Regularity of a Free Boundary for Viscosity Solutions of Nonlinear Elliptic Equations

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Our objective in this paper is to analyze the regularity of the boundary of a set Ω that admits a solution u to the following overdetermined problem:

$$F(D^2u) = \chi_\Omega \quad \text{in } B\,, \qquad u = |\nabla u| = 0 \quad \text{in } B \setminus \Omega\,.$$

Here F is a convex, uniformly elliptic operator of homogeneity one, B is the unit ball in $\mathbb{R}^n$, and the equation is satisfied in the viscosity sense. We also consider the case of concave F in $\mathbb{R}^2$.

1 Introduction

In this paper we study the regularity properties of the boundary of a set $\Omega \in \mathbb{R}^n$ and a corresponding function u, coupled through the following overdetermined boundary value problem:

$$F(D^2u) = \chi_\Omega \quad \text{in } B\,, \qquad u = |\nabla u| = 0 \quad \text{in } B \setminus \Omega\,,$$

where F is uniformly elliptic with the elliptic constants $\lambda, \Lambda > 0$ and homogeneous of degree one; at the end of this section we'll define in exact terms what conditions the operator F is to fulfill.

This problem for the Laplacian case has been studied by several people; see [3, 4] and the references therein. When F is the Laplacian operator, the problem appears naturally in linear potential theory and is related to harmonic continuation of the Newtonian potential of $B(0,1) \cap \Omega$ from the free space $B(0,1) \setminus \Omega$ into $B(0,r) \cap \Omega$ for some $0 < r < 1$. We refer the reader to the above-mentioned papers for a background.

The nonlinear problem considered in this paper is motivated by similar nonlinear phenomena in mathematical physics and natural extension of the results of [3] to nonlinear operators. The latter, however, is only partly achieved in this paper.

The case of the nonlinear obstacle problem, i.e., when the solution u is assumed to be nonnegative, has already been studied by the first author [5, 6], where he achieves the regularity of the free boundary under some standard thickness conditions of the complement of the set Ω.

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The main results of our paper generalize, to some extent, the recent results of L. A. Caffarelli et al. [3], where the case of the Laplacian with lower-order parts are considered. See also the pioneering work of L. A. Caffarelli [1]. Here we consider fully nonlinear operators of certain types. The main technique used in [3] is a monotonicity-type formula, which is applicable to divergence-type operators. The lack of the monotonicity formula for nondivergence operators forces us to use other, less effective techniques. As a result, we obtain only some partial results, which may be considered as a first step towards understanding the nature of the problem in the nonlinear setting.

We show that if the set $\{u < 0\}$ is small enough (near the origin), then $|u|$ grows quadratically near the origin (Theorem 1.2). Moreover, if $\{u < 0\}$ has zero density at the origin, then we also show that $u \geq 0$ in some uniform neighborhood of the origin (Theorem 1.6). Hence earlier results of the first author apply to deduce the $C^{1,\alpha}$-regularity of the free boundary in some uniform neighborhood of the origin. In proving Theorem 1.6, we need to show that global solutions, i.e., solutions in the entire space, of the equation

$$F(D^2u) = \chi_\Omega\,, \qquad u = |\nabla u| = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega\,,$$

are convex. The reason for this is the standard technique of blowup of local solutions, which results in a global solution and the analyzing and characterization of the latter. However, in the special case of Theorem 1.6, global solutions obtained by the procedure of the blowup (see below) are nonnegative due to the zero-density assumption on the set $\{u < 0\}$ at the origin.

In Theorem 1.4, the convexity of nonnegative global solutions is taken care of. Indeed, this can be achieved by applying local convexity results of the first author. Here we present a simple direct proof of the convexity of nonnegative global solutions. However, the classification of global solutions without any sign assumption (in nonlinear setting) remains an open and challenging problem. As the reader may note, prior to this classification, the question of quadratic growth of the solution without any sign assumption (or zero-density assumption for $\{u < 0\}$) is the first step towards a complete theory for the regularity of the free boundaries in nonlinear problems.

The main difficulty in achieving the full regularity result in the nonlinear case, without any a priori conditions on the solution, lies in the lack of an established technique of local analysis of such problems. We hope this paper paves the way for future analysis and gives some new insights into the problem.

To formulate our main theorems we need some definitions. First, we define the notion of thickness. In order to stress that Ω is related to u, we set $\Omega(u) = \Omega$; we also set

$$\Lambda(u) = \{x \in B(0,1) : u(x) = |\nabla u(x)| = 0\}$$

and

$$\Omega^+(u) = \{x \in \Omega(u) : u(x) > 0\}\,, \qquad \Omega^-(u) = \{x \in \Omega(u) : u(x) < 0\}\,.$$

Define the minimum diameter

$$\mathrm{MD}(\Lambda \cap B(z, r))$$

as the infimum of the distances between pairs of parallel hyperplanes such that $\Lambda \cap B(z, r)$ is contained in the strip determined by them.

We also define the density functions δ_r and V_r as

$$\delta_r(z, u) = \frac{\mathrm{MD}(\Lambda(u) \cap B(z, r))}{r}, \qquad V_r(z, u) = \frac{\mathrm{vol}(\Omega^-(u) \cap B(z, r))}{r^n}.$$

If there is no ambiguity in the context, then we suppress the dependence on z and u. Observe that δ_r is the normalized distance of the parallel planes discussed above and V_r is the normalized volume of the set where u is negative. Also, by continuity of u, if $V_r = 0$ then $u \geq 0$, and we are back to the case of nonnegative u.

In order to state our main results, let us define a local solution.

DEFINITION 1.1 We say a continuous function u belongs to the class $P_r(z, M)$ if u satisfies

(i) $F(D^2u) = \chi_\Omega$ in $B_r(z)$ (in the viscosity sense),
(ii) $u = |\nabla u| = 0$ in $B_r(z) \setminus \Omega$,
(iii) $\|u\|_{\infty, B_r(z)} \leq M$ (supremum norm), and
(iv) $z \in \partial\Omega$.

We also denote by $P_\infty(z, M)$ all "global solutions" with quadratic growth; i.e., solutions in the entire space $\mathbb{R}^n$ with quadratic growth $|u(x)| \leq M(|x|^2 + 1)$. Here $z \in \partial\Omega$.

Our first result asserts that local solutions have quadratic growth if the set Ω^- is reasonably small. See [3, 4] for a similar type of result for the Laplacian case.

THEOREM 1.2 *There is a universal constant $C_0 = C_0(n, \lambda, \Lambda)$ such that if, for all $r < 1$ and $u \in P_1(z, M)$, we have $V_r(z, u) \leq C_0$, then*

$$\sup_{B_r(z)} |u| \leq \frac{M}{C_0} r^2 \quad \textit{for } r < 1.$$

Remark 1.3. If $V_r(y, u) \leq C_0$ for all $y \in B(0, \frac{1}{2}) \cap \partial\Omega$, then it follows by Theorem 1.2 that

$$|u(x)| \leq \frac{M}{C_0} d^2(x, \partial\Omega),$$

where $d(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. This along with the interior $C^{2,\alpha}$ estimates (see [2]) imply that $u \in C^{1,1}(B(0, \frac{1}{4}))$ uniformly for the class $P_1(0, M)$ provided the assumption $V_r(y, u) \leq C_0$ above holds for all $y \in B(0, \frac{1}{2}) \cap \partial\Omega$. For details we refer to the proof of theorem 1.1 in [3].

The approach we have chosen in this paper is slightly different from the classical ones, where one derives the $C^{1,1}$ property of the solution first and then attacks the regularity problem of $\partial\Omega$. Since in our main regularity result (Theorem 1.6) we only assume $V_r(z, u) \leq C_0$ for $z = 0$, we cannot prove that u is $C^{1,1}$ near the origin. However, Theorem 1.2 still allows us to rescale u quadratically $(u(rx)/r^2)$ and remain in the class as r tends to zero. This is all we need. Since by assuming, as we'll do, that the origin is a point of zero density for the set Ω^-, the blowup (limit) solution will be nonnegative and hence $C^{1,1}$ by earlier results.

Our second theorem in this paper is the convexity of global solutions. This is already known and can be simply derived from local convexity results of the first author; see [5]. Here, however, we choose to give a new proof of this fact, based on completely different techniques such as simple compactness and maximum principle.

THEOREM 1.4 *Let $u \in P_\infty(0, M)$ and $u \geq 0$. Then $D_{ee}u \geq 0$ for any direction e; i.e., $\Lambda(u)$ is convex.*

The relaxation of the nonnegativity condition on u seems to be infeasible for the moment with the techniques at hand. We believe, however, that Theorem 1.4 should hold with the replacement of the nonnegativity assumption on u with the assumption that

$$\overline{\Omega} \neq \mathbb{R}^n .$$

Observe that even for the Laplacian case, one can give simple examples of nonconvex solutions u, e.g.,

$$u(x) = x_1^2 - \frac{x_2^2}{2}, \qquad \Omega = \mathbb{R}^2 \setminus \{(0,0)\},$$

which is nonconvex, since $D_{x_2x_2}u = -1 \leq 0$.

Our final result in this paper gives information about the regularity of the free boundary. To formulate the theorem, we need to restrict our class of functions $P_1(0, M)$ to those for which u^- has a uniform behavior of the form $o(|x|^2)$ near the origin. This is expressed in terms of the set Ω^-.

DEFINITION 1.5 Let $\sigma(r)(\sigma(0^+) = 0)$ be a modulus of continuity. Then we define $P_1(0, M, \sigma)$ as the subset of all functions u in $P_1(0, M)$ with the property

$$V_r(0, u) \leq \sigma(r) .$$

THEOREM 1.6 *Let $M > 0$ and σ be modulus of continuity. Then there exists a modulus of continuity $\sigma_1(r)$ such that if $u \in P_1(0, M, \sigma)$, $\delta_{r_0}(u) > \sigma_1(r_0)$ for some $r_0 < 1$, then $\partial\Omega(u)$ is the graph of a C^1 function in $B(0, C_1 r_0^2)$. Here C_1 is a universal constant depending only on the class $P_1(0, M, \sigma)$.*

Before closing this section we give some explanation and definition of the viscosity solutions. We also introduce the exact condition imposed on the nonlinear operator F.

Conditions on F. In this paper the following conditions are imposed on F:

(1) F is uniformly elliptic with ellipticity constants λ and Λ, i.e.,
$$\lambda\|N\| \leq F(A+N) - F(A) \leq \Lambda\|N\|$$
where A and N are arbitrary $n \times n$ symmetric matrices with $N \geq 0$.

(2) F is homogeneous of degree one, i.e.,
$$F(tA) = tF(A)\,,$$
for all real numbers t and matrices A.

(3) In $\mathbb{R}^n$ $(n \geq 3)$, we only consider convex F; in $\mathbb{R}^2$, F can be either convex or concave.

(4) F is C^1.

In many cases, when quantities depend on the class of operators with the above conditions, we will only use F and not F_j to denote a sequence of operators. This is only for the sake of exposition.

DEFINITION 1.7 By viscosity solutions to $F(D^2u) = \chi_\Omega$, we mean continuous functions u such that if at $(x^0, u(x^0))$ the graph of u can be touched from above and below by paraboloids, then the functions representing these paraboloids (denoted by v) should satisfy the equation $F(D^2v(x^0)) = \chi_\Omega$ pointwise.

We refer to the excellent book of L. Caffarelli and X. Cabre [2] for further details. It is known that viscosity solutions to operators defined above have the usual maximum/minimum principle as well as compactness properties. Furthermore, they are uniformly $C^{1,\alpha}$ when $|F(D^2u)| \leq C$, and $C^{2,\alpha}(B(0, \frac{1}{2}))$ when $F(D^2u) = 1$ in $B(0,1)$.

2 Proof of Theorem 1.2

In this section, we show that, away from the free boundary, the solution u has at most quadratic growth. For the case of nonnegative u, cf. lemma 2.1 and theorem 2.2 in [5]. Here the main idea is to use a homogeneous stretching (blowup) of the solution. Since we don't know the exact homogeneity of the solution, we will stretch the functions by r, i.e., $u(rx)$, and then divide it by the maximum of u over the ball $B(0,r)$. This naturally controls the growth of u, and it somehow gives information in the limit as r tends to zero. This idea combined with the monotonicity formula was used in [3]. The lack of the latter forces us to restrict the class $P_1(0, M)$ to those functions satisfying the condition in the theorem.

Now, to fix the ideas, we set
$$S_j(z,u) = \sup_{B(z,2^{-j})} |u|$$
and define $\mathcal{M}(z,u)$ to be the maximal subset of natural numbers $\mathbb{N}$ satisfying the following doubling condition:
$$16S_{(j+1)}(z,u) \geq S_j(z,u) \quad \forall j \in \mathcal{M}(z,u)\,.$$

We will first show $S_j \le C2^{-2j}$ for all $j \in \mathcal{M}(z,u)$ and then extend this to all natural numbers j. This result implies the quadratic growth of the solution u away from the free boundary. An important observation at this point is that if $\mathcal{M}$ is empty (or even if it is finite), then we may easily (by iteration) obtain the desired estimate. Hence from now on we assume $\mathcal{M} \neq \varnothing$.

LEMMA 2.1 *For $u \in P_1(z,M)$, there exists uniform constant C_0 (small) such that*

$$S_j(z,u) \le \frac{M}{C_0} 2^{-2j} \quad \textit{for } j \in \mathcal{M}(z,u)\,,$$

provided $V_{2^{-2j}}(z,u) \le C_0$ for $j \in \mathcal{M}$. The constant C_0 depends on n, λ, and Λ.

PROOF: The idea is to use a contradictory argument. Suppose the conclusion fails. Then there are $\{\Omega_j\}$, $\{u_j\}$, and $\{k_j\}$ such that

$$S_{k_j}(u_j) \ge jM2^{-2k_j}$$

and

$$V_{2^{-2k_j}}(u_j) \le \frac{1}{j}$$

with $k_j \in \mathcal{M}(u_j) \neq \varnothing$. Since u_j is bounded by M, $jM2^{-2k_j}$ is bounded. Hence $k_j \to \infty$.

Now set

$$\tilde{u}_j(x) = \frac{u_j(2^{-k_j}x)}{S_{k_{j+1}}(u_j)} \quad \text{in } B(0,1)\,.$$

Then

$$\sup_{B_{1/2}} |\tilde{u}_j| = \frac{S_{k_j+1}(u_j)}{S_{k_j+1}(u_j)} = 1\,,$$

$$\sup_{B_1} |\tilde{u}_j| = \frac{S_{k_j}(u_j)}{S_{k_j+1}(u_j)} \le 16\,,$$

$$\tilde{u}_j(0) = |\nabla \tilde{u}_j(0)| = 0\,,$$

and

$$V_1(\tilde{u}_j) \le \frac{1}{j} \to 0\,,$$

since k_j goes to ∞. Also, by ellipticity and degree one homogeneity of F (where F itself may vary within the bounds of the condition stated earlier)

$$|F(D^2\tilde{u}_j(x))| \le \Lambda \frac{(2^{-k_j})^2}{S_{k_{j+1}}(u_j)} \le \frac{\Lambda S_{k_j}(u_j)}{jMS_{k_j+1}(u_j)} \le \frac{\Lambda 16}{jM} \to 0\,.$$

Standard elliptic estimates [2] imply a uniform bound for the $C^{1,\alpha}$-norms of $\tilde{u}_j$. Therefore a subsequence of $\{\tilde{u}_j\}$ converges to a function $\tilde{u}_0$ satisfying

$$F(D^2\tilde{u}_0) = 0 \ \text{ in } \mathbb{R}^n\,, \quad \tilde{u}_0 \geq 0\,,$$
$$\tilde{u}_0(0) = |\nabla\tilde{u}_0(0)| = 0, \quad \text{and} \quad \sup_{B_{1/2}} |\tilde{u}_0| = 1\,.$$

The above in particular implies that the nonzero solution $\tilde{u}_0$ of the elliptic equation $F(D^2\tilde{u}_0) = 0$ has a local minimum. Hence the minimum principle is violated and we have a contradiction. □

The next lemma completes the chain in $\mathbb{N}$.

LEMMA 2.2 *Retain the hypothesis of Lemma* 2.1*, and let* C_0 *be the constant introduced there. Then*

$$S_j \leq \frac{4M}{C_0} 2^{-2j} \quad \textit{for } j \in \mathbb{N}$$

provided $V_{2^{-2j}}(z,u) \leq C_0$ *for all positive integers* $j \in \mathcal{M}(z,u)$.

The proof of the lemma is given in [3]. From here one infers the quadratic bound for all $r < 1$. Hence Theorem 1.2 follows.

3 Some Basic Tools

In this section we'll give some detail explanation of certain facts, standard in the theory of regularity but probably nonstandard in the literature. The main ideas and lemmas here are generalized and modified versions of the already existing ones. The reader may refer to [3, 5].

Nondegeneracy. In the proof of Theorem 1.2 it became essential that the limit function $\tilde{u}_0$ be nonzero. This property, called nondegeneracy, is a basic property that is used when we consider blowup solutions. Therefore, if

$$u_{r_j} = \frac{1}{r_j^2} u(r_j x)$$

convergences uniformly to a limit function u_0, then we need to ensure that u_0 is not identically zero and that it remains in the class $P_1(0, M)$ introduced in the beginning of this paper.

Obviously Theorem 1.2 tells us that, when Ω^- is small enough, the above sequence u_j is uniformly bounded and hence some subsequence of it converges to a limit function.

The reader may easily verify that if $|u|$ behaves like $\operatorname{dist}^2(x, \partial\Omega)$ near the free boundary, then so does the scaled function; in particular,

$$u_{r_j}(x) \geq C_0 \operatorname{dist}^2(x, \partial\Omega)\,.$$

Hence the limit function is nonzero. We formulate this in the next lemma.

LEMMA 3.1 *Let* $u \in P_1(0, M)$ *and* $x \in \overline{\Omega(u)}$. *Then*

$$\sup_{y \in B(x,\rho)} [u(y) - u(x)] \geq C_n \frac{\rho^2}{\Lambda},$$

where $1/(8n) \leq C_n \leq 1/(2n)$ *and* $0 < \rho < 1 - |x|$.

PROOF: Suppose first $u(x) > 0$. Define

$$g(y) = u(y) - u(x) - \frac{|x-y|^2}{2n\Lambda}$$

in $\Omega(u) \cap B(x, \rho)$. Then

$$F(D^2 g) = F\left(D^2 u - \frac{I}{n\Lambda}\right) \geq F(D^2 u) - \Lambda \frac{1}{\Lambda} = 0 \quad \text{in } \Omega(u) \cap B(x, \rho)$$

and $g(x) = 0$. Therefore, by the maximum principle, g attains its positive maximum on $\partial(\Omega(u) \cap B(x, \rho))$. Since $g < 0$ on $\partial\Omega$, we'll have

$$\sup_{y \in \partial B(x,\rho)} g(y) \geq 0,$$

i.e.,

$$\sup_{y \in \partial B(x,\rho)} \left[u(y) - u(x) - \frac{|x-y|^2}{2n\Lambda}\right] \geq 0,$$

which amounts to

$$\sup_{y \in B(x,\rho)} [u(y) - u(x)] \geq \sup_{y \in \partial B(x,\rho)} [u(y) - u(x)] \geq \frac{\rho^2}{2n\Lambda}.$$

Next suppose $u(x^0) = \sup_{y \in B(x,\rho/2)} u(y) > 0$ for some $x^0 \in B(x, \rho/2)$. Then we apply the above estimate to the point x^0 to obtain

$$\sup_{B(x,\rho)} u \geq \sup_{B(x^0,\rho/2)} u \geq u(x^0) + \frac{\rho^2}{8n\Lambda} \geq u(x) + \frac{\rho^2}{8n\Lambda}.$$

Now the third possibility is that $u < 0$ in $B(x, \rho/2)$. In this case $F(D^2 u) = 1$ in $B(x, \rho/2)$. Hence $F(g) \geq 0$ there, where g is as above. Therefore the same reasoning as that in the first case applies to g in $B(x, \rho/2)$, and we derive the desired result. This finishes the proof. □

Next we'll define some terminology that is frequently used in the sequel.

Half-Space Solution. A global solution of the form $((x \cdot e)^+)^2/2$ for a fixed unit vector is called a half-space solution. If the boundary of the set Ω is smooth, then one may easily verify that any blowup of a solution (to our problem) will be a half-space solution.

ε-close. Two functions f and g are said to be ε-close to each other in a domain D if $\sup_{x\in D}|f(x)-g(x)|<\varepsilon$. The reader should notice that if u satisfies an elliptic inequality $|F(D^2u)|\le C$, then the ε-closeness of u to a half-space solution $h=((x_1)^+)^2/2$ implies, by elliptic estimates, that the gradient of u is also $(C\sqrt{\varepsilon})$-close to the ∇h.

Flat Points. $x^0\in\partial\Omega(u)$ is called a flat point if some blowup Ω_0 of Ω is a half-space. In other words, the quadratic scaling at x^0, $u_\varepsilon=1/(\varepsilon^2)u(x^0+\varepsilon x)$, converges for some subsequence to a half-space solution $u_0=(x_1^+)^2/2$ in some coordinate system, and $\Omega_0=\Omega(u_0)$ is a half-space.

The nondegeneracy will give us measure theoretical stability. We use the following simple version.

LEMMA 3.2 *Suppose $u\in P(0,M)$ is ε-closed to a half-space solution $h=(x_1^+)^2/2$ in $B(0,1)$. Then $u=0$ in $B(0,1)\cap\{x_1<-C\sqrt{\varepsilon}\}$ for some universal constant C.*

PROOF: Take a point $x^0\in B(0,\frac{1}{2})\cap\{x_1<0\}$. Choose $r=|x_1^0|$. Then by Lemma 3.1 and the closeness of u to the half-space solution h, we have

$$\varepsilon\ge\sup_{B(x,r)}u\ge u(x)+\frac{r^2}{8n\Lambda}\ge-\varepsilon+\frac{r^2}{8n\Lambda}\,,$$

i.e.,

$$\varepsilon\ge\frac{r^2}{16n\Lambda}\,;$$

observe that $|u|<\varepsilon$ in $B(x^0,r)\subset\{x_1<0\}$. Hence if $x_1^0=-r<-\sqrt{16n\Lambda\varepsilon}$, then x^0 cannot be a point of $\overline{\Omega}$. This implies that $u\equiv0$ on the set $\{x_1<-\sqrt{16n\Lambda\varepsilon}\}$. □

LEMMA 3.3 *Let $u\in P_1(0,M)$, and suppose $CD_1u-u\ge-\varepsilon_0$ in $B(0,1)$ for some $\varepsilon_0>0$ and $C>0$. Then $CD_1u-u\ge0$ in $B(0,\frac{1}{2})$, provided $\varepsilon_0<1/(8\Lambda n)$. In particular, if u is ε_0-close to a half-space solution $(x_1^+)^2/2$ in $B(0,1)$, say, then (by integration) $u\ge0$ in $B(0,\frac{1}{2})$.*

The proof is a modification of lemma 4.2 in [3]; we give some details.

PROOF: Since F is homogeneous of degree one, i.e., $F(tD^2u)=t\chi_\Omega$ for $t>0$, we can take the derivative with respect to t and set $t=1$, i.e.,

$$Lu=F_{ij}(D^2u)D_{ij}u=\chi_\Omega\,.$$

Next we differentiate $F(D^2u)=1$ in direction e in $\Omega(u)$ to obtain

$$Lu_e=F_{ij}(D^2u)D_{ij}u_e=0\,.$$

Now suppose the conclusion of the lemma fails. Then there is a $u\in P_1(0,M)$ with

$$CD_1u(x^0)-u(x^0)<0$$

for some $x^0 \in B(0, \frac{1}{2})$. Let

$$w(x) = CD_1u(x) - u(x) + \frac{|x - x^0|^2}{2n\Lambda}.$$

Then w is a supersolution of the linearized equation $Lw = 0$ in $\Omega \cap B(x^0, \frac{1}{2})$ and $w(x^0) < 0$. Hence by the minimum principle

$$\inf_{\partial B(x^0,1/2)\cap\Omega} w \leq w(x^0) < 0.$$

In particular,

$$-\varepsilon_0 \leq \inf_{\partial B(x^0,1/2)\cap\Omega} (CD_1u - u) \leq -\frac{1}{8\Lambda n},$$

which is a contradiction as soon as $\varepsilon_0 < 1/(8\Lambda n)$. The second statement follows by Lemma 3.2 and integration. □

4 Proof of Theorem 1.4

In this section we give a proof of Theorem 1.4. The convexity of nonnegative global solutions actually will follow, with some minor modifications, from the local estimates of the second pure derivatives; see [5]. However, here we choose to give a new proof that is very elementary, dealing merely with the global problem. The advantage of the technique presented in this section is its simplicity and flexibility.

We split the proof of the theorem into two cases: convex F in $\mathbb{R}^n$ and concave F in $\mathbb{R}^2$. The reason that we cannot prove the theorem in its full generality in $\mathbb{R}^n$ for concave F is the fact that when F is convex the function $D_{ee}u$ will be a supersolution of a linearized elliptic equation. This is not the case when F is concave. Therefore, we use a slightly different approach in the concave case that works only in two dimensions. The authors believe, however, that the theorem should also hold for concave F in $\mathbb{R}^n$.

We also remark that if u is a nonnegative global solution, then it is locally $C^{1,1}$; see the remark following Theorem 1.2.

From here one can infer that the second derivatives of u are uniformly bounded. This actually follows by the simple fact that if y is any point in $\mathbb{R}^n$, then we may replace u by $u(rx)/r^2$ (with $r = 2|y|$) to obtain a function in the class $P_1(0, M)$ that is also nonnegative. Now one can use the remark following Theorem 1.2. Observe that the quadratic bound at the infinity and local $C^{1,1}$ estimates give a uniform bound for this function.

4.1 Case 1

When F is convex, $D_{ee}u$ is a supersolution of the linearized equation, as can be easily verified. Hence $D_{ee}u$ has the minimum principle. Let's assume u is not convex. Then there is a direction e such that

$$-\infty < \inf_{\Omega(u)} D_{ee}u = -C < 0.$$

Let $\{x^j\}$ be a minimizing sequence, i.e.,

$$D_{ee}u(x^j) \to -C \quad \text{as } j \to +\infty .$$

Set $d_j = \operatorname{dist}(x^j, \partial\Omega) < +\infty$ and consider the following blowup sequence:

$$u_j(x) = \frac{1}{d_j^2} u(x^j + d_j x) .$$

Now the nonnegative solutions u_j are uniformly $C^{1,1}$ (see Remark 1.3 or theorem 2.2 in [5]) and satisfy

$$|D^2 u_j(x)| = |D^2 u(x^j + d_j x)| \leq M , \qquad \min_{B(0,1)} u_j(x) = \frac{1}{d_j^2} u(x^j + d_j x^0) = 0 ,$$

for some $y^j \in \partial B(0,1)$ such that $x^j + d_j y^j \in \partial\Omega$. By the standard compactness argument, there is a convergent subsequence (we use the same notation) $u_j \to u_0$ uniformly on compact subsets of $\mathbb{R}^n$. Moreover, the convergence being in $C^{2,\alpha}(B(0, \frac{1}{2})$ (observe that $F(D^2 u_j) = 1$ in $B(0,1)$) implies

$$D_{ee}u_0(0) = \lim_j D_{ee}u_j(0) = \lim_j D_{ee}u(x^j) = -C$$

and

$$D_{ee}u_j(x) = D_{ee}u(x^j + d_j x) \geq -C ,$$

which implies

$$D_{ee}u_0(x) \geq -C \quad \text{in } B\left(0, \frac{1}{2}\right).$$

Hence $D_{ee}u_0$ obtains its minimum at the interior of $\Omega(u_0)$. Therefore by the minimum principle $D_{ee}u_0 \equiv -C$ in Ω', the connected component of $\Omega(u_0)$ containing the origin.

Rotate the coordinate system such that e is directed in the direction of the positive x_1-axis. Then by the above $D_{11}u_0 \equiv -C$ in Ω'. Hence $D_1 u_0$ is nonincreasing in the x_1-direction, and by integration $D_1 u_0 = -Cx_1 + g(x_1')$, and $u_0 = -Cx_1^2/2 + g(x_1')x_1 + f(x_1')$, where $x_1' = (x_2, \ldots, x_n)$. Since $u_0 \geq 0$, we must have $x_1^2 \leq (g(x_1')x_1 + f(x_1'))/C$ for $x \in \Omega'$. Hence $|x_1| \leq G(x_1')$ for some function G and all $x \in \Omega'$. In particular, for fixed x there holds that $u_0(x^m) = |\nabla u_0(x^m)| = 0$ for all $x^m = (x_1 + m, x_2, \ldots, x_n)$, where $m = m(x_1')$ and $|m|$ is large enough. Hence for large m, we have $D_1 u_0(x^m) = 0$. This in combination with the fact that $D_1 u_0$ is nonincreasing gives that $D_1 u_0 \leq 0$ in Ω'. This contradicts the nondegeneracy, Lemma 3.1.

The next case considers concave F in $\mathbb{R}^2$, with the further restriction that $F(D^2 u)$ depends only on the eigenvalues of $D^2 u$.

4.2 Case 2

Next we consider the case when F is concave in two space dimensions. If u is not convex, the smaller eigenvalue is negative at some point. Let $\lambda_i = \lambda_i(x)$ $(i = 1, 2)$ be two eigenvalues of D^2u such that

$$\lambda_1 \le \lambda_2\,, \qquad \inf_{x\in\Omega(u)} \lambda_1(x) = -C > -\infty\,.$$

Then there is a sequence x^j such that $\lambda_1(x^j) \to -C$. Without loss of generality, we assume $F(0, 1) = 1$ and

$$\sup_{x\in\Omega(u)} \lambda_2(x) = 1 + C > 1\,, \quad \lambda_2(x^j) \to 1 + C\,.$$

We can find a direction e such that $D_{ee}u(x^j) \to 1+C$. Next we consider a blowup similar to that of the previous case,

$$u_j = \frac{1}{d_j^2} u(x^j + d_j x)$$

where $d_j = \mathrm{dist}(x^j, \partial\Omega)$. Then as before (for a subsequence)

$$u_j \to u_0$$

uniformly on compact sets of $\mathbb{R}^2$ and $D_{ee}u_j \to D_{ee}u_0$ in $B(0, \frac{1}{2})$ with

$$D_{ee}u_0(0) = 1 + C \ge D_{ee}u_0(x)\,.$$

Since $D_{ee}u_0$ is a subsolution when F is concave, $D_{ee}u_0$ satisfies the strong maximum principle and achieves its maximum in the interior. Therefore $D_{ee}u_0 = 1+C$ in $\Omega(u_0)$. The same reasoning as in the previous case gives a contradiction.

5 Proof of Theorem 1.6

In this section, we prove Theorem 1.6 by showing that under conditions of Theorem 1.6, the solution u will become nonnegative in some uniform (with respect to the class) neighborhood of the origin. The idea (cf. [3]) is a simple contradictory argument that uses the convexity of nonnegative global solutions.

First we formulate a basic lemma that exhibits the regularity properties of nonnegative global solutions.

LEMMA 5.1 *Given $\varepsilon > 0$, there exists t_ε such that if $u \in P_\infty(0, M)$, $u \ge 0$, and $\delta_1(u) \ge \varepsilon$, then in $B(0, t_\varepsilon)$, $\partial\Omega$ is the graph of a C^1 function (uniformly for the class) and u is $(\varepsilon t_\varepsilon^2)$-close to a half-space solution there.*

A proof of this lemma can be given by the compactness in $P_\infty(0, M)$ and main results in [5]. A simple modification of Lemma 3.3 will give us the following lemma:

LEMMA 5.2 *Let $\varepsilon, s > 0$, and suppose $u \in P_1(0, M)$ is (εs^2)-close to a half-space solution $((x \cdot e)^+)^2/2$ in $B(0, s)$. Then in $B(0, s/2)$ we have*

$$s D_e u - u \geq 0$$

provided ε is small enough. In particular, by integration, we have $u \geq 0$ in $B(0, s/4)$, again if ε is small enough.

To begin our proof of Theorem 1.6, we claim that for given $\varepsilon > 0$, there exists r_ε ($0 < r_\varepsilon < t_\varepsilon$, where t_ε is as in Lemma 5.1) such that if $u \in P_1(0, M, \sigma)$ and $\delta_{r_0}(u) \geq \varepsilon$ for some $r_0 < r_\varepsilon$, then u is $(2\varepsilon r_0^2 t_\varepsilon^2)$-close to a half-space solution $(x_1^+)^2/2$ in $B(0, r_0 t_\varepsilon/2)$. If this claim is true, Lemma 5.2 says that $u \geq 0$ in $B(0, r_0 t_\varepsilon/8)$. Hence we can apply the main theorem in [5] to this nonnegative solution u to show the C^1-regularity of $\partial\Omega(u)$ in $B(0, r_0 t_\varepsilon/16)$. The modulus of continuity $\sigma_1(r)$ is defined by the inverse relation $\varepsilon \to r_\varepsilon$.

Now let's prove the claim. Suppose the claim fails. Then for each $r_j \searrow 0$, there is $u_j \in P(0, M, \sigma)$ such that $\delta_{r_j}(u_j) \geq \varepsilon$, and

$$\sup_{B(0,t_\varepsilon/2)} |u_j - h| > 2\varepsilon r_j^2 t_\varepsilon^2$$

for all half-space solutions h.

Let now $v_j = u_j(r_j x)/r_j^2$. Then $\delta_1(v_j) \geq \varepsilon$, and

$$\sup_{B(0,t_\varepsilon/2)} |v_j - h| > 2\varepsilon t_\varepsilon^2$$

for all half-space solutions h. Also, by Theorem 1.2, $|v_j|$ is uniformly bounded since $V_r(0, u) \leq \sigma(r)$ (small) when r is small. Hence there is a convergent subsequence $v_j \to v_0$. This v_0 is a nonnegative global solution, and

$$\sup_{B(0,t_\varepsilon/2)} |v_0 - h| \geq 2\varepsilon t_\varepsilon^2 \, .$$

By Lemma 5.1, this is a contradiction. This completes the proof of the theorem.

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