REGULARITY OF A FREE BOUNDARY WITH APPLICATION TO THE POMPEIU PROBLEM

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Abstract. In the unit ball $B(0,1)$, let u and Ω (a domain in $\mathbb{R}^N$) solve the following overdetermined problem:

$$\Delta u = \chi_\Omega \quad \text{in } B(0,1), \qquad 0 \in \partial\Omega, \qquad u = |\nabla u| = 0 \quad \text{in } B(0,1)\setminus\Omega,$$

where χ_Ω denotes the characteristic function, and the equation is satisfied in the sense of distributions.

If the complement of Ω does not develop cusp singularities at the origin then we prove $\partial\Omega$ is analytic in some small neighborhood of the origin. The result can be modified to yield for more general divergence form operators. As an application of this, then, we obtain the regularity of the boundary of a domain without the Pompeiu property, provided its complement has no cusp singularities.

1. Introduction

In this paper we study the regularity properties of solutions to a certain type of free boundary problems, resembling the obstacle problem but with no sign assumption, i.e., with no obstacle. Mathematically the problem is formulated as follows. Let $\Omega \subset \mathbb{R}^N$ and suppose there is a function u, solving the following overdetermined problem

$$\Delta u = \chi_\Omega \ \text{ in } B, \qquad u = |\nabla u| = 0 \ \text{ in } B\setminus\Omega, \tag{1.1}$$

where B is the unit ball.

The question we ask is whether $\partial\Omega$ is smooth. Indeed, if $\partial\Omega$ is an analytic surface then by Cauchy-Kowalewski theorem, we can always solve the above overdetermined problem locally. We thus ask the reverse of the Cauchy-Kowalewski theorem.

The problem has also a potential theoretic interpretation which is as follows. Denote by U the Newtonian potential of Ω (bounded set) with constant density (i.e.,

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the convolution of the fundamental solution with χ_Ω) and with $0 \in \partial\Omega$. Suppose there exists a harmonic function w in $B(0,r)$ (r small) such that $w = U$ in Ω^c (the complement of Ω); observe that U is harmonic in Ω^c. This property is called harmonic continuation.

Next, up to a normalization constant, $w - U$ satisfies equation (1.1). Once again the question is whether possession of such a property, harmonic continuation, for Ω will result in the regularity of $\partial\Omega$ near the origin. For the interested reader we refer to [Sa1–3] for similar type of problems.

It is noteworthy that this kind of problem has been in focus of attention in mathematical physic, specially in geophysics [St], [Ma], and in inverse potential theory [I1]. Also newly developed problems in operator theory reduce the analysis of the spectrum of certain hyponormal operators to the study of the solutions in problem (1.1), in the complex plane [MY], [P]. We also refer to an excellent book by H.S. Shapiro [Shap2], for some basic aspects.

In order to state our main results, let us define a local solution.

Definition 1.1. We say a function u belongs to the class $P_r(z,M)$ if u satisfies (in the sense of distributions):

(1) $\Delta u = \chi_\Omega \quad$ in $B_r(z)$,
(2) $u = |\nabla u| = 0$ in $B_r(z) \setminus \Omega$,
(3) $\|u\|_{\infty, B_r(z)} \leq M$,
(4) $z \in \partial\Omega$.

We also denote by $P_\infty(0,M)$ "global solutions" with quadratic growth, i.e., solutions in the entire space $\mathbb{R}^N$ with quadratic growth $|u(x)| \leq M(|x|^2 + 1)$.

Remark. If $u \in P_r(z,M)$, then

(1) $u(z+x) \in P_r(0,M)$,
(2) $u(z+rx)/r^2 \in P_1(0, M/r^2)$,
(3) Also if $\|D_{ij}u\| \leq M$, then $u(z+rx)/r^2 \in P_1(0,M)$.

Obviously (1) implies that the class is point independent, so we can always consider the class $P_r(0,M)$. Our class P differs from that of [Ca2] in two ways. We don't restrict the function u to be non-negative and we replace the uniform $C^{1,1}$-norm with a uniform C^0-norm.

This new feature introduces new difficulties, and as a first task we have to cope with the optimal regularity of the function u itself before attacking the regularity problem for the boundary $\partial\Omega$. The main tools in our study of problem (1.1) will be a monotonicity lemma due to [ACF], see Lemma 2.1 below.

Definition 1.2. (Minimal diameter) The minimum diameter of a bounded set D, denoted $\mathrm{MD}(D)$, is the infimum of distances between pairs of parallel planes such that D is contained in the strip determined by the planes. We also define the density function

$$\delta_r(u) = \frac{\mathrm{MD}(\{u = |\nabla u| = 0\} \cap B(0,r))}{r}.$$

Now we can state our main results. In Section 7, we apply Theorems I–III to obtain the regularity of a domain without the Pompeiu property (see Section 7 for details).

Theorem I. *There is a constant* $C_1 = C_1(N)$ *such that if* $u \in P_1(z, M)$ *then*

$$\sup_{B(z,1/2)} \|D_{ij}u\| \leq C_1 M.$$

Theorem II. *Let* $u \in P_\infty(0, M)$ *and suppose* $\{u = 0\}$ *has non-empty interior, or* $\delta_r(u) > 0$ *for some* $r > 0$. *Then* $u \geq 0$ *in* $\mathbb{R}^N$ *and* $D_{ee}u \geq 0$ *in* Ω, *for any direction* e, *i.e.* Ω^c *is convex. Moreover if* $\overline{\lim}_{R\to\infty} \delta_R(u) > 0$ *then* u *is a half-space solution, i.e.,* $u = (\max(x_1, 0))^2/2$ *in some coordinate system.*

Theorem III. *There exists a modulus of continuity* σ $(\sigma(0^+) = 0)$ *such that if* $u \in P_1(0, M)$ *and* $\delta_{r_0}(u) > \sigma(r_0)$ *for some* $r_0 < 1$, *then* $\partial\Omega$ *is the graph of a* C^1 *function in* $B(0, c_0 r_0^2)$. *Here* c_0 *is a universal constant, depending only on* M *and dimension.*

Subject to the condition in Theorem III, the analyticity of the free boundary now follows by classical results [KN], [I2]. We thus have the following corollary.

Corollary to Theorem III. *Under the thickness condition in Theorem III, the free boundary in (1.1) is analytic, in some neighborhood of the origin.*

Theorems I–III are known in 2-space dimensions [Sa2-3]. Indeed, M. Sakai [Sa2-3] gives a complete description of the boundaries of all such domains in $\mathbb{R}^2$. Recently a different approach to this problem was made by B. Gustafsson and M. Putinar [GP], where they proved that $\partial\Omega$ in (1.1) is contained in an analytic arc in $\mathbb{R}^2$. This question is still open in higher dimensions. For the case $u \geq 0$ (the original obstacle problem) the first author has recently proved the following:

Structure of the singular set. *[Ca4] Let* $u \in P_1(0, M)$ *and* $u \geq 0$. *Let also* y *be a singular point of the free boundary in (1.1), i.e., the free boundary does not satisfy the condition in Theorem III, near* y. *Then there exists a unique non-negative quadratic polynomial (and a unique matrix* A_y*)*

$$Q_y = \frac{1}{2}x^T A_y x$$

with trace$A_y = 1$ *and such that*

1. $|(u - Q_y)(x)| \leq |x - y|^2 \sigma(|x - y|)$, *for some universal modulus of continuity* σ, *depending on* M *only.*
2. A_y *is continuous on* y.
3. *If* dim(kernel(A_y)) $= k$ *then there exists a* k*-dimensional* C^1 *manifold* $\mathcal{T}_{y,u}$, *such that*

$$S_u \cap B(y, r) \subset \mathcal{T}_{y,u},$$

for some small r. *Here* S_u *indicates the singular points of the free boundary, i.e., points which don't fall under the hypothesis of Theorem III.*

This fact with no positivity assumption, is studied in a forthcoming paper by the first and the third authors. In this paper, however, we will prove the analyticity of the free boundary only in the case of "thick" complement described in Theorem III.

Plan of the paper. Section 2 is devoted to some technical tools, which are known, but probably not well known in the context used in this paper. In Section 3 we carry out the proof for Theorem I, using the monotonicity lemma (Lemma 2.1) and the blow-up technique.

In Section 4 we introduce further lemmas, which will somehow exhaust properties of the monotonicity lemma. These are used to prove Theorems II–III in Sections 5–6, respectively. In Section 7 we generalize the monotonicity lemma to yield for divergence type operators

$$\sum D_i(a_{ij}D_ju) + a(x)u.$$

As a result we obtain the regularity of a domain without the Pompeiu property, which we explain to some extent in Section 7.

2. The Monotonicity Formula

In this section we will gather all basic tools, used to prove Theorems I–III. A fundamental tool, however, is the following monotonicity lemma.

Lemma 2.1. *[ACF] Let h_1, h_2 be two non-negative continuous sub-solutions of $\Delta u = 0$ in $B(x^0, R)$ $(R > 0)$. Assume further that $h_1h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$. Then the following function is monotone in r $(0 < r < R)$*

$$\varphi(r) = \frac{1}{r^4}\left(\int_{B(x^0,r)} \frac{|\nabla h_1|^2}{|x-x^0|^{N-2}}\right)\left(\int_{B(x^0,r)} \frac{|\nabla h_2|^2}{|x-x^0|^{N-2}}\right). \tag{2.1}$$

For a fixed direction e, set

$$(D_eu)^+ = \max(D_eu, 0), \qquad (D_eu)^- = -\min(D_eu, 0).$$

Then we have the following lemma.

Lemma 2.2. *Let $u \in P_1(0, M)$ and consider the monotonicity formula (2.1) for the non-negative subharmonic functions $(D_eu)^\pm$ where e is any fixed direction, and denote this by $\varphi(r, D_eu)$. Then*

$$\varphi'(r, D_eu) \geq \frac{2}{r}\varphi(r, D_eu)(\gamma(\Gamma_+) + \gamma(\Gamma_-) - 2), \tag{2.2}$$

where

$$\gamma(\Gamma_\pm)(\gamma(\Gamma_\pm) + N - 2) = \lambda(\Gamma_\pm), \tag{2.3}$$

with

$$\lambda(\Gamma_\pm) = \inf\left(\frac{\int_{\Gamma_\pm} |\nabla' w|^2 d\sigma}{\int_{\Gamma_\pm} |w|^2 d\sigma}\right). \tag{2.4}$$

Here, ∇' is the gradient on $\mathbb{S}^{N-1}$, $\Gamma_\pm = \Gamma_\pm(r)$ is the projection of $\partial B(0, r) \cap \{(D_eu)^\pm > 0\}$ onto the unit sphere, and the infimum has been taken over all non-zero functions with compact support in $\Gamma_\pm$.

For a proof of this lemma see (the proof of) Lemma 5.1 in [ACF]. The "set function" $\gamma(E)$ as a function of $E \subset \mathbb{S}^{N-1}$ is called the characteristic constant of

E; see e.g. the paper by Friedland and Hayman [FH]. A result of Sperner [Spn] states that, among all sets with given $(N-1)$-dimensional surface area $s\omega_N$ on the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$, a spherical cap, i.e. a set of the form $-1 \le c < x_1 \le 1$, has the smallest characteristic constant $\gamma(s, N)$, where ω_N is the area of the unit sphere in $\mathbb{R}^N$, $0 < s < 1$, and c and s are coupled by the relations

$$s = \frac{\omega_{N-1}}{\omega_N}\int_0^{\theta_0} (\sin t)^{N-2} dt, \qquad c = \cos\theta_0, \qquad 0 < \theta_0 < \pi.$$

It thus suffices to consider the function γ as a function of $s = area(E)/\omega_N$, with s as above. From this [FH; Theorem 2] deduce that for fixed s, γ is a monotone decreasing function in N (the space dimension), and that the limit exists as N tends to infinity

$$\gamma(s, N) \ge \gamma(s, \infty).$$

On the other side as $N \to \infty$ we'll have, for some $h \ge -\infty$,

$$s = \frac{1}{\sqrt{2\pi}}\int_h^\infty \exp\left(-t^2/2\right) dt, \tag{2.5}$$

see e.g. [FH, page 149].

Hence for any set $E \in \mathbb{S}^{N-1}$ and s as above

$$\gamma(E, N) \ge \gamma(E^\star, N) = \gamma(s, N) \ge \lim_{N\to\infty} \gamma(s, N) = \gamma(s, \infty), \tag{2.6}$$

where $E^\star$ is the above described symmetrization of E.

Now from (2.3) (cf. [ACF; page 441]) one has

$$\gamma = \frac{\lambda}{(N-2)} + O(N^{-3}). \tag{2.7}$$

In (2.7) if we let N tend to infinity we'll obtain, by some tedious calculations (cf. [FH page 149]),

$$\frac{\lambda}{(N-2)} + O(N^{-3}) \to \Lambda, \tag{2.8}$$

where $-\Lambda$ is the first Dirichlet eigenvalue of the one-dimensional Ornstein-Uhlenbeck operator $\Delta - x\cdot\nabla$ on the set (h, ∞), with h as in (2.5). Now (2.2)–(2.8) imply that

$$\varphi'(r, D_e u) \ge \frac{2}{r}\varphi(r, D_e u)(\Lambda(h_+) + \Lambda(h_-) - 2), \tag{2.9}$$

where $h_\pm$ are the corresponding constants, in (2.5), for $\Gamma_\pm$. Obviously $h_+ + h_- \ge 0$. Also by results of Beckner-Kenig-Pipher [BKP] (cf. also [CK, Section 2.4]) Λ is convex and $\Lambda(0) = 1$. Hence

$$\Lambda(h_+) + \Lambda(h_-) \ge 2\Lambda(0) = 2. \tag{2.10}$$

In [BKP] it is actually proved that the convex function Λ satisfies

$$\Lambda''(0) = 4(1 - \ln 2)/\pi > 0.$$

For convenience we now set

$$\gamma(r) = \gamma(\Gamma_+(r)) + \gamma(\Gamma_-(r)) - 2.$$

Then inserting (2.10) in (2.9) we obtain the following lemma.

Lemma 2.3. *There holds $\gamma(r) \geq 0$ for all r. Moreover the strict inequality holds unless $\Gamma^\star_\pm(r)$ are both half spheres. In particular if any of the $\Gamma^\star_\pm(r)$ digresses from being a half spherical cap by an area-size of ϵ, say, then*

$$\gamma(r) \geq C\epsilon^2. \tag{2.11}$$

Remark. The reader may verify by elementary calculus that for $E \subset \mathbb{S}^{N-1}$ with

$$\text{area}(E) = (\frac{1}{2} - \epsilon)\omega_N,$$

i.e., an ϵ digression from the half-spherical cap, we have

$$\frac{1}{2} - \epsilon = \frac{1}{\sqrt{2\pi}} \int_h^\infty \exp(-t^2/2)dt = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^h \exp(-t^2/2)dt \approx \frac{1}{2} - h.$$

Hence $\epsilon \approx h$ and therefore in Lemma 2.3 we'll have

$$\Lambda(h_+) + \Lambda(h_-) - 2 \geq C\Lambda''(0)(h_+^2 + h_-^2) \approx C(h_+^2 + h_-^2) \approx C\epsilon^2,$$

which gives (2.11).

3. Proof of Theorem I

First we need some definitions and notations.

Definition 3.1. Set

$$S_j(z,u) = \sup_{B(z,2^{-j})} |u|,$$

and define $\mathbb{M}(z,u)$ to be the maximal subset of $\mathbb{N}$ (natural numbers) satisfying the following doubling condition

$$4S_{(j+1)}(z,u) \geq S_j(z,u) \qquad \forall\, j \in \mathbb{M}(z,u). \tag{3.1}$$

Our aim is to prove that $S_j \leq C2^{-2j}$, for all $j \in \mathbb{M}(z,u)$ and some positive constant C. An important observation at this point is that if $\mathbb{M}$ is empty then we may easily (by iteration) obtain the desired estimate. Hence from now on we assume $\mathbb{M} \neq \emptyset$.

Lemma 3.2. *Let $u \in P_1(z,M)$. Then there exists a constant $C_0 = C_0(N)$ such that*

$$S_j(z,u) \leq C_0 M 2^{-2j} \qquad \forall j \in \mathbb{M}(z,u). \tag{3.2}$$

Proof. By the remark following Definition 1.1 we may assume z is the origin. Suppose the conclusion in the lemma fails. Then there exist $\{u_j\}$, $\{k_j\}$ such that

$$S_{k_j}(0,u_j) \geq j2^{-2k_j}, \tag{3.3}$$

with $k_j \in \mathbb{M}(0,u_j) \neq \emptyset$.

Now define $\tilde{u}_j$ as

$$\tilde{u}_j(x) = \frac{u_j(2^{-k_j}x)}{S_{k_j+1}(0,u_j)} \qquad \text{in } B(0,1).$$

Then $\tilde{u}_j$ satisfies the following properties:

$$\|\Delta \tilde{u}_j\|_{\infty,B} \leq \frac{2^{-2k_j}}{S_{k_j+1}(0,u_j)} \leq \frac{S_{k_j}(0,u_j)}{jS_{k_j+1}(0,u_j)} \leq \frac{4}{j} \to 0, \tag{3.4}$$

where the second inequality follows from (3.3), and the last inequality follows from (3.1),

$$\sup_{B(1/2)} |\tilde{u}_j| = 1, \tag{3.5}$$

$$\|\tilde{u}_j\|_{\infty,B} \leq \frac{S_{k_j}(0,u_j)}{S_{k_j+1}(0,u_j)} \leq 4. \tag{3.6}$$

$$\tilde{u}_j(0) = |\nabla \tilde{u}_j|(0) = 0 \tag{3.7}$$

Now by (3.4)–(3.7) we will have a subsequence of $\tilde{u}_j$ converging in $C^{1,\alpha}(B)$ (see [GT]) to a non-zero harmonic function u_0, satisfying $u_0(0) = |\nabla u_0|(0) = 0$. For any fixed direction e define

$$v = D_e u_0, \qquad v_j = D_e u_j, \qquad \tilde{v}_j = D_e \tilde{u}_j.$$

Then, for a subsequence, $\tilde{v}_j$ converges in $C^{1,\alpha}(B)$ to v, where $\Delta v = 0$. Now according to Lemma 2.1 (since v_j is harmonic in Ω_j)

$$\frac{1}{r^{2N}} \int_{B(0,r)} |\nabla v_j^+|^2 \int_{B(0,r)} |\nabla v_j^-|^2 \leq C \qquad \forall r, j \tag{3.8}$$

where C depends on the $W^{2,2}$ norm of u_j over the unit ball; by elliptic estimates this is uniformly bounded for all j. Making change of the variables in (3.8) and letting $r = 2^{-k_j}$, we'll obtain

$$\int_{B(0,1)} |\nabla \tilde{v}_j^+|^2 \int_{B(0,1)} |\nabla \tilde{v}_j^-|^2 \leq C\left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4 \leq C\left(\frac{2^{-2k_j}}{S_{k_j}}\right)^4 \qquad \forall j,$$

where in the last inequality we have used (3.1). Here, and in the sequel, C is a generic constant. Next, invoking the Poincare inequality we may reduce (improve) the above to

$$\int_{B(0,1)} |\tilde{v}_j^+ - M_j^+|^2 \int_{B(0,1)} |\tilde{v}_j^- - M_j^-|^2 \leq C\left(\frac{2^{-2k_j}}{S_{k_j}}\right)^4 \qquad \forall j,$$

where $M_j^\pm$ is the mean-value for the functions $\tilde{v}_j^\pm$, on the unit ball. From here using (3.3) we obtain, by letting j tend to infinity,

$$\int_{B(0,1)} |v^+ - M^+|^2 \int_{B(0,1)} |v^- - M^-|^2 = 0, \tag{3.9}$$

where $M^\pm$ is the corresponding mean-value for $v^\pm$. Obviously (3.9) implies that either of $v^\pm$ is constant. Since also $v(0) = 0$, the constant must be zero. In particular v does not change sign. But then the maximum principle gives $v \equiv 0$, i.e., $D_e u_0 \equiv 0$. Since e is arbitrary we also have u_0 is constant. Next using $u_0(0) = 0$ we will have $u_0 \equiv 0$ which contradicts (3.5). This proves the lemma. □

Next we will complete the chain in $\mathbb{N}$ for the estimate (3.2).

Lemma 3.3. *Let $C_0 = C_0(N)$ be the constant in Lemma 3.2. Then*

$$S_j \leq 4C_0 M 2^{-2j} \qquad \forall j \in \mathbb{N}.$$

Proof. Let $u \in P_1(z, M)$. Then obviously $S_1 \leq M$. Let $j > 1$ be the first positive integer such that the statement of the lemma doesn't hold, i.e.,

$$S_j > 4C_0 M 2^{-2j}. \tag{3.10}$$

Then

$$S_{j-1} \leq 4C_0 M 2^{-2(j-1)} = 16 C_0 M 2^{-2j} < 4S_j.$$

Hence $j-1 \in \mathbb{M}(u)$. By Lemma 3.2, then,

$$S_j \leq S_{j-1} \leq C_0 M 2^{-2(j-1)} = 4C_0 M 2^{-2j},$$

which contradicts (3.10). The result follows. □

From Lemma 3.3 we infer a uniform $C^{1,1}$ estimate for the class $P_1(z, M)$.

Proof of Theorem I. Let u be in $P_1(z, M)$ and set $d(x) = \operatorname{dist}(x, \partial\Omega)$. Then by Lemma 3.3

$$|u(x)| \leq CMd(x)^2 \qquad \forall\ x \in B(0, 1/2). \tag{3.11}$$

Define now

$$v(y) = \frac{u(x + yd(x))}{d(x)^2} \qquad \text{in } B(0,1).$$

By (3.11) v is bounded on the unit ball, and by its definition it satisfies $\Delta v = 1$ in $B(0,1)$. Hence by elliptic estimates $D_{ij}v(0) = D_{ij}u(x)$ is uniformly bounded (independent of x). This gives the result. □

4. Further Auxiliary Lemmas

In this and the next sections we will frequently use the blow-up of functions, i.e. for a given u we consider $u_r(x) = u(rx)/r^2$ and let r tend to zero, through some subsequence. It is, however, not clear whether the blow-up (the limit function) will not be the zero function. Indeed, if $u(x) = o(|x|^2)$, then any blow-up will be identically zero. To prevent this we need a non-degeneracy from below, asserted in the following remark.

Remark. The function u in (1.1) satisfies

$$\sup_{B(x^0,r)} u \geq u(x^0) + C_N r^2, \tag{4.1}$$

for all $x^0 \in \overline{\Omega}$. Here $C_N = 1/2N$, if $u(x^0) \geq 0$, and C_N is somewhat smaller if $u(x^0) < 0$. Also r is small enough so that $B(x^0, r) \subset B(0,1)$.

The proof of this is given in [Ca2] in the case $u > 0$ or $x^0 \in \partial\Omega$. The general case is proven as follows. Suppose $u(x^0) \leq 0$ and $x^0 \in \Omega$. Then if there is $x^1 \in B(x^0, r/2) \cap \partial\Omega$ then we apply the above to u in $B(x^1, r/2)$ to obtain the same estimate with $C_N = 1/8N$. So suppose the set $B(x^0, r/2)$ contains no free boundaries. we can apply the mean value theorem for harmonic functions to $u(x) - |x - x^0|^2/2N$ to obtain

$$\int_{B(x^0,r/2)} (u - u(x^0))\, dx = cr^{2+N},$$

where c depends on N only. From here one obtains (4.1).

Now, by (4.1) and Theorem I, we may assume that any blow up of functions in $P_1(0, M)$ remain in the class. Our next definition will be used frequently here and later in Section 6.

ϵ-Close. We say two functions f and g are ϵ-close to each other in a domain D if

$$\sup_{x \in D} |f(x) - g(x)| < \epsilon.$$

Blow-up limit. A blow-up limit u_0 is a uniform limit on compact subsets of $\mathbb{R}^N$

$$u_0(x) = \lim_{j \to \infty} \frac{u(r_j x)}{r_j^2}$$

where $u \in P_1(0, M)$ and $r_j \to 0$. The function u may even change for different j.

Flat points. We say $\partial\Omega$ is flat at the origin if there is a blow up u_0 such that the set where $u = 0$ is a half-space.

Half-space solutions. A half-space solution u_0 is a global solution that has the representation $(\max(x_1, 0))^2/2$ in some coordinate system.

For the readers convenient and for the future references we will recall and explain some general (known) facts. These will be crucial in the rest of the paper. We recommend that a reader unfamiliar with such problems carefully verify these facts.

General Remarks.

a) By (4.1) and techniques of [Ca4; Lemma 6] one may show that the set

$$\Omega \cap \{x : \ |\nabla u(x)| < \epsilon \}$$

has volume less than $C\epsilon$, with universal C. One uses only the $C^{1,1}$ property of the solution, and not the non-negativity of the function.

This implies, in particular, that an ϵ-neighborhood

$$K_\epsilon = \{x : \ \mathrm{dist}(x, \partial\Omega) < \epsilon\}$$

of the free boundary has Lebesgue measure less than $C\epsilon$, i.e. volume$(K_\epsilon) < C\epsilon$, with universal C.

From here, using covering arguments, we conclude that the free boundary $\partial\Omega$ has locally finite $(N-1)$-Hausdorff measure; see [Ca4] for details.

b) Let u_j be a blow-up of u and suppose u_j converges to u_0 in $C^{1,\alpha}_{loc}(\mathbb{R}^N)$. Then it follows that

$$\{u_0 = |\nabla u_0| = 0\} \supset \overline{\lim}\{u_j = |\nabla u_j| = 0\},$$

see e.g. ([Ca2], [KS1]); here $\overline{\lim}$ denotes the limit set of all sequences $\{x_j\}, x_j \in \{u_j = |\nabla u_j| = 0\}$.

Next, let u_0 be a blow-up of a sequence u_j, and suppose $u_0 = 0$ in $B(x^0, r_0)$ for some x^0, and r_0. Then uniform convergence and (4.1) imply that $u_j = 0$ in $B(x^0, r_0/2)$, for large j.

A consequence of this is the following

$$\text{interior}(\{u_0 = |\nabla u_0| = 0\}) \subset \overline{\lim}\{u_j = |\nabla u_j| = 0\}.$$

c) From a) and b) above we infer an L^p convergence of the second derivatives of u_j to u_0, i.e.

$$D_{ik}u_j \to D_{ik}u_0 \qquad \text{in } L^p_{loc} - \text{norm},$$

for $1 < p < \infty$. This depends on the fact that Δu_j and Δu_0 differ only inside an ϵ-neighborhood of the free boundary $\partial\Omega_0$, i.e. on a set of Lebesgue measure ϵ.

d) From a)–b) we may also deduce that if $u \in P_1(0, M)$ is ϵ-close to a half-space solution $h = (\max(x_1, 0))^2/2$, say, then $u \equiv 0$ in $B(0, 1/2) \cap \{x_1 < -C\sqrt{\epsilon}\}$, for some constant $C > 0$. We sketch some details. Let us suppose $x^0 \in B(0, 1/2) \cap \{x_1 < 0\}$. Choose $r = |x^0_1|$. Then by (4.1) and the closeness of u to the half-space solution h we have

$$2\epsilon \geq C_N r^2;$$

observe that $|u| < \epsilon$ in $B(x^0, r) \subset \{x_1 < 0\}$. Hence if $x^0_1 < -\sqrt{2\epsilon/C_N}$ then x^0 cannot be a point of $\overline{\Omega}$. This implies that $u \equiv 0$ on the set $\{x_1 < -\sqrt{2\epsilon/C_N}\}$.

e) A consequence of this is the following simple fact: Let $u \in P_1(0, M)$ and suppose the origin is a flat point, with respect to some blow-up sequence u_{r_j}. Then in $B(0, 1/2)$, u_{r_j} is ϵ-close to a half-space solution for small enough r_j.

Lemma 4.1. *(Essentially due to Spruck [Spk]) Let $u \in P_1(0, M)$. Then any blow-up u_0 of u is a homogeneous function of degree two, and the set $\{u_0 = |\nabla u_0| = 0\}$ is a cone.*

The proof of Lemma 4.1 in the global case ($P_\infty(0, M)$) is given in details in [KS2]. The proof of the local case is similar and therefore omitted.

Lemma 4.2. *Let $u \in P_1(0, M)$ and suppose $CD_1u - u \geq -\epsilon_0$ in $B(0, 1)$, for some $\epsilon_0 > 0$ and $C > 0$. Then $CD_1u - u \geq 0$ in $B(0, 1/2)$, provided ϵ_0 is small enough. In particular if u is close to the half-space solution $\max(x_1, 0)^2/2$ in $B(0, 1)$, then (by integration and d) in General Remarks) $u \geq 0$ in $B(0, 1/2)$.*

Proof. Suppose the conclusion of the lemma fails. Then there is a $u \in P_1(0, M)$ with

$$CD_1u(x^0) - u(x^0) < 0, \tag{4.2}$$

for some $x^0 \in B(0, 1/2)$. Let

$$w(x) = CD_1u(x) - u(x) + \frac{1}{2N}|x - x^0|^2.$$

Then w is harmonic in $\Omega \cap B(x^0, 1/2)$, $w(x^0) < 0$ (by (4.2)) and $w \geq 0$ on $\partial\Omega$. Hence by the maximum principle the negative infimum of w is attained on $\partial B(x^0, 1/2)$. We thus obtain

$$-\epsilon_0 \leq \inf_{\partial B(x^0,1/2)\cap\Omega}(CD_1u - u) \leq -\frac{1}{8N},$$

which is a contradiction as soon as $\epsilon_0 < 1/(8N)$. The second part, that $u \geq 0$ in $B(0, 1/2)$ for u near to half-space solution, follows by d) in General Remarks and integration. The reader should notice that ϵ-closeness for u to a half-space solution implies, by elliptic estimates, that the gradient of u also becomes $(C\sqrt{\epsilon})$-close to the gradient of the half-space solution. Here C is a universal constant. We leave the details to the reader. This proves the lemma. $\square$

In the next lemma we'll apply the following simple fact, which we formulate as a remark.

Remark. Suppose u_0 is a blow-up solution of u, and $\delta_1(u_0) > 0$. Then u_0 is a degree two homogeneous global solution and the interior of $\mathbb{R}^N \setminus \Omega$ is a non-void cone.

Indeed, by Lemma 4.1 u_0 has the mentioned properties and $\mathbb{R}^N \setminus \Omega$ is a cone. If the cone has empty interior, then General Remark a), $\mathbb{R}^N \setminus \Omega$ has Lebesgue measure zero. Therefore by Liouville's theorem u_0 is a polynomial of degree two and hence for all r we'll have $\delta_r(u_0) = 0$, which is a contradiction.

Lemma 4.3. *Let $u \in P_1(0, M)$, and suppose $\overline{\lim}_{r \to 0} \delta_r(u) > 0$. Then $u \geq 0$ in some neighborhood of the origin.*

Proof. Let $\{r_j\}$ be a decreasing sequence such that

$$\lim_{j \to \infty} \delta_{r_j}(u) > 0.$$

We blow up the function u through r_j to obtain, by Lemma 4.1, a global homogeneous solution u_0 of degree 2 in $\mathbb{R}^N$, with

$$\delta_1(u_0) > 0.$$

Hence by the discussion preceding this lemma $\mathbb{R}^N \setminus \overline{\Omega}_0$ is a non-void cone.

Now choose a direction e, and consider the monotonicity formula for $D_e u_0$. By degree two homogeneity of u_0 we have $|\nabla D_e u_0|$ is homogeneous of degree zero. By scaling, this implies that $\varphi(r, D_e u_0)$ must be a positive constant, unless one of the functions $(D_e u_0)^{\pm}$ is zero. To exclude the first case, observe that by Lemmas 2.2–2.3 if $\varphi \neq 0$ then it is strictly monotone, since (by the condition $\delta_1(u_0) > 0$) at least one of the sets $\Gamma_{\pm}$ cannot be a half-sphere. In the second case we will have $D_e u_0 \geq 0$ (or ≤ 0), for any fixed direction e.

Let x^0 be a fixed point in Ω_0 and suppose $|\nabla u_0(x^0)| \neq 0$. Set

$$\nu = \frac{\nabla u_0(x^0)}{|\nabla u_0(x^0)|}.$$

Then for any directional vector e orthogonal to ν we'll have $D_e u_0(x^0) = 0$. Moreover $D_e u_0 \geq 0$ in Ω_0 (or ≤ 0), it is harmonic there, and it takes a local minimum (or maximum). Hence by the minimum (or maximum) principle $D_e u_0 = 0$ in Ω_0. This implies that u_0 is independent of the directions e orthogonal to ν and depends only on the direction ν. Therefore u_0 is one dimensional in each connected component. Since the only one dimensional solutions are half-space solutions (in each connected components) there must be at most two connected components, which are half-spaces. Now the assumption $\delta_1(u_0) > 0$ implies that there must be at most one connected components, which is a half-space.

In particular, near the origin, u is close to a half-space solution. Therefore the origin is a flat point. Hence Lemma 4.2, and part c) in General Remarks gives the result. This proves the lemma. $\square$

Our next result will not be used in this paper. However, we include this for future references. First we notify the reader of the following obvious fact.

Remark. Suppose $u \in P_1(0, M)$ and the origin is a point of zero upper Lebesgue density for the complement of Ω. Then any blow-up of u at the origin is a polynomial of degree two.

Lemma 4.4. *Let $u \in P_1(0, M)$, satisfy $\overline{\lim}_{r\to 0}\,\delta_r(u) = 0$, and suppose for some blow-up sequence with limit u_0, $D_e u_0 \neq 0$. Then there is a constant C_e such that $\varphi(r, D_e u) \geq C_e > 0$, for all $r < 1$.*

Proof. Blow up u to obtain a global solution u_0, in $\mathbb{R}^N$ with $int(\Omega_0^c) = \emptyset$ (see the discussion preceding this lemma). Hence $u_0 = P$, a degree two polynomial. For any direction e non-parallel to the kernel of P we have $D_e u_0 = D_e P$ is a non-constant linear function. Next

$$\varphi(r) \geq \frac{1}{r^{2N}} \int_{B_r} |\nabla(D_e u)^-|^2 \int_{B_r} |\nabla(D_e u)^+|^2.$$

Scaling and letting r tend to zero we'll have (see c) in General Remarks)

$$\varphi(0) \geq \int_{B_1} |\nabla(D_e P)^-|^2 \int_{B_1} |\nabla(D_e P)^+|^2 \geq C_e, \tag{4.5}$$

where in the last inequality we have used the fact that $D_e P$ is a non-constant linear polynomial. Now (4.5) together with the monotonicity formula gives the result. $\square$

Remark. Lemma 4.4 can be stated in a more accurate way. We can, with suitable choice of e, make the constant C_e uniformly bounded from below for the whole class $P_1(0, M)$. Here is how: Rearrange the coordinate system such that the polynomial P in the proof of Lemma 4.4 has the representation

$$P = \sum_{i=1}^{m} a_i x_i^2 \qquad \sum_{i=1}^{m} a_i = \frac{1}{2},$$

where $m \leq N$. Let now e_j be the standard coordinate system and set

$$e = \frac{(e_1 + \cdots + e_m)}{\sqrt{m}}.$$

Then

$$D_e P = \frac{2}{\sqrt{m}} \sum_{i=1}^{m} a_i x_i,$$

and

$$|\nabla D_e P|^2 = \frac{4}{m} \sum_{i=1}^{m} a_i^2 \geq \frac{4}{m^2} (\sum_{i=1}^{m} a_i)^2 = \frac{1}{m^2} \geq \frac{1}{N^2}.$$

Hence for e as above

$$\varphi(0, D_e u) \geq \left(\frac{\mathrm{vol}(B_1)}{2N^2} \right)^2.$$

5. Proof of Theorem II

We remark that in the definition of the global solutions $P_\infty(0, M)$, we require that the functions have quadratic growth with uniform constant M. It is noteworthy that this restriction is not superfluous, as there are examples of solutions to (1.1) in the entire space, with $\mathbb{R}^N \setminus \overline{\Omega}$ non-void, and with u of polynomial or even exponential growth; see [shap1].

However, we will use Theorem II, in connection with blow-up functions, i.e., we consider blow-up of functions in $P_1(0, M)$ and these, by Theorem I, have quadratic growth near the origin. Hence any blow-up of such functions will also be of quadratic growth in the entire space.

The reader should observe that, by the assumption $\delta_r(u) > 0$ for some $r > 0$ in Theorem II, and the discussion in the remark preceding Lemma 4.2, $\mathbb{R}^N \setminus \Omega$ has nonempty interior. It suffices to show that $u \geq 0$, since then by [Ca2] it follows that $D_{ee}u \geq 0$ on $\partial\Omega$, and by scaling we also have this property in Ω (see the details of this scaling argument in [KS2]). We will prove

Theorem II'. *If $u \in P_\infty(0, M)$ and $\mathbb{R}^N \setminus \Omega$ has nonempty interior, then $u \geq 0$.*

We split the proof in three cases.
Case 1: $\mathbb{R}^N \setminus \Omega$ is bounded.
Case 2: $\overline{\lim}_{R\to\infty} \delta_R(u) > 0$.
Case 3: $\overline{\lim}_{R\to\infty} \delta_R(u) = 0$.

Remark. The reader should observe that, by the assumption $\delta_r(u) > 0$ (for some $r > 0$) in Theorem II, and the discussion in the remark preceding Lemma 4.2, $\mathbb{R}^N \setminus \Omega$ has non-empty interior. The second part of Theorem II, the fact that $\overline{\lim}\, \delta_R(u) > 0$ implies u is a half-space solution, is included in the proof of Case 2.

Proof of Case 1. Let U be the Newtonian potential of the complement of Ω, with constant density, and such that $\Delta U = \chi_{\Omega^c}$, i.e. ($U = c|.|^{2-N} * \chi_{\Omega^c}$; for $N = 2$ we take the logarithmic kernel). Then, since $\partial\Omega$ has zero Lebesgue measure, $\Delta(U + u) = 1$ a.e. in $\mathbb{R}^N$, and it has quadratic growth. Hence by Liouville's theorem it is a second degree polynomial P.

By translation and rotation we may assume $P = \sum a_j x_j^2 + d$; observe that, by the imposed rigid motion, the origin is not necessarily on $\partial\Omega$ any more. From here we'll have $u = P - U$ in Ω. Let now $v = x \cdot \nabla u - 2u$. Then by homogeneity of $P - d$ we'll have

$$v = -2d - x \cdot \nabla U + 2U \quad \to \quad C \qquad\qquad \text{as } x \to \infty,$$

and $v = 0$ on $\partial\Omega$. If $N = 2$ then $C = +\infty$ and if $N \geq 3$, then $C = -2d$.

Suppose first $C < 0$. Then by the maximum principle $v < 0$. Now elementary calculus we'll imply $u(rx)/r^2$ is decreasing in r. Then, on the unbounded cone-like set $K = \{rx : \ x \in \mathbb{R}^N \setminus \Omega, \ r \geq 1 \}$ we'll have $u \leq 0$. By subharmonicity, this implies

$$0 = u(x^0) \leq \mathrm{vol}(B) \int_B u, \qquad\qquad \text{for all } \ x^0 \in \partial\Omega \cap \mathrm{int}(K),$$

where B is a ball in K with center x^0.

Since also $u \leq 0$ in K, we will have $u \equiv 0$ in B, i.e. $\partial\Omega \cap \mathrm{int}(K) = \emptyset$. Therefore $K \subset \mathbb{R}^N \setminus \Omega$. But then $\mathbb{R}^N \setminus \Omega$ is unbounded, which is a contradiction. Therefore $C > 0$.

Now $C > 0$ implies, by the maximum principle, that $v > 0$, i.e., $u(rx)/r^2$ is increasing in r. Then $u \leq 0$ on the truncated cone $K = \{rx : \; x \in \mathbb{R}^N \setminus \Omega, \; r \leq 1 \}$. A similar argument as above then shows that $u = 0$ on K. Hence we conclude that the set $\mathbb{R}^N \setminus \Omega$ has positive Lebesgue density at the origin.

Now two cases arise: (a) the origin is in the interior of $\mathbb{R}^N \setminus \Omega$, (b) the origin is on $\partial\Omega$. The first case, along with the monotonicity of $u(rx)/r^2$ will result in the positivity of u. As to the second case, we observe that since $K \subset \{u = 0\}$, $\overline{\lim}_{r\to 0} \delta_r(u) > 0$ and by Lemma 4.3 $u \geq 0$ near the origin. Again by the monotonicity of $u(rx)/r^2$ in r we'll have $u \geq 0$ in $\mathbb{R}^N$. □

A simple alternate, but indirect, proof for Case 1 goes as follows: Since the Newtonian potential U is a polynomial in Ω^c, it follows from [DF] that Ω is the exterior of an ellipsoid and from [Shah] it follows that $u \geq 0$.

Proof of Case 2. We first blow up u through a sequence $\{R_j\}$ $(R_j \to \infty)$ and such that $\lim \delta_{R_j}(u) \neq 0$, to obtain a global homogeneous solution u_0 of degree two (see [KS2; Lemma 2.5]). A similar argument as that in the proof of Lemma 4.3 then implies that u_0 is a half-space solution. This in particular implies

$$\varphi(\infty, D_e u) := \lim_{R_j\to\infty} \varphi(R_j, D_e u) = \lim_{R_j\to\infty} \varphi(1, D_e u_{R_j}) = \varphi(1, D_e u_0) = 0,$$

for all directions e. The last equality depends on the fact that a half-space solution u_0 is monotone in every direction, and thus either $(D_e u_0)^+ \equiv 0$ or $(D_e u_0)^- \equiv 0$. Observe also that we have used the convergence in $W^{2,2}$; see c) in General Remarks.

Now by the monotonicity formula,

$$\varphi(r, D_e u) \leq \varphi(\infty, D_e u) = 0,$$

for all vectors e, and hence $D_e u \geq 0$ (or the reverse), i.e. u is monotone in all directions. As in Lemma 4.3 it follows that ∇u is parallel at any two points of a component of Ω, and hence that u is a half-space solution. □

Proof of Case 3. Since $\mathbb{R}^N \setminus \Omega$ is unbounded and $\overline{\lim}_{R\to\infty} \delta_R(u) = 0$ we may conclude that there is a blow-up $u_j(x) = u(R_j x)/R_j^2$ at infinity $(R_j \to \infty)$ with a subsequence converging to a polynomial P in $\mathbb{R}^N$, and that P is independent of some of the variables. Indeed, by the assumptions in this case there is an unbounded sequence $x^j \in \partial\Omega$. Therefore we may take $R_j = |x^j|$ and obtain that P vanishes, along with its gradient, at the origin and at some other point on the unit sphere. Hence by homogeneity, Lemma 4.1, on the whole line generated by these points, and thus the above conclusion.

Suppose $D_1 P = 0$. Then for any point $x^0 \in \partial\Omega$ we may consider the monotonicity formula $\varphi(r, D_1 u, x^0)$, which by Lemma 2.1 is non-decreasing in r. Hence for r fixed we choose $R_j \geq r$ to obtain

$$\varphi(r, D_1 u, x^0) \leq \varphi(R_j, D_1 u, x^0) = \varphi(1, D_1 u_j, x^0) \to \varphi(1, D_1 P, x^0) = 0,$$

as $R_j \to \infty$. Here we have used that $D_1 u_j \to D_1 P = 0$. This will imply that $\varphi \equiv 0$, i.e. either of $(D_1 u)^+$ or $(D_1 u)^-$ is zero. Suppose $D_1 u \geq 0$ (the other case is treated similarly). From here we want to deduce that $D_{11} u \geq 0$. To do this we set

$$-C := \inf_{\Omega} D_{11} u,$$

which is bounded because u is in $P_\infty(0, M)$.

Let x^j be a minimizing sequence, i.e.,

$$-C = \lim_j D_{11}u(x^j).$$

Then we consider a blow up at x^j and with $d_j = \text{dist}(x^j, \partial\Omega)$. Hence we define

$$u_j(x) = \frac{u(d_jx + x^j)}{d_j^2} \qquad \text{in } B(0,1).$$

Now by compactness, for a subsequence we'll have $u_j \to u_0$, where u_0 is a global solution. Also $D_{11}u_j \to D_{11}u_0$, uniformly in $B(0, 1/2)$. The latter depends on the fact that $\Delta u_j = 1$ in $B(0,1)$. Hence we'll have a global solution u_0 with $\Delta u_0 = \chi_{\Omega_0}$ where $\Omega_0 = \Omega(u_0)$. From the minimal properties of the sequence x^j we also deduce that for all $x \in B(0, 1/2)$.

$$D_{11}u_0(0) = -C \le \lim_j D_{11}u_j(x) = \lim_j D_{11}u(d_jx + x^j) = D_{11}u_0(x).$$

Hence by the maximum principle $D_{11}u_0 \equiv -C$ in the connected component of Ω_0 that contains the unit ball, we call this Ω'. Observe also that the free boundary $\partial\Omega_0$ is non-empty. Therefore we assume that $z = (z_1, \cdots, z_N) \in \partial\Omega_0$. Now integration gives that in Ω_0 we have the following representation

$$D_1u_0(x) = -Cx_1 + g(x_2, \cdots, x_N).$$

Since also $D_1u_0 = \lim_j D_1u_j \ge 0$ we'll have $x_1 \le g(x')/C$ (of course we assume that $C > 0$ otherwise there is nothing to prove). This in particular means that any ray l_x emanating at $x \in \Omega$ and parallel to the x_1-axis hits $\partial\Omega$. Now the component Ω' of Ω is the under-graph of the function $x_1 = g/C$, in the x_1-direction. Now $D_1u_0 \ge 0$ implies that in the negative x_1 direction (inwards to Ω') u_0 is decreasing. Since it is also zero on $\partial\Omega'$ it becomes non-positve in Ω'. This contradicts the non-degeneracy (4.1).

Therefore we conclude $C \ge 0$ and hence

$$D_{11}u \ge 0 \qquad \text{in } \Omega.$$

This along with $D_1u \ge 0$, implies that $u \ge 0$ on lines, which hit $\mathbb{R}^N \setminus \Omega$ and are parallel to the x_1-axis. Our goal will be to prove that for any x in $\mathbb{R}^N$,

$$\lim_{m\to\infty} u(x_1 - m, x_2, \cdots, x_N) \ge 0,$$

which together with $D_1u \ge 0$ implies $u(x) \ge 0$. Now let

$$u_m(x) = u(x_1 - m, x_2, \cdots, x_N) \qquad (m = 1, 2, \cdots)$$

be a family of translations of u. Since the negative x_1-axis is in $\mathbb{R}^N \setminus \Omega$ (observe that $D_1u \ge$ and $u(0) = 0$) and $|u(x)| \le M(|x|^2 + 1)$ we deduce that $|u_m(x)| \le C_0R^2$ on $B(0, R)$ (C_0 independent of m). This in particular implies that u_m is a bounded sequence and hence there is converging subsequence with a limit function u_0 in $\mathbb{R}^N$. It is then elementary to see that the function u_0 is also a global solution.

Next using the non-negativity of $D_{11}u$ and $D_1 u$ we infer that the non-negative monotone function $D_1 u$ has a limit at $-\infty$, i.e.,

$$D_1 u_0(x) = \lim_{m\to\infty} D_1 u(x_1 - m, x_2, \cdots, x_N) = \text{ constant} = 0,$$

and hence u_0 is cylindrical ($(N-1)$-dimensional) in the x_1-direction. If $u_0 \equiv 0$ there is nothing to prove, therefore we assume that u_0 is non-trivial, and one lower dimensional. Also the set $\{u_0 \equiv 0\}$ has non-empty interior since it contains the projection of the interior of the set $\{u \equiv 0\}$. But then one of the cases may occur for this new lower dimensional function, and we repeat the argument (if the third case occurs) until we obtain a one dimensional problem.

Now the reader may verify, using elementary calculus, that the one dimensional global solution is non-negative. This proves $u \geq 0$. $\square$

6. Proof of Theorem III

In this section we will give a proof of Theorem III. The proof is based on two lemmas that are consequences of already established results, namely Theorem II, the techniques in Lemma 4.2, and the main result in [Ca2]. We first use Theorem II in conjunction with [Ca2] to obtain uniform flatness for the free boundary of $u \in P_\infty(0, M)$ provided $\delta_1(u) \geq \epsilon$. Next we show that for $u \in P_1(0, M)$, there is a uniform neighborhood of 0 such that $u \geq 0$ in that neighborhood provided u is close to a half-space solution (ϵ close as in Section 4).

The first author's original result for non-negative solutions in the class $P_1(0, M)$, provides us with the following lemma; see [Ca2] or [Ca4].

Lemma 6.1. *Given a positive number ϵ, there exists t_ϵ such that if $u \in P_\infty(0, M)$ and $\delta_1(u) \geq \epsilon$, then in $B(0, t_\epsilon)$ the boundary of Ω is the graph of a C^1 function (uniformly for the class) and u is (ϵt_ϵ^2)-close to a half-space solution there.*

Observe that by Theorem II, $u \in P_\infty$ is non-negative. Therefore [Ca2] applies.

Now a simple proof of Lemma 6.1 can be given based on compactness, a contradictory argument, and the main result in [Ca2].

Observe that by Theorem II, $u \in P_\infty$ is non-negative. Therefore [Ca2] applies.

The proof of the next lemma follows the same lines as that of the proof of Lemma 4.2. See also General Remarks. We omit the proof.

Lemma 6.2. *Let $\epsilon, s > 0$, and suppose $u \in P_1(0, M)$ is (ϵs^2)-close to a half-space solution* $(\max(x \cdot e, 0))^2/2$ *in $B(0, s)$. Then in $B(0, s/2)$ we have*

$$sD_e u - u \geq 0, \tag{6.1}$$

provided ϵ is small enough. In particular, by integration and d) in General Remarks, we have $u \geq 0$ in $B(0, s/4)$; again if ϵ is small enough.

Proof of Theorem III. First we claim that for given $\epsilon > 0$ there exists

$$0 < r_\epsilon < t_\epsilon, \qquad \text{(where } t_\epsilon \text{ is as in Lemma 6.2)}$$

such that if for $u \in P_1(0, M)$ we have $\delta_{r_0}(u) \geq \epsilon$ for some $r_0 < r_\epsilon$ then u is $(2\epsilon r_0^2 t_\epsilon^2)$-close to a half-space solution $(\max(x \cdot e, 0))^2/2$ in $B(0, r_0 t_\epsilon/2)$. Suppose, for the moment, this is true. Then it follows by Lemma 6.2 that $u \geq 0$ in $B(0, r_0^2/4)$. Moreover $(2\epsilon r_0^2 t_\epsilon^2)$-closeness to a half-space solution also implies (see d) in General Remarks) that $u \equiv 0$ in $B(0, r_0^2/4) \cap \{x_1 < -C\sqrt{\epsilon} r_0^2\}$. Hence, for small ϵ, [Ca2] or [Ca4] applies to conclude that in $B(0, c_0 r_0^2)$, $\partial\Omega$ is the graph of a C^1-function, with a uniform C^1-norm. Here c_0 depends only on M and N.

The modulus of continuity $\sigma(r)$ is then defined by the inverse of the relation $\epsilon \to r_\epsilon$.

Now to complete the proof we need to show the $(2\epsilon r_0^2 t_\epsilon^2)$-closeness of u to a half-space solution in $B(0, r_0 t_\epsilon/2)$. Suppose this fails. Then, for each $r_j \searrow 0$ there exists $u_j \in P_1(0, M)$ with $\delta_{r_j}(u_j) \geq \epsilon$, and such that

$$\sup_{B(0, r_j t_\epsilon/2)} |u_j - h| > 2\epsilon r_j^2 t_\epsilon^2,$$

for all half-space solutions h.

Set $v_j = u_j(r_j x)/r_j^2$. Then $v_j \in P_{1/r_j}(0, M)$, $\delta_1(v_j) \geq \epsilon$, and

$$\sup_{B(0, t_\epsilon/2)} |v_j - h| > 2\epsilon t_\epsilon^2,$$

for all half-space solutions h.

Now, for a subsequence and in an appropriate space, v_j converges to a global solution v_0 which on one side satisfies the hypotheses of Lemma 6.1, but on the other side

$$\sup_{B(0, t_\epsilon/2)} |v_0 - h| \geq 2\epsilon t_\epsilon^2,$$

for all half-space solutions h. By Lemma 6.1, this is a contradiction. Hence the claim holds.

This completes the proof of the theorem. □

7. Application to the Pompeiu type problem

Theorem I and Theorem III, above, apply to more general operators of the form

$$L(u) = \sum D_i(a_{ij} D_j u) + a(x)u,$$

with a_{ij} and a in the C^α-class, and $a_{ij}(0) = \delta_{ij}$ (Kroneker's delta function). We refer to [Ca3] for this more general operators, and with $a(x) \equiv 0$. Here we treat the case of $\Delta + a(x)$ with $a(x)$ constant.

To show this we need mainly to make sure that the monotonicity formula works for these operators. However, this is not true in general, as one may easily observe. On the other hand, the full strength of the monotonicity formula is not used in our analysis.

The monotonicity formula is used either locally near the origin or globally when the blow-up functions are considered. For the first case we simply will show that the operator L will admit a monotonicity formula which is almost increasing.

The second case is even simpler, as the blow-up of any function in the class $P_1(0, M, L)$ (L denotes the dependent on the operator) will converge to $P_\infty(0, M, \Delta)$, i.e., after rotation the blow-up will satisfy $\Delta u_0 = \chi_{\Omega_0}$, and we are back to the same situation as before.

As to the technique in Lemmas 4.2 (and 6.2) we observe that for the case $\Delta u + u = 1$ we will have $\Delta w = -CrD_e u - 1 + u + c_0 \leq 0$ if c_0 is small enough, and the lemmas work. Spruck's Theorem also works with almost no changes. We leave the details to the reader.

A particular problem, where the Laplacian is replaced with the Helmholtz operator, is the classical and well studied Pompeiu problem which can be stated as follows. Suppose there exists a bounded domain $\Omega \subset \mathbb{R}^N$ and a function u satisfying

$$\Delta u + u = \chi_\Omega \quad \text{in } \mathbb{R}^N, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{7.1}$$

Does it follow that Ω is a ball. This question, put forward somewhat differently by Dimitrie Pompeiu, has puzzled many mathematicians during the past 50 years. There are many partial results for this problem, which we will not discuss. For backgrounds in the Pompeiu problem and for further references and how the formulation of Pompeiu is related to the one here see [W].

From our point of perspective it is interesting to know whether the boundary of such domains are analytic. Williams [W] proved that if the boundary of Ω in (7.1) is Lipschitz then it is analytic.

It does follow from our results, with the modification below, that under the thickness condition of Theorem III, the boundary of such domains are analytic. We formulate this in the following theorem.

Theorem IV. *Under the thickness assumptions of Theorem III, any domain Ω which admits a solution u to (7.1) has an analytic boundary.*

Now to show the almost monotonicity of the function φ we need to define the following functions,

$$I(r, v) = \int_{\partial B_r} \frac{|\nabla v|^2}{|x|^{N-2}}, \qquad J(r, v) = \int_{B_r} \frac{|\nabla v|^2}{|x|^{N-2}}.$$

Then

$$\varphi' = \frac{\varphi}{r}\left[-4 + \frac{rI(r, (D_e u)^+)}{J(r, (D_e u)^+)} + \frac{rI(r, (D_e u)^-)}{J(r, (D_e u)^-)}\right]. \tag{7.2}$$

Now let F be the fundamental solution of $(\Delta + 2)$. Then

$$F = F_1 |x|^{2-N} + F_2 \log |x|,$$

where F_2 is zero for odd N, and F_1 and F_2 are regular functions not vanishing at the origin. Next as in [ACF; Lemma 5.1] (cf. also [Ca3]) we will have, for a nonnegative subsolution v to (7.1),

$$\begin{aligned}
J(r, v) &\leq (1 + Cr)\int_{B(0,r)} (\Delta(v^2/2)F + v^2 F)\\
&= (1 + Cr)\int_{B(0,r)} (\Delta(v^2/2)F - \frac{v^2}{2}\Delta F)\\
&= (1 + Cr)\int_{\partial B(0,r)} v v_n F - \frac{v^2}{2}F_n\\
&\leq (1 + Cr)\int_{\partial B(0,r)} (v|v_n| r^{2-N} + (N-2)\frac{v^2}{2} r^{1-N})\\
&\leq (1 + Cr)\frac{rI(r, v)}{2\gamma(\Gamma_v(r))}, \qquad (7.3)
\end{aligned}$$

where $\gamma(\Gamma_v(r))$ is the corresponding r-dependent value for v, as that in Lemma 2.2, and v_n is the normal-directional derivative of v on the sphere $\partial B(0,r)$. Now let us consider the new function

$$\psi(r, D_e u) = \varphi(r, D_e u) e^{Cr},$$

with C to be chosen later, so as to make ψ non-decreasing. Observe also that $(D_e u)^{\pm}$ are sub-solutions to (7.1). Differentiating we obtain

$$\psi'(r) = \frac{\psi(r)}{r}\left[\frac{rI(r,(D_e u)^+)}{J(r,(D_e u)^+)} + \frac{rI(r,(D_e u)^-)}{J(r,(D_e u)^-)} - 4 + Cr\right]. \tag{7.4}$$

Plugging (7.3) in (7.4) we obtain

$$\psi' \geq \frac{\psi(r)}{r}\left[2\gamma(\Gamma_+)(1+O(r)) + 2\gamma(\Gamma_-)(1+O(r)) - 4 + Cr\right].$$

Now choosing C large the extra terms will be taken care of, and the result follows as in [ACF, Lemma 5.1].

References

[ACF] H.W. Alt, L.A. Caffarelli, A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. AMS **282** (1984), 431–461.

[AC] I. Athanasopoulos, L.A. Caffarelli,, *A theorem of real analysis and its application to free boundary problems*, Comm Pure applied mathematics **38** (1985), 499–502.

[BKP] W. Beckner, C. Kenig, J. Pipher, *In preparation.*

[Ca1] L.A. Caffarelli, *The regularity of free boundaries in higher dimension*, Acta Math. **139** (1977), 155-184.

[Ca2] L.A. Caffarelli, *Compactness methods in free boundary problems*, Comm. P.D.E. **5** (1980), 427-448.

[Ca3] L.A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries*, Scoula Norm. Sup., Pisa **15** (1988), 583–602.

[Ca4] L.A. Caffarelli, *The obstacle problem revisited*, journal of harmonic analysis and applications (1998 to appear).

[CK] L.A. Caffarelli, C. Kenig, *Gradient estimates for variable coefficient parabolic equations and singular perturbation problems*, Amer. J. of Math. **120(2)** (1998), 391–440.

[DF] E. DiBenedetto and A. Friedman, *Bubble growth in porous media*, Indiana Univ. Math. J. **35** (1986), 573-606.

[FH] S. Friedland, W.K. Hayman, *Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions*, Comment. Math. Helv. **51** (1976), 133–161.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag 2^{nd} edition, Berlin Heidelberg New York Toko, 1983.

[GP] B. Gutafsson, M. Putinar, *The exponential transform and regularity of free boundaries in two dimensions*, Scoula Norm. Sup., Pisa (to appear).

[I1] V. Isakov, *Inverse Source Problems*, AMS Math. Surveys and Monographs 34, Providence, Rhode Island, 1990.

[I2] V. Isakov, *Inverse theorems on the smoothness of potentials*, Differential Equations **11** (1976), 50-57 [translated from Russian].

[KS1] L. Karp and H. Shahgholian, *Regularity of a free boundary problem*, J. Geometric Analysis; to appear.

[KS2] L. Karp and H. Shahgholian, *Regularity of a free boundary at the infinity point*, manuscript.

[KN] D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary value problems*, Ann. Scu. Norm. Sup. Pisa **4** (1977), 373-391.

[Ma] A.S.Margulis, *Potential theory for L^p-densities and its applications to inverse problems of gravimetry*, Theory and Practice of Gravitational and Magnetic Fields Interpretation in USSR, Naukova Dumka Press, Kiev, 1983, pp. 188-197 (Russian).

[MY] J. McCarthy, Liming Yang, *Subnormal operators and quadrature domains*, Adv. Math. **127 (1)** (1997), 52–72.

[P] M. Putinar, *Extremal solutions of the two dimensional L-problems of moments*, J. Funct. Anal. **136** (1996), 331–364.

[Sa1] M. Sakai, *Quadrature Domains*, Lecture Notes in Math., vol. 934, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[Sa2] M. Sakai, *Regularity of a boundary having a Schwarz function*, Acta Math. **166** (1991), 263-297.

[Sa3] M. Sakai, *Regularity of boundaries of quadrature domains in two dimensions*, SIAM J. Math. Anal. **24** (1993), 341-364.

[Shah] H. Shahgholian, *On quadrature domains and the Schwarz potential*, J. Math. Analys. **171(1)** (1992), 61–78.

[Shap1] H.S. Shapiro, *Global geometric aspects of Cauchy's problem for the Laplace operator*, Geometrical and algebraical aspects in several complex variables (Cetraro 1989), pp. 309–324 sem. conf. 8.

[Shap2] H.S. Shapiro, *The Schwarz Function and Its Generalization to Higher Dimensions*, Univ. of Arkansas Lecture Notes in the Math. Sci., vol. 9, Wiley, New York, 1992.

[Spn] E. Sperner, *Zur symmetrisierung von funktionen auf sphären*, Math. Z. **134** (1973), 317–327.

[Spk] J. Spruck, *Uniqueness in diffusion model of population biology*, Comm. P.D.E. **8** (1983), 1605-1620.

[St] V.N. Strakhov, *The inverse logarithmic potential problem for contact surface*, Physics of the Solid Earth **10** (1974), 104-114 [translated from Russian].

[W] S. Wiliams, *Analyticity of the boundary for Lipschitz domains without the Pompeiu property*, Indiana Univ. Math. J. **30** (1981 no. 3), 357–369.

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