REGULARITY OF A FREE BOUNDARY AT THE INFINITY POINT

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Abstract. Suppose there is a nonnegative function u and an open set $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, satisfying

$$\begin{cases} \Delta u = \chi_\Omega & \text{in} \quad B_r^e, \\ u = |\nabla u| = 0 & \text{on} \quad B_r^e \setminus \Omega, \end{cases}$$

where $B_r^e = \{x : |x| > r\}$. Under a certain thickness condition on $\mathbb{R}^n \setminus \Omega$, we prove that $\partial\{x : x/|x|^2 \in \Omega\}$ is the graph of a C^1–function in a neighborhood of the origin.

As a by-product of the method of the proof, we also obtain the following result: Replace χ_Ω by $f\chi_\Omega$, with certain assumptions on f. Then for any solution u which is asymptotically nonnegative at infinity, there holds

$$\lim_{r\to\infty} \frac{|\Omega \cap B_r|}{|B_r|} \in \{\frac{1}{2}, 1\}.$$

0. Introduction

The regularity of free boundaries, arising in the obstacle problem, was studied by L.A. Caffarelli. This paper deals with the infinity-point version of Caffarelli's result. The problem can be formulated as follows.

Suppose there are C^1–functions u, f, and an open set $\Omega := \Omega(u) \subset \mathbb{R}^n$ satisfying, for some $r > 0$,

$$\text{(0.1)} \qquad \begin{cases} \Delta u = f\chi_\Omega & \text{in} \quad B_r^e, \\ u = |\nabla u| = 0 & \text{on} \quad \Lambda(u) := B_r^e \setminus \Omega, \end{cases}$$

where the Laplacian is in the sense of distributions, $\Lambda(u) := \{x : u(x) = |\nabla u(x)| = 0\}$, $B_r^e := \{x : |x| > r\}$ and χ_D is the characteristic function of D, a subset of $\mathbb{R}^n$.

Our main result asserts that if $\Lambda(u)$ is sufficiently "thick" at the infinity point, $u \geq 0$, and $f \equiv 1$, then

$$\partial\Omega_{inv} := \partial\{x : \frac{x}{|x|^2} \in \Omega\}$$

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is the graph of a C^1–function in a neighborhood of the origin.

The above free boundary problem appears in several areas in mathematical physics. For example, certain types of Hele-Shaw flow lead to equation (0.1); see [BF], [EE] [EV], [Ho]. Also in Geophysics, in the problem of gravimetrical prospecting, one studies this type of equation; see [Ma] and [St] and the literatures cited therein. In the present paper, except from the below example, due to Sakai, we are not concerned with the existence of solutions to (0.1).

Example. (Sakai) A trivial example of solution to (0.1) is

$$u(x) = \frac{1}{2}(\max\{x_n, 0\})^2 \qquad \text{with } \Omega(u) = \{x : \ x_n > 0\}.$$

Below we present Sakai's method [Sa1; §11] for construction of nontrivial solutions to (0.1). The free boundary in this construction is, in a certain sense, asymptotically a hyperplane.

Let $g \geq 0$ be a bounded function with compact support in $\Pi_+ := \{x : x_n > 0\}$. Let $\{D_m\}$ be a monotone exhaustion of Π_+ such that $\mathrm{supp}(g) \subset D_1$. Then for each m there is $u_m \geq 0$ and a bounded domain $\Omega_m \supset D_m$ such that

$$\begin{cases} \Delta u_m = \chi_{\Omega_m} - \chi_{D_m} - g & \text{in } \ \mathbb{R}^n, \\ u_m = |\nabla u_m| = 0 & \text{on } \ \mathbb{R}^n \setminus \Omega_m, \end{cases} \tag{0.2}$$

(see [Sa2; Theorem 7.5]). Furthermore, by monotonicity (see e.g. Lemma 6.1 in [Sa2]), $\Omega_m \subset \Omega_{m+1}$. Letting $\Omega := \cup_m \Omega_m$ and using standard compactness arguments, we'll have $u_m \to u_0$ uniformly on compact sets of $\mathbb{R}^n$, where u_0 satisfies

$$\begin{cases} \Delta u_0 = \chi_\Omega - \chi_{\Pi_+} - g & \text{in } \ \mathbb{R}^n, \\ u_0 = |\nabla u_0| = 0 & \text{on } \ \mathbb{R}^n \setminus \Omega. \end{cases}$$

Defining

$$u = u_0 + \frac{1}{2}(\max\{x_n, 0\})^2$$

and taking the inclusion $\Pi_+ \subset \Omega$ into account, we'll have

$$\begin{cases} \Delta u = \chi_\Omega - g & \text{in } \ \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{on } \ \mathbb{R}^n \setminus \Omega. \end{cases} \tag{0.3}$$

Since $\mathrm{supp}(g) \subset B_r$ for some $r > 0$, equation (0.3) is of type (0.1) in B_r^e. Moreover, (0.2) implies

$$|\Omega_m \setminus D_m| = \int \Delta u_m dx + \int g dx = \int g dx,$$

which by Fatou's Lemma reduces to

$$|\Omega \setminus \Pi_+| \leq \liminf_m |\Omega_m \setminus D_m| = \int g dx.$$

Hence

$$\lim_{r \to \infty} \frac{|\Omega \cap B_r|}{|B_r|} = \frac{1}{2}.$$

We return now to the general problem. For u in $C^1(\mathbb{R}^n)$, we set $\Lambda(u) = \{x : u(x) = |\nabla u(x)| = 0\}$, $\Omega(u) = \mathbb{R}^n \setminus \Lambda(u)$ and $\Omega^-(u) = \{x : \ u < 0\}$. For convenience, we also define two groups of conditions.

Condition A. Let $m, M_0 > 0$, $\gamma > 1$ and $x \in \mathbb{R}^n$. Then we refer to the following as Condition A:

$$u(x) \leq C(1 + |x|^m), \tag{A.1}$$

$$\lim_{x\to\infty} f(x) = 1, \tag{A.2}$$

$$|\nabla f(x)| \leq \frac{M_0}{|x|^\gamma}, \tag{A.3}$$

$$\lim_{r\to\infty} \frac{|\Omega^-(u) \cap B_r|}{|B_r|} = 0. \tag{A.4}$$

Condition B.

$$f \equiv 1, \qquad \text{in } B^e_{r_0}, \quad (\text{ for some } r_0 > 0), \tag{B.1}$$

$$u \geq 0, \qquad \text{in } B^e_{r_0}, \tag{B.2}$$

$$\limsup_{r\to\infty} \frac{\mathrm{MD}(\Lambda(u) \cap B_r)}{r} > 0 \tag{B.3}$$

where $\mathrm{MD}(\cdot)$ is the minimum diameter; see Notation.

The main purpose of this paper is to show the following theorems:

Theorem A. *Any solution to (0.1), subject to Condition A, satisfies*

$$\lim_{r\to\infty} \frac{|\Omega \cap B_r|}{|B_r|} \in \{\frac{1}{2}, 1\}.$$

Theorem B. *For any solution to (0.1), subject to Condition B,*

$$\partial\Omega_{inv} := \partial\left\{x : \frac{x}{|x|^2} \in \Omega\right\}$$

is a C^1 hypersurface in a neighborhood of the origin.

Theorem B is the infinity-point counterpart of Caffarelli's local regularity theory for the free boundary in the obstacle problem; see [Ca1], [Ca2]. Theorem A is a weaker version of the regularity theory, and its proof requires only that the function u to be asymptotically nonnegative at infinity; see (A.4).

In $\mathbb{R}^2$ and for $f \equiv 1$, Theorems A and B were proved by Shapiro [Shp] and Sakai [Sa4], respectively; in both cases no restriction on the sign of u was considered.

The advantage of the two dimensional space, when $f \equiv 1$, is simply the access to Function Theory. Because in this case

$$S(z) = 4\frac{\partial u}{\partial z} - \overline{z}, \qquad\qquad (z = x_1 + ix_2)$$

serves as the Schwarz function of $\Omega \cap B_r^e$, i.e. $S(z)$ is analytic in $\Omega \cap B_r^e$ and satisfies the boundary condition $S(z) = \bar{z}$ on $\partial\Omega \cap B_r^e$ (see e.g. [AS], [D]). Now the Schwarz function enjoys a very useful property: if $S(z)$ is the Schwarz function for $\Omega \cap B_r^e$, then $1/S(z^{-1})$ is the Schwarz function of $\Omega_{inv} \cap B_{(1/r)} := \{z : 1/z \in \Omega\} \cap B_{(1/r)}$. Thus, in the two dimensional case, the investigation of the regularity of free boundaries near the infinity point is transformed to the corresponding local one. Also, a complete description of these free boundaries was given by Sakai [Sa3].

Contrary to the two dimensional case, the inversion in the higher dimensional space produces new difficulties. Indeed, let $f \equiv 1$ and set

$$\tilde{u}(x) = |x|^4 u(\frac{x}{|x|^2}).$$

Then tedious, but elementary, calculations show that

$$\begin{cases} \Delta\tilde{u} = \tilde{f}\chi_{\Omega_{inv}} & \text{in} \quad B_{(1/r)}, \\ \tilde{u} = |\nabla\tilde{u}| = 0 & \text{on} \quad B_{(1/r)} \setminus \Omega_{inv}, \end{cases} \tag{0.4}$$

where

$$\tilde{f}(x) = 1 + 2(n+2)(2|x|^2 u(\frac{x}{|x|^2}) - x \cdot \nabla u(\frac{x}{|x|^2})).$$

Hence $\tilde{u}$ solves a local version of (0.1), and one may thus expect that Caffarelli's regularity theory would work. But then, a "small" problem occurs. Namely, the function $\tilde{f}$ is not necessarily C^α, which is a prerequisite for the application of Caffarelli's technique to (0.4). We are thus forced to study the problem near the infinity point and in its original setting, i.e. (0.1).

Technical Remark. For convenience, both for the reader and for us, we would like to consider equation (0.1) in the entire space $\mathbb{R}^n$. This can be done by taking a cut-off function φ such that $\varphi(x) = 0$ for $|x| \leq r$ and $\varphi(x) = 1$ for $|x| \geq 2r$. Then the function $u\varphi$ will satisfy

$$\Delta(u\varphi) = (\varphi f + 2\nabla\varphi \cdot \nabla u + (\Delta\varphi)u)\,\chi_{\Omega(u\varphi)} \quad \text{in } \mathbb{R}^n, \qquad 0 \in \Lambda(\varphi u).$$

Also

$$\Omega(u\varphi) \setminus B_{2r} \equiv \Omega(u) \setminus B_{2r}.$$

Thus, hence forward we will consider the equation

$$\begin{cases} \Delta u = f\chi_{\Omega(u)} & \text{in} \quad \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{on} \quad \Lambda(u), \\ 0 \in \Lambda(u), \end{cases} \tag{0.5}$$

where $\Omega(u) := \mathbb{R}^n \setminus \Lambda(u)$ and $\Lambda(u) := \{x : u(x) = |\nabla u(x)| = 0\}$.

Although the present work much owes to Caffarelli's pioneering work [Ca2], it should be emphasized that the "infinity-point" version of the regularity theory leads to new substantial difficulties.

The paper is divided into eight sections. In Section 1 we show that solutions to (0.5) (or (0.1)) satisfying (A.1) and (A.4) grow quadratically, and that nonnegative solutions have bounded second order derivatives. Section 2 deals with the properties of the shrinking down (see the definition below); we prove a modified version of a theorem of Spruck (Theorem 2.5), which asserts that any shrinking down is a homogeneous function of degree two. In Section 3 we show the convexity of the shrinking down u_0. In Section 4, we show certain flatness property ((4.2a) and (4.2b)) of the free boundary. This property is enough for proving Theorem A; and the proof is carried out in Section 5. Section 6 is the heart of this work, several new ingredients are invoked, estimates of the second order derivatives are achieved, and the flatness property, in Section 4, is improved and replaced by the cone property (6.12a) and (6.12b). Theorem B is proven in Section 7. Section 8 (Appendix) is devoted to a technical lemma which is needed to show Lemma 6.4.

Notation

C	a generic constant
χ_D	the characteristic function of the set $D, (D \subset \mathbb{R}^n, n \geq 2)$
$\overline{D}$	the closure of D
∂D	the boundary of a set D
D_{inv}	$\{x : x/\lvert x\rvert^2 \in D\}$
$\lvert D\rvert$	$n -$ dimensional volume of the set D
$B_r(x)$	$\{y \in \mathbb{R}^n : \lvert y - x\rvert < r\}$
B_r^e	$\mathbb{R}^n \setminus \overline{B_r} = \{y \in \mathbb{R}^n : \ \lvert y\rvert > r\}$
$\mathbb{S}^{n-1}$	the unit sphere in $\mathbb{R}^n$
ω_n	area of the unit sphere in $\mathbb{R}^n$
$d(x)$	distance function
$\Lambda(u)$	$\{x : u(x) = \lvert\nabla u(x)\rvert = 0\}$, for any C^1–function u
$\Omega(u)$	$\mathbb{R}^n \setminus \Lambda(u)$
$\Omega^+(u)$	$\{x : u(x) > 0\}$
$\Omega^-(u)$	$\{x : u(x) < 0\}$
$\Gamma(u)$	$\partial\Omega(u)$
$\lVert \cdot \rVert_\infty$	Supremum norm
MD	minimum diameter; se the definition below
$\mathbf{e}$	$(e_1, \cdots, e_n)$, orthonormal basis
$\mathbb{K}_\delta^-(\mathbf{e})$	$\{x : x_n < -\lvert x\rvert \sin\delta\}$ in the base $\mathbf{e}$
$\mathbb{K}_\delta^+(\mathbf{e})$	$\{x : x_n > \lvert x\rvert \sin\delta\}$ in the base $\mathbf{e}$
ν	arbitrary unit vector
$D_\nu, D_{\nu\nu}$	first and second $\nu -$ directional derivatives, respectively
$\alpha(x, y)$	$\arccos(\frac{x \cdot y}{\lvert x\rvert\lvert y\rvert})$, the angle between x and y as vectors.

Minimum Diameter. The minimum diameter of a set D, denoted MD(D), is the infimum of distances between pairs of parallel planes such that D is contained in the strip determined by the planes.

Scaling and Shrinking Down. For any sequence $\rho_m \nearrow \infty$, we define

$$u_m(x) = u_{\rho_m}(x) = \frac{u(\rho_m x)}{\rho_m^2}.$$

If for a subsequence $\{\rho_{m_j}\}$ the sequence $u_{\rho_{m_j}}$ converges to a function u_0, then we call u_0 a "shrinking down" of u.

1. The optimal regularity of solutions

The results of this section concern the quadratic growth (Theorem 1.1), and bounds for the second order derivatives (Lemma 1.3) for solutions to (0.5).

1.a. The quadratic growth. We introduce the following notation

$$L(r,u) := \frac{|B_r \cap \Omega^-(u)|}{|B_r|}.$$

Theorem 1.1. *Suppose $\Delta u = f$ in $\mathbb{R}^n$, $f \in L^\infty(\mathbb{R}^n)$, $|u(x)| \le C_0(1+|x|)^m$ $(m \ge 2)$ and*

$$\limsup_{r\to\infty} L(r,u) < \frac{4^{2-m}}{\kappa_n}, \tag{1.1}$$

where $\kappa_n = 2^{(2n+6)}$. Then

$$|u(x)| \le C\|f\|_\infty (1+|x|)^2, \qquad \forall x \in \mathbb{R}^n, \tag{1.2}$$

where C depends on the limit in (1.1), C_0, and the space dimension.

Proof. We may assume $f(x) \ge 0$ and $\|f\|_\infty \le 1$, or else we replace u by

$$\tilde u(x) = \frac{u(x)}{2\|f\|_\infty} + \frac{1}{4n}|x|^2,$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Since $\{x : \tilde u(x) < 0\} \subset \{x : u(x) < 0\}$, we'll have $L(r,\tilde u) \le L(r,u)$. Thus $\tilde u$ will satisfy the hypotheses of the theorem.

Define M_r by the relation

$$M_r r^2 = \sup_{B_r} u.$$

Then according to [KS1, Lemma 1.5],

$$M_r \le \kappa_n (1 + L(2r,u) M_{4r}). \tag{1.3}$$

Let $\theta = \theta(r_0) := \sup_{r\ge r_0} L(r,u)$. Iterating (1.3) for $r \ge r_0$, we obtain

$$M_r \le (\theta\kappa_n)^k M_{4^k r} + \kappa_n \sum_{j=0}^{k-1} (\theta\kappa_n)^j, \qquad k = 1,2,...$$

By the growth assumption on u, we have $M_r \le C_0 r^{(m-2)}$, which in conjunction with the last estimate gives

$$M_r \le C_0 \beta^k r^{m-2} + \kappa_n \sum_{j=0}^{k-1} (\theta\kappa_n)^j,$$

where $\beta = \theta\kappa_n 4^{m-2}$. In view of (1.1), we can take r_0 sufficiently large so that $\beta < 1$. Hence, letting $k \to \infty$, we obtain for $r \geq r_0$

$$M_r \leq \frac{\kappa_n}{1-\theta\kappa_n},$$

that is

$$\sup_{B_r} u = M_r r^2 \leq \frac{\kappa_n}{1-\theta\kappa_n} r^2, \qquad \forall r \geq r_0. \tag{1.4}$$

Now by [KS1; Lemma 1.3],

$$\sup_{B_r} |u| \leq C(\sup_{B_{2r}} u + r^2), \tag{1.5}$$

which together with (1.4) gives the desired result. □

Remark. The formulas (1.3) and (1.5) are proved in [KS1] for $r \in (0,1)$. However, they are valid for any $r > 0$. Also, it follows from the proof of Theorem 1.1 that if u is nonnegative for $|x| \geq r$, then the constant in (1.2) only depends on $\|f\|_\infty$, r and the space dimension.

1.b. Bounds for the second order derivatives.

It is our aim here to show that nonnegative solutions to (0.5), have bounded second order derivatives and the bound is uniform.

Lemma 1.2. *In (0.5), suppose f is nonnegative in B_2^e and $u \geq 0$ in $\mathbb{R}^n$. Then*

$$u(x) \leq C\|f\|_\infty d^2(x) \qquad \textit{for } x \in B_4^e, \tag{1.6}$$

where $d(x) := dist(x, \Lambda(u))$ and the constant C depends only on the dimension.

Proof. Fix $x \in B_4^e$. If $d(x) \geq |x|/2$, then by Theorem 1.1 (and the above Remark), we have

$$u(x) \leq C\|f\|_\infty |x|^2 \leq 4C\|f\|_\infty d^2(x).$$

If $2d(x) < |x|$, then one readily verifies that $B_{d(x)}(x) \subset B_2^e$. It thus follows that u is subharmonic in $B_{d(x)}(x)$. Hence

$$u(x) \leq \frac{1}{|B_{d(x)}|}\int_{B_{d(x)}(x)} u dy \leq \frac{1}{|B_{d(x)}|}\int_{B_{2d(x)}(x^0)} u dy,$$

where $x^0 \in \Gamma(u)$ is a point such that $d(x) = |x - x^0|$.

Now to estimate the last integral, observe that $\frac{\|f\|_\infty |y-x^0|^2}{2n} - u(y)$ is subharmonic in $\mathbb{R}^n$ and vanishes at x^0. Hence

$$0 \leq \int_{B_{2d(x)}(x^0)} \left(\frac{\|f\|_\infty |y-x^0|^2}{2n} - u(y) \right) dy,$$

which amounts to

$$\int_{B_{2d(x)}(x^0)} u(y)dy \leq \frac{\|f\|_\infty}{2n}\int_{B_{2d(x)}(x^0)} |y-x^0|^2 dy = \frac{2^{n+1}|B_1|}{(n+2)}\|f\|_\infty (d(x))^{n+2}.$$

This gives the desired result. □

Lemma 1.3. *In (0.5), let f be nonnegative in B_2^e and $u \geq 0$ in $\mathbb{R}^n$. Suppose, moreover,*

$$|\nabla f(x)| \leq \frac{M_0}{1+|x|}, \qquad \forall\, x \in \mathbb{R}^n. \tag{1.7}$$

Then

$$|D_{ij} u(x)| \leq M, \qquad \forall\, x \in B_5^e, \tag{1.8}$$

where M depends on $\|f\|_\infty$, M_0 and the space dimension.

Proof of Lemma 1.3. Let first $x \in \Lambda(u) \cap B_5^e$. Then by Lemma 1.2, for any $h \in \mathbb{R}^n$ with $|h| \leq 1$, we have

$$|u(x+h) - 2u(x) + u(x-h)| \leq C|h|^2.$$

Hence (1.8) follows in this case. For $x \in \Omega(u) \cap B_5^e$, we have by standard elliptic estimate (see e.g., [GT; Theorem 3.9]),

$$\rho\, |\nabla u(x)| \leq C \left(\sup_{B_\rho(x)} |u| + \rho^2 \sup_{B_\rho(x)} |\Delta u| \right) \tag{1.9}$$

Now for any $y \in B_{d(x)}(x)$, there holds $d(y) \leq 2d(x)$. Hence, by Lemma 1.2 and (1.9) with $\rho = d(x)$,

$$|D_i u(x)| \leq C\|f\|_\infty d(x). \tag{1.10}$$

Since $\Delta(D_i u) = D_i f$, we can apply (1.9) with $\rho = d(x)/2$, which in conjunction with (1.7) gives

$$\frac{d(x)}{2} |D_{ij} u(x)| \leq C \left(\sup_{B_{(\frac{d(x)}{2})}} |D_i u| + \left(\frac{d(x)}{2}\right)^2 M_0 \sup_{B_{(\frac{d(x)}{2})}} \frac{1}{1+|y|} \right).$$

Since $d(x) \leq |x|$,

$$\frac{d(x)^2}{4} \sup_{B_{(\frac{d(x)}{2})}} \frac{1}{1+|y|} \leq \frac{d(x)|x|}{2(2+|x|)} \leq \frac{d(x)}{2}.$$

Combining the last two inequalities with (1.10) we obtain (1.8). $\square$

Remark. It follows from the proof of Lemmas 1.2 and 1.3 that if $f \geq 0$ in $\mathbb{R}^n$, then the estimate (1.6) and (1.8) hold for all $x \in \mathbb{R}^n$.

2. Shrinking down

Let $\{\rho_m\}$ be any sequence of positive numbers and set

$$u_m(x) = u_{\rho_m}(x) = \frac{u(\rho_m x)}{\rho_m^2}.$$

In this section we will derive several properties of the limit function $u_0 = \lim_m u_m$, where u is a solution to (0.5) . The case when $\lim_m \rho_m = \infty$ is of particular interest, and it is referred to as a " shrinking down" of u (blow up when $\lim \rho_m = 0$).

We first show that the quadratic growth of u suffices for the existence of a convergent subsequence of $\{u_m\}$. As a rule in this paper, we will not distinguish between sequences and subsequences.

Proposition 2.1. *Suppose $\Delta u = f$ in $\mathbb{R}^n$, where $f \in L^\infty(\mathbb{R}^n)$, and*

$$|u(x)| \le C|x|^2, \qquad \forall\ x \in \mathbb{R}^n. \tag{2.1}$$

Then for any sequence $\{\rho_m\}$ with $\rho_m \nearrow \infty$, there is a function u_0 such that $u_m \to u_0$ and $\nabla u_m \to \nabla u_0$ uniformly on compact sets of $\mathbb{R}^n$.

Proof. By (2.1),

$$\sup_{B_r} |u_m| \le Cr^2$$

for any $r > 0$. Hence, the $L^p(B_r)$-norm of u_m is bounded by a constant depending only on r. Then, for r fixed and $1 < p < \infty$, the elliptic estimate (see e.g. [GT])

$$\|u\|_{W^{2,p}(B_r)} \le C\left(\|u\|_{L^p(B_{2r})} + \|\Delta u\|_{L^p(B_{2r})}\right)$$

implies that $\{u_m\}$ is bounded in $W^{2,p}(B_r)$. Also for $0 < \sigma < 1 - n/p$ (p large), the embedding $W^{2,p}(B_r) \hookrightarrow C^{1,\sigma}(\bar{B}_r)$ is compact. Hence the final conclusion follows now by standard arguments. □

Remark. For nonnegative u in (0.5) we have $\Lambda(u_m) = \{x : \ u_m = 0\}$. Hence it suffices that $u_m \to u_0$ uniformly on compact sets.

Next we introduce the notation $\Omega^+ := \{x : \ u(x) > 0\}$.

Proposition 2.2. *Let u be a solution to (0.5) satisfying (2.1) and (A.2). Let also D be a set with the following property*

$$D \subset \Omega^+(u_m) \cap B^e_{(r_0/\rho_m)}, \qquad \forall\ m.$$

Then $D \subset \overline{\Omega^+(u_0)}$.

Proof. Observe that by (A.2), $f(x) \ge 0$ for large $|x|$. So the proposition follows from [Ca2; Corollary 1]. □

Proposition 2.3. *Under the hypotheses of Proposition (2.2), the function u_0 solves the complementary problem*

$$u_0(\Delta u_0 - 1) = 0 \qquad in \quad \mathbb{R}^n, \tag{2.2}$$

with $0 \in \Lambda(u_0)$.

Proof. Clearly $0 \in \Lambda(u_m)$ for all m. Hence $0 \in \Lambda(u_0)$. If $x^0 \in \Omega(u_0)$, then $u_0(x^0) \neq 0$ or $|\nabla u_0(x^0)| \neq 0$. Hence for some $\varepsilon > 0$, $B_\varepsilon(x^0) \subset \Omega(u_m)$ for all large enough m. Also, by the assumption on f, $\Delta u_m(x) = f(\rho_m x) \to 1$, as $m \to \infty$ and $x \in B_\varepsilon(x^0)$. Thus (2.2) holds and the proof is completed. □

The following result (Theorem 2.5, below) is an essential ingredient in the regularity theory. It asserts that the shrinking down u_0, is a homogeneous function of degree two, which implies that $\Omega(u_0)$ is a cone. The proof is essentially due to Spruck [Sp]. However, to adapt his method to our case we need Ahlfors-Bers mollifier [Ah], [B]; see also [He].

Lemma 2.4. *(Ahlfors-Bers) For any open set $D \subset \mathbb{R}^n$, there is a sequence of functions $\{w_k\} \subset C_0^\infty(D)$ such that: $0 \leq w_k(x) \leq 1$, $\lim_k w_k(x) = 1$ for all $x \in D$, and*

$$|\partial^\alpha w_k(x)| \leq \frac{C}{k}\frac{1}{d^{|\alpha|}(x)|\log(d^{-1}(x))|}, \qquad |\alpha| \geq 1, \tag{2.3}$$

where C depends on α and ∂^α denotes the standard multi-index notation for partial derivatives.

Theorem 2.5. *(Essentially due to Spruck) In equation (0.5) assume f satisfies (A.3). Then, for any solution u having quadratic growth (2.1), there holds that any shrinking down u_0, is a homogeneous function of degree two.*

Proof. Denote by (r, Θ) the spherical coordinates, and set

$$\Psi(t, \Theta) := \frac{u(r,\Theta)}{r^2}, \qquad t = \log r.$$

Then, by the quadratic growth of u, both Ψ and $\partial_t \Psi$ are bounded. Computations show

$$\partial_t^2 \Psi + (n+2)\partial_t \Psi + 2n\Psi + \Delta_\Theta \Psi = f\chi_{\Omega(u)} \qquad \text{in } \mathbb{R}^n, \tag{2.4}$$

and

$$\Psi = |\nabla \Psi| = 0 \qquad \text{on } \Lambda(u).$$

Multiplying both sides of (2.4) by $\partial_t \Psi$ and integrating over the unit sphere $\mathbb{S}^{n-1}$ and the interval $[T_0, T]$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{S}^{n-1}} (\partial_t \Psi)^2 d\Theta \,\Big|_{T_0}^T + (n+2)\int_{T_0}^T \int_{\mathbb{S}^{n-1}} (\partial_t\Psi)^2 d\Theta dt + n\int_{\mathbb{S}^{n-1}} \Psi^2 d\Theta \,\Big|_{T_0}^T \\ &-\frac{1}{2}\int_{\mathbb{S}^{n-1}} |\nabla_\Theta \Psi|^2 d\Theta \,\Big|_{T_0}^T = \int_{T_0}^T \int_{\mathbb{S}^{n-1}} (\partial_t \Psi) f\chi_{\Omega(u)} d\Theta dt. \end{aligned} \tag{2.5}$$

Our aim is to bound
$$\int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}(\partial_t\Psi)^2 d\Theta dt.$$
Since all other terms of the left hand side of (2.5) are bounded, it remains to bound the right hand side. One way of doing this is through integration by parts, which requires that the function $\partial_t\chi_\Omega$ be integrable. To circumvent this difficulty we are forced to use the Ahlfors-Bers mollifier (Lemma 2.4). Hence we'll have

$$\int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}(\partial_t\Psi)f\chi_{\Omega(u)}d\Theta dt = \lim_k \int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}(\partial_t\Psi)fw_k d\Theta dt.$$

Next
$$\begin{aligned}&\int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}(\partial_t\Psi)fw_k d\Theta dt = \int_{\mathbb{S}^{n-1}}\Psi f w_k d\Theta\ |_{T_0}^{T}\\ &-\int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}\Psi(\partial_t f)w_k d\Theta dt - \int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}\Psi f(\partial_t w_k)d\Theta dt\\ &=I_k^1+I_k^2+I_k^3.\end{aligned}$$

We now estimate each of this integrals separately. The first term is clearly bounded. Also, by (A.3) we have $|\partial_t f(e^t,\Theta)|\le M_0e^{t(1-\gamma)}$, which gives

$$|I_k^2|\le CM_0\int_{T_0}^{T}\int_{\mathbb{S}^{n-1}}e^{t(1-\gamma)}d\Theta dt \le C\frac{M_0}{\gamma-1}e^{T_0(1-\gamma)} =: C(T_0,\gamma,M_0).$$

Note that we do not assume $T_0\ge 0$.

Since $f\chi_{\Omega(u)}\in L^\infty$, $\partial_i u$ belongs to the Zygmund class. Therefore,

$$|u(x)|\le C|x|d^2(x)\log(1/d(x)) \tag{2.6}$$

for all $x\in\Omega(u)$ with $d(x)<e^{-2}$ (see e.g. [Ka]). Combining (2.3) with (2.6) we have $|u\partial_r w_k|\le Crk^{-1}$, i.e. $|\Psi\partial_t w_k|\le Ck^{-1}$; observe that $\partial_r w_k(x)=0$ for $d(x)\ge e^{-2}$. Hence for large enough k,

$$|I_k^3|\le\frac{C}{k}\|f\|_\infty\int_{T_0}^{T}dt\le\frac{C\|f\|_\infty(T-T_0)}{k}.$$

Letting k and T tend to infinity, respectively, we obtain

$$\int_{T_0}^{\infty}\int_{\mathbb{S}^{n-1}}(\partial_t\Psi)^2 d\Theta dt\le C(T_0,\gamma,M_0). \tag{2.7}$$

For any sequence $\rho_m\nearrow\infty$, we now consider the scaling $u_m(x):=\rho_m^{-2}u(\rho_m x)$ and its corresponding Ψ_m. Each u_m satisfies (0.5) with the right hand side $f_m(x):=f(\rho_m x)$. By the assumptions (A.3) and (2.1), the bound (2.7) holds for Ψ_m uniformly in m. Also, by Proposition 2.1, u_m tends to u_0 and $\nabla u_m\to\nabla u_0$ uniformly on compact sets. Hence $\partial_t \Psi_m \to \partial_t \Psi_0$ in $L^2((T_0, N) \times \mathbb{S}^{n-1})$ for any N. Since $\Psi_m(t) = \Psi(t + \log \rho_m)$, we have by (2.7),

$$\int_{T_0}^{\infty} \int_{\mathbb{S}^{n-1}} (\partial_t \Psi_0)^2 d\Theta dt = \lim_m \int_{T_0}^{\infty} \int_{\mathbb{S}^{n-1}} (\partial_t \Psi_m)^2 d\Theta dt$$
$$= \lim_m \int_{T_0 + \log \rho_m}^{\infty} \int_{\mathbb{S}^{n-1}} (\partial_t \Psi)^2 d\Theta dt = 0.$$

Thus, $\partial_t \Psi_0 = 0$ in (T_0, ∞) for any $-\infty < T_0 < \infty$. Therefore $\Psi_0(t, \Theta) = \Psi_0(\Theta)$, i.e. $u_0(r, \Theta) = r^2 \Psi_0(\Theta)$. □

Remark. A similar result as that in Theorem 2.5 can be obtained for the local problem (0.4). One should in this case set $t = -\log r$. Then, essentially, the same proof would give that the blow up is a homogeneous function of degree two.

We now show that if in addition f satisfies (A.2), then $\Omega(u_0)$ is a non-empty cone.

Lemma 2.6. *Let u be a solution to (0.5), and suppose conditions (2.1), (A.2) and (A.3) hold. If $\Omega(u)$ is unbounded, then for any shrinking down u_0, $\Omega(u_0)$ is a non-empty open cone with vertex at the origin.*

Proof. We need to prove a nondegeneracy condition with respect to the infinity point, that is

$$\sup_{|x|=r} |u(x)| \geq \frac{1}{16n} r^2, \qquad \forall\ r \text{ large enough.} \tag{2.8}$$

Assuming (2.8), we have for m large

$$\sup_{B_1} |u_m| = \sup_{B_{\rho_m}} \frac{|u|}{\rho_m^2} \geq \frac{1}{16n}.$$

Therefore there is a $\Theta_0 \in \mathbb{S}^{n-1}$ such that $|u_0(1, \Theta_0)| > 0$, and by continuity it holds in some neighborhood of Θ_0. By Theorem 2.5, u_0 is homogeneous of degree two, and hence $\Omega(u_0)$ contains an open cone.

Note that by (A.2) there is $r_0 > 0$ such that $f(x) \geq 1/2$ for $|x| \geq r_0$. Now we split the proof of (2.8) into two cases.

Case 1: Suppose there is an unbounded sequence $\{x^j\}$ with $u(x^j) > 0$. Then the function $v(x) = u(x) - u(x^j) - \frac{1}{4n}|x - x^j|^2$, is subharmonic in $(B_r \setminus \overline{B}_{r_0}) \cap \Omega$, and vanishes at x^j. Hence it takes its positive maximum on the boundary of $(B_r \setminus \overline{B}_{r_0}) \cap \Omega$. On $\partial\Omega$, $v < 0$ and on ∂B_{r_0},

$$\sup_{|x|=r_0} v(x) \leq \sup_{|x|=r_0} \left(u(x) - \frac{1}{4n}|x - x^j|^2 \right) \leq \left(\sup_{|x|=r_0} u(x) \right) - \frac{1}{4n}(|x^j| - r_0)^2,$$

which can be made negative by choosing $|x^j|$ large. It thus follows that v takes its positive maximum on $|x| = r$. Taking $r \geq 2|x^j|$, we obtain

$$\sup_{|x|=r} u \geq \frac{1}{4n}(r - |x^j|)^2 \geq \frac{1}{16n} r^2,$$

and (2.8) follows in this case.

Case 2: Suppose now $u(x) \leq 0$ in $B^e_{r_1}$, for some $r_1 > r_0$. Then it is not hard to see that $u < 0$ in $B^e_{r_1}$. Indeed, if there is any x^0 such that $u(x^0) = 0$, then by subharmonicity, $0 = |B_r|u(x^0) \leq \int_{B_r(x^0)} u \leq 0$; hence $u = 0$ in $B_r(x^0)$, where r is such that $B_r(x^0) \subset B^e_{r_1}$. We thus conclude that the set $\{u = 0\}$ is relatively open in $B^e_{r_1}$. Since this is also a closed set, it must either be the empty set or coincides with $B^e_{r_1}$. The latter is excluded by the assumption that $\Omega(u)$ is unbounded, and hence we must have $u < 0$ in $B^e(0, r_1)$.

Now let x^0 be any fixed point in $B^e_{2r_1}$. Then the function $v(x) = u(x) - u(x^0) - \frac{1}{4n}|x - x^0|^2$ is subharmonic in $B^e_{r_1}$ and $\limsup_{x\to\infty} v(x) = -\infty$. Therefore, since $v(x^0) = 0$, the positive maximum of v is attained on $|x| = r_1$. We thus have

$$-u(x^0) \geq \sup_{|x|=r_1} (u(x) - u(x^0)) > \inf_{|x|=r_1} \frac{1}{4n}|x - x^0|^2 \geq \frac{1}{4n}(|x^0| - r_1)^2 \geq \frac{1}{16n}|x^0|^2,$$

where we've used that $u \leq 0$ on ∂B_{r_1} in the first inequality and in the last equality we've used that $|x^0| \geq 2r_1$. $\square$

3. The cone problem

The results in the previous section imply that any shrinking down u_0 of a solution u to (0.5), satisfies the global equation

$$\begin{cases} \Delta v = \chi_K & \text{in } \mathbb{R}^n \\ v = |\nabla v| = 0 & \text{on } \mathbb{R}^n \setminus K, \end{cases} \tag{3.1}$$

where K is a cone with vertex at the origin.

Equation (3.1) has an interesting potential theoretic interpretation; namely harmonic continuation of potentials. We explain this in the following example.

Example. Let us denote by U the Newtonian potential of the function $\chi_{B_1\cap K}$ i.e.,

$$U(x) = C\int_{B_1\cap K} \frac{dx}{|x - y|^{n-2}},$$

where the constant C is chosen such that $\Delta U = \chi_{B_1\cap K}$. Now if there is a solution v to (3.1), then the function $w = U - v$ is harmonic in B_1 and coincides with U in $B_1 \setminus K$. This means that the Newtonian potential U, here, can be harmonically continued from $B_1 \setminus K$ into $B_1 \cap K$.

The classification of the cones in (3.1), is an indispensable step for both local and global regularity theory. For positive solutions it follows from Caffarelli's work that any solution v to (3.1), with $\overline{K} \neq \mathbb{R}^n$ is a half-space solution. We will treat this case in more detail here below. There are also other examples of classifications where sign restriction is replaced by geometric restriction. Namely, if $K \cap \mathbb{S}^{n-1}$ or $(\mathbb{R}^n \setminus \bar{K}) \cap \mathbb{S}^{n-1}$ is starlike, then the solution of (3.1) is a half-space solution; see Corollary 3.9 in [KM]. In $\mathbb{R}^2$, however, the solution to (3.1) must be a half-space. This follows by the uniqueness part of Cauchy-Kovalewski theorem. It was conjectured in [KM] that the cone K in (3.1) must be a hyperplane if $\overline{K} \neq \mathbb{R}^n$. Spruck's theorem gives a new insight and a stronger motivation for that conjecture which we restate here. We also refer to this as the "Cone Problem".

Conjecture. (The Cone Problem.)

$$v \text{ solves (3.1)}, \quad \overline{K} \neq \mathbb{R}^n \qquad \Longleftrightarrow \qquad K \text{ is a half-space.}$$

The main result of this section, Corollary 3.2 also follows from [Ca2]. However, since the solution of the Cone Problem (for $v \geq 0$) is one of the steps in Caffarelli's proof [Ca2], we would like to give a different proof which does not rely on the deep result of Caffarelli. The convexity result appears as a remark in a paper of B. Ou; see [O]. Since we could not follow his proof, we repeat Caffarelli's argument.

Lemma 3.1. *Let v be a nonnegative solution to (2.2). Then $\Lambda(v)$ is convex.*

Corollary 3.2. *(To Lemma 3.1.) Let v be a nonnegative solution to (3.1) with $\overline{K} \neq \mathbb{R}^n$. Then K is a half-space.*

Proof. Lemma 3.1 gives that $\mathbb{R}^n \setminus K$ is convex. Thus the result follows from [KM; Corollary 3.9]. $\square$

Proof of Lemma 3.1. It suffices to show

$$D_{\nu\nu} v \geq 0, \tag{3.2}$$

for any direction ν. Since, then, the convexity of $\Lambda(u)$ follows by elementary calculus. In order to show (3.2), we set

$$M_k = -\inf_{B_{2^k}} D_{\nu\nu} v, \qquad k = 0, 1, 2, \ldots$$

and $M = \lim_k M_k$. By Lemma 1.3, $M < \infty$. If $M = 0$ then we are done. Otherwise, we follow Caffarelli's argument (see the proof of Lemma 3 in [Ca2]) where he shows that for any $0 < \delta < 1$, there are positive constants c and C such that

$$D_{\nu\nu} v(x) \geq -M_{k+1} + c\delta^{n-1} \left(M_{k+1} - C\sqrt{\delta} \right) \qquad \text{for all } x \in B_{2^k}. \tag{3.3}$$

Let $\varepsilon < \min\{\frac{C}{M}, 1\}$ and take $\delta_k = \left(\frac{\varepsilon M_k}{C}\right)^2$. Insert $\delta = \delta_{k+1}$ in (3.3) to obtain

$$M_k \leq M_{k+1} - C M_{k+1}^{2n-1}. \tag{3.4}$$

Letting $k \to \infty$ in (3.4), we'll have $M \leq M - CM^{2n-1}$, which gives $M \leq 0$. $\square$

4. Flatness at the infinity point

In this section we will show that the free boundary $\Gamma(u)$ satisfies a certain flatness property near the infinity point. In the local regularity problem, Caffarelli first shows the regularity for the class of convex solutions, and there after he shows the flatness condition [Ca2]. This method is not available for the infinity point version. Instead, we have by Theorem 2.5, that the first shrinking down is a cone, and hence under certain assumptions, the cone is a half-space.

For a given system of coordinates $\mathbf{e}$, we define the cones

$$\mathbb{K}_\delta^-(\mathbf{e}) = \{x : x_n < -|x| \sin\delta\} \qquad \mathbb{K}_\delta^+(\mathbf{e}) = \{x : x_n > |x| \sin\delta\}.$$

Lemma 4.1. *Let u be a solution to (0.5) and suppose Condition A holds. Then for any $\varepsilon > 0$ and $0 < \delta < 1$, there exists $\rho_0 = \rho_0(\varepsilon, \delta)$ such that: if $MD(\Lambda(u) \cap B_\rho)/\rho \geq \varepsilon$ for some $\rho \geq \rho_0$, then in an appropriate system of coordinates* $\mathbf{e}$,

$$\Omega^+(u) \supset B^e_{2\rho} \cap \overline{B_{(2\rho/\delta)}} \cap \mathbb{K}^+_\delta(\mathbf{e}), \tag{4.1a}$$

$$\Lambda(u) \supset B^e_{2\rho} \cap \overline{B_{(2\rho/\delta)}} \cap \mathbb{K}^-_\delta(\mathbf{e}). \tag{4.1b}$$

Proof. Suppose the conclusion in the lemma fails. Then there is an unbounded sequence $\{\rho_m\}$ such that $\mathrm{MD}(\Lambda(u) \cap B_{\rho_m}) \geq \rho_m \varepsilon$, and for which either (4.1a), or (4.1b) fails in any coordinate system. Let us assume (4.1b) fails. Due to Condition A, Theorem 1.1 can be applied to deduce that u has quadratic growth. Set now $u_m(x) := \rho_m^{-2} u(\rho_m x)$, then Proposition 2.1 implies $u_m \to u_0$ and $\nabla u_m \to \nabla u_0$ uniformly on compact sets. Moreover, by Proposition 2.3, Theorem 2.5 and Lemma 2.6, u_0 satisfies equation (3.1) with the non-empty cone $K := \Omega(u_0)$. We claim $u_0 \geq 0$. Since, otherwise $u_m < 0$ in $B_r(x^0)$ for some x^0, $0 < r < 1$ and all large enough m. Hence $u < 0$ in $B_{\rho_m r}(\rho_m x^0)$. This implies

$$\frac{\left|\Omega^-(u) \cap B_{(|x^0|+\rho_m)}\right|}{(|x^0| + \rho_m)^n} \geq \frac{(r\rho_m)^n}{(|x^0| + \rho_m)^n} \to r^n \quad \text{as } \rho_m \to \infty.$$

But this contradicts (A.4). Now, since

$$\mathrm{MD}(\Lambda(u_0) \cap B_1) \geq \liminf_m \mathrm{MD}(\Lambda(u_m) \cap B_1) = \liminf_m \frac{\mathrm{MD}(\Lambda(u) \cap B_{\rho_m})}{\rho_m} \geq \varepsilon,$$

we have $\overline{\Omega(u_0)} \neq \mathbb{R}^n$. Hence, by Corollary 3.2, $\Omega(u_0) = \{x : x_n > 0\}$ in the coordinate system $\mathbf{e} = (e_1, ..., e_n)$.

Now, (4.1b) fails also in this particular coordinates system. Hence, for any m we can find $x^m \in \left(B^e_{2\rho_m} \cap B_{(2\rho_m/\delta)} \cap \mathbb{K}^-_\delta(\mathbf{e})\right) \cap \Omega^+(u)$. Setting $y^m = x^m/\rho_m$, we have $2 \leq |y^m| \leq 2/\delta$ and thus $y^m \to y^0$. By Proposition 2.2, we'll have $y^0 \in \overline{\Omega(u_0)} = \{x : x_n \geq 0\}$. Also, since $y^m \in \mathbb{K}^-_\delta(\mathbf{e})$ for any m, $y^0 \in \overline{\mathbb{K}^-_\delta(\mathbf{e})}$, which is a contradiction. In case (4.1a) fails in any coordinate system, we obtain a contradiction in the same way. □

Lemma 4.1 gives that if $\mathrm{MD}(\Lambda(u) \cap B_\rho) \geq \rho\varepsilon$, then

$$\{x : |x| = 2\rho\} \cap K^-_\delta(\mathbf{e}) \subset \Lambda(u).$$

Hence,

$$\frac{\mathrm{MD}(\Lambda(u) \cap B_{2\rho})}{2\rho} \geq \frac{\mathrm{MD}\left(\{x : |x| = 2\rho\} \cap \mathbb{K}^-_\delta(\mathbf{e})\right)}{2\rho} \geq 1 - \sin\delta \geq 1 - \delta.$$

Therefore if $\max\{\varepsilon, \delta\} < 1/2$, then we apply Lemma 4.1 inductively to obtain the next lemma.

Lemma 4.2. *Let u be a solution to (0.5) and suppose Condition A holds. Then for any $0 < \varepsilon < 1/2$, and $0 < \delta < 1/2$, there exists $\rho_0 = \rho_0(\varepsilon, \delta)$ such that: if $MD(\Lambda(u) \cap B_\rho)/\rho \geq \varepsilon$ for some $\rho \geq \rho_0$, then in an appropriate system of coordinates* $\mathbf{e}^k$,

$$\Omega^+(u) \supset B^e_{(2^k\rho)} \cap \overline{B_{(2^k\rho/\delta)}} \cap \mathbb{K}^+_\delta(\mathbf{e}^k), \tag{4.2a}$$

$$\Lambda(u) \supset B^e_{(2^k\rho)} \cap \overline{B_{(2^k\rho/\delta)}} \cap \mathbb{K}^-_\delta(\mathbf{e}^k). \tag{4.2b}$$

for $k = 1, 2, ...$

5. Proof of Theorem A

Proof. We first consider the case $\lim_{r\to\infty} \mathrm{MD}(\Lambda(u)\cap B_r)/r = 0$. Then by the inequality

$$\frac{|\Lambda(u)\cap B_r|}{|B_r|} \le C\frac{\mathrm{MD}(\Lambda(u)\cap B_r)}{r},$$

we have $\lim_{r\to\infty} (|\Lambda(u)\cap B_r|)/|B_r| = 0$. Here C is a constant depending only on the space dimension. Since $f(x) > 0$ in B_r^e with large r, $\Gamma(u)\cap B_r^e$ has zero Lebesgue measure for large r (Lemma 2.11 in [KM]). Therefore $\lim_{r\to\infty} |\Omega(u)\cap B_r|/B_r = 1$.

Next, in case $\limsup_{r\to\infty} \mathrm{MD}(\Lambda(u)\cap B_r)/r = \varepsilon_0 > 0$ we apply Lemma 4.2 with $2\varepsilon = \min\{\varepsilon_0, 1\}$ and for any $0 < \delta < 1/2$. Now there is $\rho > \rho_0(\varepsilon,\delta)$ such that $\mathrm{MD}(\Lambda\cap B_\rho) > \varepsilon\rho$, therefore (4.2b) holds. Rewriting it, we have

$$\Lambda(u)\setminus B_{2\rho} \supset \bigcup_{k=1}^{\infty} \left(\mathbb{K}_\delta^-(\mathbf{e}^k)\cap\{x:\ 2^k\rho\le|x|\le 2^{k+1}\rho\}\right).$$

Note that for any coordinate system $\mathbf{e}^k$,

$$|K_\delta^-(\mathbf{e}^k)\cap\{x:\ r_1\le|x|\le r_2\} = (1-\delta')\frac{\omega_n}{2n}(r_2^n - r_1^n),$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$ and $\delta' = \int_0^\delta (\sin\theta)^{n-2}d\theta$ tends to zero as $\delta\to 0$. Therefore, the above inclusion yields

$$|\Lambda(u)\cap B_r| \ge |\Lambda(u)\cap B_{2\rho}| + (1-\delta')\frac{\omega_n}{2n}\left(r^n - (2\rho)^n\right)$$

for $r > 2\rho$. Hence

$$\liminf_{r\to\infty} \frac{|\Lambda(u)\cap B_r|}{|B_r|} \ge \frac{1}{2}(1-\delta'). \tag{5.1}$$

Repeating the same argument, but with (4.2a) instead of (4.2b), we obtain

$$\liminf_{r\to\infty} \frac{|\Omega(u)\cap B_r|}{|B_r|} \ge \liminf_{r\to\infty} \frac{|\Omega^+(u)\cap B_r|}{|B_r|} \ge \frac{1}{2}(1-\delta'). \tag{5.2}$$

Since $\Gamma(u)$ has zero Lebesgue measure, we may deduce by (5.1) and (5.2) that

$$\frac{1}{2}(1-\delta') \le \liminf_{r\to\infty} \frac{|\Omega(u)\cap B_r|}{|B_r|} \le \limsup_{r\to\infty} \frac{|\Omega(u)\cap B_r|}{|B_r|} \le \frac{1}{2}(1+\delta'). \tag{5.3}$$

Since (5.3) holds for all small δ' (this can be done by choosing δ small), we'll have $\lim_{r\to\infty} |\Omega(u)\cap B_r|/B_r = 1/2$ which completes the proof of Theorem A. □

Remark. The proof of Theorem A depends essentially on Lemma 4.2, and hence on the solution of the Cone Problem, and the quadratic growth. Therefore in $\mathbb{R}^2$, by using the uniqueness part of Cauchy-Kovalewski theorem, we can obtain the same result under weaker assumptions. Namely, Condition (A.4) can be dispensed with. Also, the quadratic growth follows from [HKS], or [KS2].

6. The cone condition at the infinity point

Our aim is to improve the flatness properties (4.2a), (4.2b) and to replace them by (6.12a), (6.12b) below. In doing this, we need to analyze the second order derivatives. This analysis, however, is not easily performed without further restrictions on the solutions to (0.5). Therefore throughout this section we are assuming that Condition B holds. Having $f \equiv 1$ in a neighborhood of the infinity, in the two dimensional space, enables us to use Sakai's results [Sa4]. Thus in the remaining sections, we will assume $n \geq 3$. This restriction is also needed for a certain technical reason. Namely, in the "almost convexity" lemma we need $\mu < 1/2$ (see the appendix in Section 8).

Lemma 6.1. *([Ca2; Lemma 3]) Let u be any solution to (0.5) satisfying Condition B. Then for any direction ν,*

$$D_{\nu\nu}u(x) \geq \frac{-C}{|\log d(x)|^{\beta}} \qquad \textit{for } x \in B^e_{r_0} \cap \{y : d(y) < 1\}, \tag{6.1}$$

where $d(x) = dist(x, \Gamma(u))$ and $\beta = 2n - 2$.

It follows from (6.1) that for any finite $y \in \Gamma(u)$,

$$\liminf_{x \to y} D_{\nu\nu}u(x) \geq 0. \tag{6.2}$$

We will now extend (6.2) to the case $x \to \infty$.

Lemma 6.2. *Let u be any solution to (0.5), subject to Condition B. Then for any arbitrary direction ν*

$$\liminf_{x \to \infty} D_{\nu\nu}u(x) \geq 0. \tag{6.3}$$

Proof. Assuming the contrary, we find a sequence $\{x^k\} \subset \Omega(u)$ such that $x^k \to \infty$ and

$$D_{\nu\nu}u(x^k) \leq -\varepsilon_0 < 0. \tag{6.4}$$

Let $\rho_k = d(x^k, \Gamma(u))$, $y^k \in \Gamma(u)$ be the nearest point to x^k, and consider the family of functions

$$u_k(x) := \frac{u(y^k + \rho_k x)}{\rho_k^2}.$$

By Lemma 1.2 and by Proposition 2.1, $u_k \to u_0$ uniformly on compact sets. We show that u_0 is a nonnegative solution to equation (2.2). Assuming this for the moment, we obtain the following contradiction. For each k, $B_1(z^k) \subset \Omega(u_k)$, where $z^k = (x^k - y^k)/\rho_k$. Since $|z^k| = 1$, it converges to a point $z^0 \in \mathbb{S}^{n-1}$. Hence $B_{(1/2)}(z^0) \subset B_1(z^k)$ for large k. Now the uniformly convergence in $B_{(1/2)}(z^0)$ implies the convergence in the C^2-norm in $B_{(1/4)}(z^0)$. Hence

$$D_{\nu\nu}u_0(z^0) = \lim_{k\to\infty} D_{\nu\nu}u_k(z^k) = \lim_{k\to\infty} D_{\nu\nu}u(x^k) \leq -\varepsilon_0 < 0,$$

which contradicts (3.2) in the proof of Lemma 3.1.

To show that u_0 solves (2.2), we define $f_0(x) := \lim_k f(y^k + \rho_k x)$. Then it suffices to verify $f_0(x) = 1$ for almost all $x \in \mathbb{R}^n$. Note that by (6.1) and (6.4), $\rho_k \geq \lambda_0 > 0$. Therefore, by going over to subsequences, we are left with the following two possibilities: (i) $\lim_k \rho_k = \infty$; (ii) $0 < \lambda_0 \leq \rho_k \leq \lambda_1 < \infty$. In the first case, since $\{x : f(x) \neq 1\} \subset B_{r_0}$, $f_0(x) \neq 1$ for at most one point of $\mathbb{R}^n$. In the second case, since ρ_k is bounded, y^k is unbounded and therefore, by (B.2), $\lim_k f(y^k + \rho_k x) = 1$ for all $x \in \mathbb{R}^n$. The proof of Lemma 6.2 is completed. □

Following Caffarelli's argument we now improve (6.3). It is also in the proof of next lemma where we use that $n \geq 3$.

Lemma 6.3. *Suppose Condition B holds. Then under the hypotheses of Lemma (4.2) there is* $0 < \mu < 1/2$ *such that for any direction* ν,

$$D_{\nu\nu}u \geq -M\mu^{k-1} \quad \text{in } \Omega(u) \cap B^e_{(2^k\rho)} \quad (k = 1, 2, ...), \tag{6.5}$$

where M is the bound for the second order derivatives.

Proof. We first observe that by Lemma 1.3 the second order derivatives of u are bounded in $\Omega \cap B^e_\rho$. Next, let h be a harmonic function in B_1 such that

$$\begin{aligned} h &= -1 \quad \text{on } \partial B_1 \cap \{x_n \geq -\sin(1/2)\}, \\ h &= 0 \quad \text{on } \partial B_1 \cap \{x_n < -\sin(1/2)\}, \end{aligned}$$

and set

$$\hat{h}(x) = |x|^{2-n} h(\frac{x}{|x|^2}).$$

Since $n \geq 3$,

$$-\mu \equiv \inf_{B^e_2} \hat{h} > -1/2.$$

Suppose, by induction,

$$D_{\nu\nu}u \geq -M\mu^{k-1} \quad \text{in } \Omega(u) \cap B^e_{(2^k\rho)}. \tag{6.6}$$

Now choose e^k_n in the vertical direction and define

$$w(x) := D_{\nu\nu}u(x) - M\mu^{k-1}\hat{h}\left(\frac{x}{2^k\rho}\right) \qquad \text{in } B^e_{(2^k\rho)} \cap \Omega(u).$$

The function w is harmonic in $B^e_{(2^k\rho)} \cap \Omega(u)$, and nonnegative on $\partial B_{(2^k\rho)} \cap \Omega(u)$ (by (6.6) and (4.2b)) and on $\partial\Omega(u) \cap B^e_{(2^k\rho)}$ (by 6.2)). Also, Lemma 6.2 gives

$$\liminf_{x \to \infty} w(x) \geq 0. \tag{6.7}$$

Now (6.7) can be used to apply the maximum principle to conclude $w(x) \geq 0$ in $\Omega(u) \cap B^e_{(2^k\rho)}$. Hence, for $x \in \Omega(u) \cap B^e_{(2^{k+1}\rho)}$,

$$D_{\nu\nu}u(x) \geq M\mu^{k-1} \inf_{B^e_2} \hat{h}(x) \geq -M\mu^k. \qquad \square$$

We improve now (4.2b). The angle between two vectors $x^1, x^2 \in \mathbb{R}^n$ is denoted by $\alpha(x^1, x^2)$.

Lemma 6.4. *Assume u is a solution to (0.5), subject to Condition B. Then for a given $0<\delta<1/2$, there is $r_1 = r_1(\delta)$ and $\mathbf{e}^0 = \mathbf{e}^0(\delta)$ such that*

$$\Lambda(u) \supset B^e_{r_1} \cap \mathbb{K}^-_{3\delta}(\mathbf{e}^0). \tag{6.8}$$

Proof. Applying Lemma 4.2 with

$$\varepsilon = \frac{1}{2}\limsup \frac{\mathrm{MD}\,(\Lambda(u)\cap B_r)}{r},$$

we obtain the existence of a sequence $\{\mathbf{e}^k\}$ and ρ, such that (4.2b) holds. Obviously the sequence $\{\mathbf{e}^k\}$ has a converging subsequence. We prove the lemma by showing that the entire sequence converges. So for the moment, we have $N_1 \subset \mathbb{N}$ such that

$$\mathbf{e}^k \to \mathbf{e}^0 \qquad N_1 \ni k \to \infty,$$

and (4.2b) holds, for $\mathbf{e}^k$. Let $k_0 \in N_1$ be sufficiently large such that $\alpha(\mathbf{e}^k, \mathbf{e}^0) < \delta$ for $k \in N_1$ with $k \geq k_0$, then

$$\mathbb{K}^-_{2\delta}(\mathbf{e}^0) \subset \mathbb{K}^-_{\delta}(\mathbf{e}^k) \qquad \text{for } k \in N_1,\ k \geq k_0. \tag{6.9}$$

Take $k_1 \in N_1$, $k_1 \geq k_0$ such that

$$\sqrt{10n(1-2\mu)^{-1}M}\rho \sum_{i=k_1}^{\infty} \mu^{(i/2)} < \delta, \tag{6.10}$$

where M and μ are the constants of (6.5). We claim that (6.8) holds with $r_1 = \rho 2^{k_1}$. If not, then for some $k > k_1$ there is

$$x^k \in \Omega(u) \cap B^e_{\rho 2^{k_1}} \cap \mathbb{K}^-_{3\delta}(\mathbf{e}^0)$$

and

$$\rho 2^k < |x^k| \leq \rho 2^{k+1}.$$

Set $A_m = (1-2\mu)^{-1}M\rho^2 2^{2m}\mu^m$ and $\Theta_m = \sqrt{10nA_m}$, where $k_1 \leq m \leq k$. Starting with $m = k$, there is by [Ca2; Lemma 1], y^k such that $|y^k - x^k| = \Theta_k$ and $u(y^k) \geq \Theta_k^2/2n$. Define now

$$g(t) = u(\nu\rho 2^k t), \qquad \nu = \frac{y^k}{|y^k|}.$$

Since $|y^k| \leq \rho 2^{k+2}$ and $|\alpha(x^k, y^k)| \leq \sqrt{10n(1-2\mu)^{-1}M}\rho\mu^{(k/2)}$, we may conclude by (6.9), (6.10) and (4.2b) that there is $N \geq 3$ such that $g(2^N) = g'(2^N) = 0$. By Lemma 6.3, we may now apply the almost convexity lemma (Lemma 8.1) with $A := (1-2\mu)A_k/\mu$, and $1 < \tau := |y^k|/(\rho 2^k) \leq 4$. Since $g(2^N) = 0$, (8.2)–(8.3) yields,

$$\lambda g(1) \geq g(\tau) - \frac{4A\mu}{1-2\mu} = \frac{\Theta_k^2}{2n} - 4A_k = A_k > 0.$$

Thus, setting $x^{k-1} = \frac{y^k}{|y^k|}\rho 2^k$, we have:

$$|x^{k-1}| \le \rho 2^k, \qquad u(x^{k-1}) > 0 \quad \text{and} \quad |\alpha(x^k, x^{k-1})| \le \sqrt{10n(1-2\mu)^{-1}M}\rho\mu^{(k/2)}.$$

Proceeding in this way, we get $x^k,\ x^{k-1}, ..., x^{k-j}$, where $x^{k-j} \in \Omega(u)$, $|x^{k-j}| \le \rho 2^{k-j+1}$ and $|\alpha(x^k, x^{k-j})| \le \sqrt{10n(1-2\mu)^{-1}M}\rho\left(\mu^{(k/2)} + \cdots + \mu^{(k-j/2)}\right)$. Taking j so that $k - j = k_1$, we obtained by (6.9) and (6.10) that

$$x^{k_1} \in [\mathbb{K}^-_{2\delta}(\mathbf{e}^0) \cap \Omega(u) \cap B^e_{\rho 2^{k_1}} \cap B_{(\rho 2^{k_1}/\delta)}] \subset [\mathbb{K}^-_{\delta}(\mathbf{e}^{k_1}) \cap \Omega(u) \cap B^e_{\rho 2^{k_1}} \cap B_{(\rho 2^{k_1}/\delta)}],$$

which contradicts (4.2b). □

Lemma 6.5. *Retain the hypotheses of Lemma 6.4. Let $\gamma > 1$ be fixed, and r_1 as in (6.8). Then, there is $r_2 = r_2(\delta) \ge r_1(\delta)$ such that*

$$\Omega(u) \supset B^e_{r_2} \cap \mathbb{K}^+_{3(\gamma\delta)}(\mathbf{e}^0). \tag{6.11}$$

Proof. If (6.11) fails, then there is a sequence $\{x^k\} \subset \Lambda(u) \cap B^e_{r_2} \cap \mathbb{K}^+_{3(\gamma\delta)}(\mathbf{e}^0)$ with $x^k \to \infty$. Set $\rho_k = |x^k|$ and let $u_k(x) := \rho_k^{-2}u(\rho_k x)$. Then $y^k := x^k/\rho_k \in \Lambda(u_k) \cap \mathbb{K}^+_{3(\gamma\delta)}(\mathbf{e}^0)$ for any k. On the other hand, by Lemma 6.4, $\mathbb{K}^-_{3\delta}(\mathbf{e}^0) \cap B^e_{(r_1/\rho_k)} \subset \Lambda(u_k)$. Thus $-y^k \in \Lambda(u_k)$. Now, $u_k \to u_0$ uniformly on compact sets, $y^k \to y^0 \in \partial B_1 \cap \Lambda(u_0)$, but also $-y^0 \in \Lambda(u_0)$. Moreover, since $\gamma > 1$, $-y^0$ is an interior point of $\Lambda(u_0)$. By Corollary 3.2, $\Lambda(u_0)$ is a half-space which contains both y^0 and $-y^0$, and in addition $-y^0 \notin \partial(\Lambda(u_0))$. This is a contradiction. □

Theorem 6.6. *Let u be a solution to (0.5), subject to Condition B. Then there is a system of coordinates $\mathbf{e}$ such that: for any $0 < \delta < 1/2$, there is $r = r(\delta)$ ($r(\delta) \to \infty$ as $\delta \to 0$) such that*

$$\Omega(u) \supset B'_{r(\delta)} \cap \mathbb{K}^+_{4\delta}(\mathbf{e}), \tag{6.12a}$$

$$\Lambda(u) \supset B'_{r(\delta)} \cap \mathbb{K}^-_{4\delta}(\mathbf{e}). \tag{6.12b}$$

Proof. We apply Lemmas 6.4 and 6.5 with $\gamma = 5/4$. By conditions (6.8) and (6.11), the system $\mathbf{e}^0(\delta)$ converges to a unique system $\mathbf{e}$ as $\delta \to 0$. Therefore for small δ, $\mathbb{K}^\pm_{4\delta}(\mathbf{e}) \subset \mathbb{K}^\pm_{3(\gamma\delta)}(\mathbf{e}^0(\delta))$ and this completes the proof. □

7. Proof of Theorem B

In the proof of Theorem B we use the full strength of Caffarelli's theorem for the regularity of local points of $\partial\Omega$, [Ca2; Theorem 1]. For this purpose we need the following definition.

Definition. A function v says belong to the class $P_1(x^0, M)$, if :

(a) $\sup_{B_1(x^0)} |D_{ij}v| \le M$;
(b) $\Delta v(x) = 1$ for $x \in \Omega(v)$;
(c) $v \ge 0$ in $B_1(x^0)$;
(d) $x^0 \in \partial\Omega(v)$.

Theorem 7.1. *(Caffarelli) There exists a positive, nondecreasing function $\sigma(t)$, with $\sigma(0^+) = 0$, such that: if $v \in P_1(x^0, M)$ and if for some $t_0 \in (0,1)$*

$$\frac{MD\left((\Lambda(v) \cap B_{t_0}(x^0)\right)}{t_0} > \sigma(t_0), \tag{7.1}$$

then $\partial\Omega(v)$ is C^1 in $B_{\tilde{t}}(x^0)$, $\tilde{t} \leq t_0$. Moreover the C^1–norm of $\partial\Omega(v)$ depends only on M, t_0, and the space dimension.

Proof of Theorem B. For $y \in \partial\Omega(u)$ we define

$$u_y(x) = \frac{u(|y|x/2)}{(|y|/2)^2}, \qquad z^y = \frac{y}{|y|}. \tag{7.2}$$

Then $u_y \in P_1(z^y, M)$. We split the proof into two steps. In the first step we use Theorem 6.6 to verify condition (7.1) for u_y with large $y \in \partial\Omega(u)$. This together with Theorem 6.6 gives that $\partial\Omega_{inv}$ has a tangent plane at any point $y \in \partial\Omega_{inv}$, in a small neighborhood of the origin. Thus, the second step consists of showing

$$\eta^y \to e_n \quad \text{as} \quad (\partial\Omega_{inv}) \ni y \to 0, \tag{7.3}$$

where η^y and e_n are the unit normal to the tangent planes at y, and the origin, respectively, and $\mathbf{e} = (e_1, ..., e_n)$ is the coordinate system of Theorem 6.6.

Step 1. Take t_1 such that $\sigma(t) < 1/4$ for $0 < t \leq t_1$. Fix δ small so that $8\tan(8\delta) \leq t_1$ and let $r(\delta)$ be of relations (6.12a)-(6.12b) in Theorem 6.6. Then, by (6.12b) we can estimate the minimal diameter of $\Lambda(u) \cap B_{(t|y|/2)}(y)$ for $y \in \partial\Omega(u)$ and $0 < t \leq 1$. Indeed, for any $y \in \partial\Omega(u)$ with $|y| \geq r(\delta)$ and any t with $4\tan(8\delta) \leq t$, we have

$$\text{MD}\left(\Lambda(u) \cap B_{(t|y|/2)}(y)\right) \geq \frac{t|y|}{2}\left(1 - (\frac{2}{t} + 1)\tan(8\delta)\right) \geq \frac{t|y|}{8}.$$

Taking $4\tan(8\delta) \leq t \leq 8\tan(8\delta) \leq t_1$, and using the last estimate we obtain,

$$\frac{\text{MD}(\Lambda(u_y) \cap B_t(z^y))}{t} = \frac{\text{MD}\left(\Lambda(u) \cap B_{(t|y|/2)}(y)\right)}{t|y|/2} \geq \frac{1}{4} > \sigma(t_1) \geq \sigma(t),$$

i.e. (7.1) holds. It also follows from the proof of Theorem 7.1 that for given positive θ there is $t_2 > 0$ (which does not depend on z^y) such that

$$\Lambda(u_y) \supset B_{t_2}(z^y) \cap \{x : \alpha(x - z^y, -\eta^y) < \frac{\pi}{2} - \theta\}. \tag{7.4}$$

Step 2. If (7.3) fails, then there is a sequence $\{y^k\} \subset \partial\Omega(u)$ such that $|y^k| \to \infty$ while

$$\alpha(\eta^{(y^k)}, e_n) \geq \tau > 0, \tag{7.5}$$

Take $\theta = \tau/2$, and let $k \to \infty$. Then by (7.4) we have

$$\Lambda(u_0) \supset B_{t_2}(z^\infty) \cap \{x : \alpha(x - z^\infty, -\eta^\infty) < \frac{\pi}{2} - \frac{\tau}{2}\}, \tag{7.6}$$

where $z^\infty = \lim z^{(y^k)}$, and $\eta^\infty = \lim \eta^{(y^k)}$. Now, by Theorem 6.6, we have $\Lambda(u_0) = \{x : \ x_n \leq 0\}$ and $z_n^\infty = 0$. By (7.5) the truncated cone to the right of (7.6) intersects $\{x : x_n > 0\}$, which is a contradiction. □

8. Appendix - Almost convex functions

The following lemma asserts that a function of one variable satisfying (8.1) below is almost a convex function provided $\mu < 1/2$.

Lemma 8.1. *Let g be a $C^{1,1}$–function defined on $[1, \infty)$, and satisfying*

$$g''(t) \geq -A\mu^i \qquad \text{for} \quad 2^{i-1} \leq t \leq 2^i, \qquad (i = 1, 2, \cdots) \tag{8.1}$$

where $0 < \mu < 1/2$ is fixed. Then for any $1 < \tau \leq 4$, and positive integer $N \geq 3$,

$$g(\tau) \leq \lambda g(1) + (1-\lambda) g(2^N) + C(A, \mu), \tag{8.2}$$

where $1/4 \leq \lambda = (2^N - \tau)/(2^N - 1) \leq 1$,

$$C(A, \mu) \leq \frac{4A\mu}{1 - 2\mu}. \tag{8.3}$$

Remark. If $\mu < 1/4$, then (8.2) holds for any $\tau \in [1, 2^N]$ with $C(A, \mu) \leq CA\mu/(1-4\mu)$.

Proof. We may assume (as we do) that $A \leq 1$. We will now construct a decreasing $C^{1,1}$ function P such that $g + P$ will be convex in the interval $[1, 2^N]$. First define the constants α_i, β_i, inductively, by $\alpha_N = \beta_N = 0$, and

$$\beta_i = -2^i \mu^{i+1} + \beta_{i+1}, \qquad \alpha_i = 2^{2i-1} \mu^{i+1} - 2^i \beta_{i+1} + \alpha_{i+1} \tag{8.4}$$

for $i = N-1, N-2, \cdots, 1$. Now define P as

$$P(t) := \frac{(t - 2^i)^2}{2} \mu^i + \beta_i (t - 2^i) + \alpha_i, \qquad \text{for } t \in [2^{i-1}, 2^i]. \tag{8.5}$$

It is easy to check that P is $C^{1,1}$, decreasing and convex with $P(2^N) = 0$ and $P(1) > 0$. By (8.1), $g + P$ is also convex. Hence

$$g(\tau) \leq \lambda g(1) + (1-\lambda) g(2^N) + \lambda P(1) - P(\tau).$$

Since P is decreasing and $P(1) > 0$, we'll have

$$\lambda P(1) - P(\tau) \leq P(1) - P(4) = [P(1) - P(2)] + [P(2) - P(4)] =: C(A, \mu).$$

Thus, the proof will be completed by showing the estimate (8.3) for $C(A, \mu)$. By (8.5), $P(1) = \mu/2 - \beta_1 + \alpha_1$ and $P(2) = \alpha_1$. So (8.4) gives

$$P(1) - P(2) = \frac{\mu}{2} - \beta_1 = \frac{\mu}{2} + 2\mu^2 - \beta_2 = \cdots = \frac{\mu}{2} + \sum_{i=1}^{N-1} (2\mu)^i. \tag{8.6}$$

In a similar way,

$$P(2) - P(4) = 2\mu^2 - 2\beta_2 = 2(\mu^2 + 2\mu^3 - \beta_3) = \cdots = 2\mu \left(\mu + \sum_{i=2}^{N-1} (2\mu)^i \right). \tag{8.7}$$

By (8.6)–(8.7) and that $2\mu < 1$ we'll have

$$C(A, \mu) = [P(1) - (P(2)] + [P(2) - P(4)] \leq \frac{4\mu}{1 - 2\mu}.$$

This completes the proof. $\square$

References

[AS] D. Aharonov and H.S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39-73.

[Ah] L.V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964), 413-429.

[BF] E. DiBenedetto and A. Friedman, *Bubble growth in porous media*, Indiana Univ. Math. J. **35** (1986), 573-606.

[B] L. Bers, *An approximation theorem*, J. Analyse Math. (1965), 1-4.

[Ca1] L.A. Caffarelli, *The regularity of free boundaries in higher dimension*, Acta Math. **139** (1977), 155-184.

[Ca2] L.A. Caffarelli, *Compactness methods in free boundary problem*, Comm. P.D.E. **5** (1980), 427-448.

[D] P. Davis, *The Schwarz Function and Its Application*, Carus Math. Monographs 17, Math. Assoc. of America, 1974.

[EE] V.M. Entov and P.I. Etingof, *Bubble contraction in Hele-Shaw cells*, Quart. J Mech Appl, Math. **44** (1991), 507-535.

[EV] P.I. Etingof and A.N. Varachenko, *Why the Boundary of a Round Drop Becomes a Curve of Order 4*, University Lecture Series, vol. 3, Amer. Math. Soc., Providence, Rhode Island, 1992.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag 2^{nd} edition, Berlin Heidelberg New York Toko, 1983.

[HKS] W. Hayman, L. Karp and H.S.Shapiro, *Newtonian capacity and quasi balayage*, In preparation.

[He] L. Hedberg, *Approximation in the mean by solutions of elliptic equations*, Duke Math. J. **40** (1973), 9-16.

[Ho] S.D. Howison, *Bubble growth in porous media and Hele Shaw cells*, Proc. Roy. Soc. Edinburgh **A102** (1985), 141-148.

[Ka] L. Karp, *Generalized Newton potential and its applications*, J. Math. Anal. Appl. **174** (1993), 480-497.

[KM] L. Karp and A.S. Margulis, *On the Newtonian potential theory for unbounded sources and its application to free boundary problems*, submitted.

[KS1] L. Karp and H. Shahgholian, *Regularity of a free boundary problem*, Research Report, Department of Mathematics, University of Stockholm, No. 2 (1996).

[KS2] L. Karp and H. Shahgholian, *Global estimates for solutions to partial differential inequalities*, manuscript.

[Ma] A.S.Margulis, *Potential theory for L^p-densities and its applications to inverse problems of gravimetry*, Theory and Practice of Gravitational and Magnetic Fields Interpretation in USSR, Naukova Dumka Press, Kiev, 1983, pp. 188-197 (Russian).

[O] Ou B., *Global solutions to a free boundary problem*, Comm. P.D.E. **19** (1994), 369-397.

[Sa1] M. Sakai, *Quadrature Domains*, Lecture Notes in Math., vol. 934, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[Sa2] M. Sakai, *Solutions to the obstacle problem as Green potentials*, J. Analyse Math. **44** (1984/85), 97-116.

[Sa3] M. Sakai, *Regularity of a boundary having a Schwarz function*, Acta Math. **166** (1991), 263-297.

[Sa4] M. Sakai, *Regularity of boundaries of quadrature domains in two dimensions*, SIAM J. Math. Anal. **24** (1993), 341-364.

[Shp] H.S. Shapiro, *Unbounded quadrature domains*, Complex Analysis I (C. Berenstein, ed.), Lecture Notes in Math., vol. 1275, Springer-Verlag, Berlin-Heidelberg-New York, 1987, pp. 287-331.

[Sp] J. Spruck, *Uniqueness in diffusion model of population biology*, Comm. P.D.E. **8** (1983), 1605-1620.

[St] V.N. Strakhov, *The inverse logarithmic potential problem for contact surface*, Physics of the Solid Earth **10** (1974), 104-114 [translated from Russian].

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