geq 0$*, then* $|\Omega \mathrm{M} D| = 0$*.*

THEOREM 5.3. *Let* $\Omega_j \in Q(T, HL^1)$ *for* $j = 1, 2$ *be bounded domains. Denote by* u_j *the MSP of* Ω_j *for* $j = 1, 2$ *and assume* $|\Omega_1 \mathrm{M} \Omega_2| \neq 0$*. Then, at least, one of the following is true:*

$$- \inf_{\Gamma^2} u_1 > \sup_{\Gamma^2} u_1,$$

$$- \inf_{\Gamma^1} u_2 > \sup_{\Gamma^1} u_2.$$

PROOF: Let $u = (u_1 - u_2)^2$ and assume neither of the conclusions stated in the theorem is true. Then u, being subharmonic in $\Omega_1 \cap \Omega_2$, admits its maximum value at x, say, on the boundary $\partial(\Omega_1 \cap \Omega_2)$ and since u is positive $x \notin \Gamma^1 \cap \Gamma^2$. So let $x \in \Gamma^1$. Then by the Hopf's maximum principle [4] the outward normal derivative of u at x is positive. Since u_1 has vanishing Cauchy-data this normal derivative at x coincides with that of $(u_2)^2$. Hence,

$$\frac{\partial (u_2)^2}{\partial \nu}(x) > 0,$$

where ν is any (fixed) outward vector to Ω_1 at x. Since $\sup_{\Gamma^1} u_2 > 0$, and (by supposition) $\sup_{\Gamma^1} u_2 > -\inf_{\Gamma^1} u_2$, $\sup_{\Gamma^1} u = (\sup_{\Gamma^1} u_2)^2$ and hence at x we have

$$\frac{\partial (u_2)^2}{\partial \nu}(x) = 2u_2(x) \frac{\partial u_2}{\partial \nu}(x) > 0,$$

while $u_2(x) > 0$. This implies $\dfrac{\partial u_2}{\partial \nu}(x) > 0$. Therefore, applying the maximum principle to u_2 in $\Omega_2 \setminus \Omega_1$, we reach a contradiction. ∎

Both theorem 5.1 and 5.3 indicate that in order to have non-uniqueness, the boundary of each domain has to encounter points where the MSP of the other domain takes negative values.

COROLLARY 5.4. *Let Ω_j and u_j for $j = 1, 2$ be as in theorem 5.3 and assume moreover that the domains are solid. Denote by $r(x)$ the radius of the ball $B(x)$ with center x and such that $B(x) \subset \mathrm{R}^n \setminus (\Omega_1 \cap \Omega_2)$. Then*

$$-\inf_{\Gamma^1} u_2(y) > \sup_{x \in (\partial\Omega_2)\setminus\Omega_1} \frac{r^2(x)}{2n},$$

or

$$-\inf_{\Gamma^2} u_1(y) > \sup_{x \in (\partial\Omega_1)\setminus\Omega_2} \frac{r^2(x)}{2n},$$

or both.

PROOF: The corollary follows from theorem 5.3 in conjunction with lemma 3.3. ∎

This corollary indicates that two solids having the same exterior gravitational potential cannot be too different in shape. This is emphasized by the next theorem. Before proceeding we want to make a slight change in the definition of a QD. Namely, here we will allow the support of the corresponding distribution for a QD to meet the boundary of the domain. To be more precise, supp $T \cap \partial\Omega$ may be non-empty. We do, however, require that supp T is bounded.

THEOREM 5.5. *Let Ω be an unbounded domain with a complement which has non-empty interior, and assume moreover*

$$\int_\Omega \frac{dx}{|x|^n + 1} < \infty. \tag{5.1}$$

Then Ω is unique in the following sense: there exists no other domain D satisfying: $\overline{D} \cap \overline{\Omega}$ is bounded and $U^D = U^\Omega$ in $\mathrm{R}^n \setminus (D \cup \Omega)$.

PROOF: Suppose there is two such domains D and Ω. Let

$$u = \begin{cases} U^\Omega - U^D & \text{in } \Omega \setminus D \\ 0, & \text{elsewhere} . \end{cases}$$

Set now $T = \chi_\Omega - \Delta u$. Then supp $T \subset \overline{\Omega} \cap \overline{D}$ and $U^\Omega = \hat{T}$ on $\mathrm{R}^n \setminus \Omega$. Thus $\Omega \in Q(T, HL^1)$. Now in view of ([12]; theorem 1.4) Ω is bounded, which is a contradiction. ∎

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