REGULARITY PROPERTIES OF A FREE BOUNDARY NEAR CONTACT POINTS WITH THE FIX BOUNDARY

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Abstract. In the upper half of the unit ball $B^+ = \{|x| < 1,\ x_1 > 0\}$, let u and Ω (a domain in $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n :\ x_1 > 0\}$) solve the following overdetermined problem:

$$\Delta u = \chi_\Omega \quad \text{in } B^+, \qquad u = |\nabla u| = 0 \quad \text{in } B^+ \setminus \Omega, \qquad u = 0 \quad \text{on } \Pi \cap B,$$

where B is the unit ball with center at the origin, χ_Ω denotes the characteristic function of Ω, $\Pi = \{x_1 = 0\}$, $n \geq 2$, and the equation is satisfied in the sense of distributions. We show (among other things) that if the origin is a contact point of the free boundary, i.e., $u(0) = |\nabla u(0)| = 0$, then $\partial\Omega \cap B_{r_0}$ is the graph of a C^1 function over $\Pi \cap B_{r_0}$. The C^1-norm depends on the dimension and $\sup_{B^+} |u|$. The result is extended to general sub-domains of the unit ball with C^3 boundary.

1. **Introduction**

The regularity properties of solutions to a certain type of free boundary problems are the main object of study in this paper. Mathematically the problem is formulated as follows. Let $\Omega \subset \mathbf{R}^n_+$ and suppose there is a function u, solving the following problem

$$\Delta u = \chi_\Omega \ \text{ in } B^+, \qquad u = |\nabla u| = 0 \ \text{ in } B^+ \setminus \Omega, \qquad u = 0 \quad \text{on } \Pi \cap B, \tag{1.1}$$

where B^+ is the upper half of the unit ball with center at the origin, and $\Pi = \{x_1 = 0\}$; see Figure 1. Let us denote the free boundary

$$\{x : u(x) = |\nabla u(x)| = 0\} \cap \partial\Omega$$

by $\Gamma(u)$.

The behavior of the free boundary near the contact points $\Gamma_0(u) = \Gamma(u) \cap \Pi$ is of main interest in this paper.

This problem was considered earlier by the second author [U] and the second author and D. Apushkinskaya [AU] but with restriction on the sign of the solution. Under the Lipschitz regularity assumption on $\partial\Omega$ the authors in [AU] could prove that the free and the fixed boundary meet tangentially. Later it was shown by [U1] that the $\Gamma(u)$ is C^1. Recently the second author has extended these results to the situation with no-sign restriction on u and with a certain thickness assumption on the set $\Pi \cap \{|\nabla u| = 0\}$ (see [U3]). For the case $u \geq 0$ any a priori assumptions on $\partial\Omega$ were removed in [U2].

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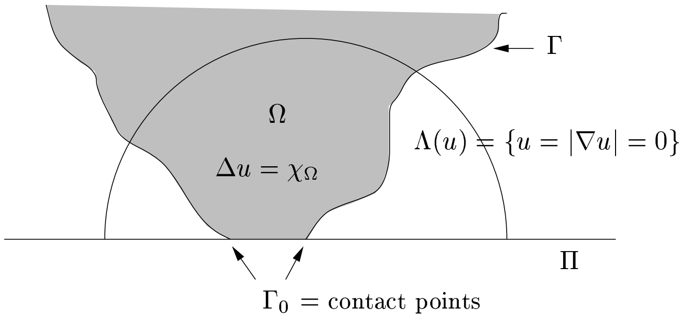

FIGURE 1

Our main objective in this paper will be to study the problems with neither sign restriction and nor Lipschitz regularity assumptions. We will still conclude that the free boundary is a graph of a C^1-function. This is now possible due to recent progress in related problems (see [CKS]), using strong tools such as the monotonicity formula. See also [Ca1–3] for backgrounds and developments in the case of the obstacle problem.

Motivation. Our motivation for studying this problem is threefold. The motivation for studying contact points is related to problems in mathematical physics, where certain data produce contact points between the free (unknown) and the prescribed boundary. Such problems appear in filtration [F], motion by mean curvature with non-convex obstacles, one or two phase problems in flame combustion describing the propagation of curved premixed flame when the fixed and the free boundary touch (see [BL], [BLN], [BCN], [Gu]).

The second reason for studying this problem, is the peculiarity of the "no-sign-assumption" on the solution. Since, generally, solutions to free boundary problems are physical quantities and such quantities are positive, in most applications, it may occur that our problem is artificially created. However, in purely mathematical (read academic) problems such as "the inverse of Cauchy-Kowalewski theorem" or "harmonic continuation of potentials" (see [KS], [CKS]) and in some applications such as chemical reaction interface [CH], or the two phase Bernoulli problem [ACF], or problems involving a system (see [KSt; Chapter VI Section 5–6]; here one takes $u = u_1 - u_2$ where u_1, u_2 are solutions of the system) one may encounter with the problem that there is no restriction on the sign of the solution. In this connection we also refer to [ACS] for a problem in inverse potential theory where the no-sign assumption is dictated by the problem. Yet another problem that has this special feature of no-sign assumption is a free boundary arising in the stationary case in the mean-field theory of superconductivity, see [ESS], [CS].

Our third, and maybe the most important, reason for this study is to develop a technique that handles problems of the above nature. This was initiated in [KS] and further investigated in [CKS]. The advantage of the technique presented in this paper, besides its flexibility, is its simplicity and clarity. The local fine analysis used in (by now) classical approach ([Ca1-2], [U1]) has been abandoned in favor of the global analysis [Ca3], [CKS]. The nature of the free boundary problems is such that this way of approach seems to be more appropriate for analyzing the free boundaries. Our efforts in this paper are most in this regard.

Notation. We will use the following notations throughout the paper.

C, C_0, C_1	generic constants
χ_D	the characteristic function of the set D, $(D \subset \mathbf{R}^n,\ n \geq 2)$
$\overline{D}$	the closure of D
∂D	the boundary of a set D
$\|D\|$	$n-$dimensional volume of the set D
x, X	$x = (x_1, \cdots, x_n)$, $\quad X = (x_2, \cdots, x_n)$
$\mathbf{R}^n_+, \mathbf{R}^n_-$	$\{x \in \mathbf{R}^n :\ x_1 > 0\}$; $\{x \in \mathbf{R}^n :\ x_1 < 0\}$
$B_r(x), B(x,r)$	$\{y \in \mathbf{R}^n : \|y - x\| < r\}$
$B^+_r(x)$	$B_r(x) \cap \mathbf{R}^n_+$
$B_r,\ B^+_r$	$B_r(0),\ B^+_r(0)$
$\Lambda(u)$	$\{x : u(x) = \|\nabla u(x)\| = 0\}$, for any C^1–function u
$\Omega(u)$	$\mathbf{R}^n_+ \setminus \Lambda(u)$
Π	$\{x :\ x_1 = 0\}$
$\Gamma(u)$	$\partial\Omega(u) \cap \Lambda(u)$; free boundary
$\tilde{\Lambda}(u)$	projection of $\Lambda(u)$ on Π
$\Gamma_0(u)$	$\Gamma(u) \cap \Pi$ contact points
$\Lambda_r(u,x)$	$\Lambda(u) \cap B_r(x)$
$\mathbf{Proj}(V_1, V_2)$	projection of the set V_1 onto V_2
$\tilde{\Lambda}_r(u,x)$	$\mathbf{Proj}(\Lambda_r(u,x), \Pi)$
$\\|\cdot\\|_\infty$	supremum norm
MD	minimum diameter; see Definition 1.2 below
$e_1, \cdots, e_n$	standard basis in $\mathbf{R}^n$
ν, e	arbitrary unit vectors
$D_\nu, D_{\nu e}$	first and second directional derivatives
$e \perp \nu$	e is orthogonal to ν
$P^+_r(M), \cdots$	see Definition 1.1
v^+, v^-	$\max(v,0)$; $\max(-v,0)$.

In order to state our main results, let us define a local solution.

Definition 1.1. We say a function u (not identically zero) belongs to the class $P^+_r(M)$ if u satisfies (in the sense of distributions):

(1) $\Delta u = \chi_\Omega \quad$ in B^+_r, $\quad$ for some open set $\Omega \subset \mathbf{R}^n_+$,
(2) $u = |\nabla u| = 0$ in $B^+_r \setminus \Omega$,
(3) $\|u\|_{\infty, B^+_r} \leq M$,
(4) $u = 0$ on $\Pi \cap B_r$.

The interior points of the free boundary is well studied by [CKS]. Here the interior points refers to the situation were $B^+_{r_0}(0)$ is replaced by $B_{r_0}(0)$ and u solves the above problem (without (4)) in the interior of $B_{r_0}(0)$, and without the boundary condition on Π. In this paper we will consider the free boundary near and on Π.

Observe that the class $P^+_r(M)$, as defined above, is invariant under rotation in Π, i.e., rotations that leave e_1 fixed. Next we define a subclass of $P^+_r(M)$ which is the main class under consideration in this paper. Namely

$$P^+_r(0, M) := \{u \in P^+_r(M) :\ 0 \in \Gamma(u)\}.$$

We also denote by $P_\infty^+(M)$, $P_\infty^+(0,M)$ the corresponding "global solutions" with quadratic growth, i.e., solutions in the entire upper half space $\mathbf{R}^n_+$ with quadratic growth $|u(x)| \le M(|x|+1)^2$. Similarly we define $P_r(M)$, $P_r(0,M)$ and $P_\infty(0,M)$ where we replace B_r^+ and $\mathbf{R}^n_+$ by B_r and $\mathbf{R}^n$ respectively and Ω is an open set in $\mathbf{R}^n$. Here there is no extra Dirichlet data assumed on the plane Π (see [CKS]).

Examples of global solutions can be given easily. For $a \ge 0$

$$u(x) = \frac{((x_1 - a)_+)^2}{2},$$

is a global solution. When $a = 0$ the free boundary coincides with the fixed boundary, and for $a > 0$ the free boundary is $\{x_1 = a\}$. Observe that the solution here is one dimensional.

Two dimensional solutions in $P_\infty^+(0,M)$ can be represented by

$$u(x) = \frac{x_1^2}{2} + \sum_{i=2}^{n} a_i x_1 x_i.$$

Here the free boundary is of dimension $(n-2)$, if $a_i \neq 0$ for some $i \ge 2$. One of the main results of this paper asserts that these are the only global solutions (Theorem B below).

Examples of local solutions are harder to give, since one has to construct them. However, there are some classical examples for the interior free boundaries, i.e., for the problem inside the unit ball. Fortunately, some of these examples can be used to produce examples of contact points. For clarity, and for readers convenience we give some details of this procedure.

Suppose $n = 2$, and recall from [KN; page 387–390] that there are examples of the free boundary in the interior of the ball, where cusps appear. These cusps are represented by the curves

$$x_2 = \pm x_1^{\mu/2}, \qquad 0 \le x_1 \le 1,$$

where $\mu = 4k+1$, $(k = 1, 2, \cdots)$ gives non-negative solutions and $\mu = 4k+3$, $(k = 0, 1, \cdots)$ gives solutions that become negative on the negative x_1-axis. The solution is defined locally by

$$u(x) = x_2^2 - \frac{2}{1+\mu/2}\rho^{1+\mu/2}\sin(1+\mu/2)\theta + \cdots, \qquad x \in \Omega,\ |x| < \varepsilon,$$

for ε small. Here we've used both real and complex notation

$$x = (x_1, x_2), \qquad z = \rho e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

Also the domain Ω is the image of the set

$$\{z : \ |z| < 1,\ \mathrm{Im}\ z > 0\}$$

under the conformal mapping $f(z) = z^2 + iz^\mu$.

Now for the case, $\mu = 4k+3$ the solutions are negative on the negative x_1-axis. However, one can show (we will not go into the details of this part, for the specific case of the cardioid one can even compute this numerically) that the set $\partial\{u < 0\}$ consists of two C^∞ curves one above and one below the negative x_1-axis, and they meet the x_1-axis tangentially. Let us denote these graphs by f_1, f_2 respectively. We also assume these graphs are identically zero on the positive x_1-axis. Now consider a conformal mapping, which takes the lower part, i.e., f_2 into the x_1 axis. Then our new free boundary is exactly the one which hits the x_1-axis tangentially, and it lies above the positive x_1-axis. The set where $\{u<0\}$ lies above the negative x_1-axis, and it has zero Lebesgue density at the origin. Moreover our transfered function is in the class $P^+_\varepsilon(0,M)$ for some small ε. By scaling we may transfer the solution to the unit upper half ball.

Our definition below requires the following notation:

$$\Lambda_r(u,x)=\Lambda(u)\cap B_r(x), \qquad \tilde\Lambda_r(u,x)=\mathbf{Proj}(\Lambda_r(u,x),\Pi),$$

where $\mathbf{Proj}(\Lambda_r(u,x),\Pi)$ means projection of $\Lambda_r(u,x)$ onto Π. We suppress the point x in $\Lambda_r(u,x)$ and $\tilde\Lambda_r(u,x)$ when it is the origin, or there is no ambiguity.

Definition 1.2. (Minimal diameter) The minimum diameter of a bounded set $D\subset\mathbb{R}^k$, denoted $\mathrm{MD}(D)$, is the infimum of distances between pairs of parallel hyperplanes in $\mathbf{R}^k$ such that D is contained in the strip determined by the planes. We also define the density function

$$\delta_r(u,x)=\frac{\mathrm{MD}(\tilde\Lambda_r(u,x))}{r}.$$

Observe that by our definition of $\tilde\Lambda_r$ we have $\tilde\Lambda_r\subset\Pi\approx\mathbf{R}^{n-1}$, i.e. $k=n-1$ in this case. Let us state our main results in this paper, i.e., Theorems A,B,C, and D below.

Theorem A. *There is a constant C_1 (depending only on the dimension), such that if $u\in P_1^+(M)$ then*

$$\sup_{B^+(0,1/2)}|D_{ij}u|\le C_1M.$$

The proof of this theorem, follows the main steps in that of [Theorem I in CKS]. We will only mention the minor changes of the new situation later in Section 2.

Theorem B. *Let $u\in P^+_\infty(M)$, then the following hold:*

$$\partial\Omega\cap\{x_1>0\}=\emptyset \qquad\Longrightarrow\qquad u(x)=\frac{x_1^2}{2}+ax_1x_2+\alpha x_1,$$

in some rotated system of Π, and for some real numbers a and α.

$$\partial\Omega\cap\{x_1>0\}\neq\emptyset \qquad\Longrightarrow\qquad u(x)=\frac{((x_1-a)_+)^2}{2}, \quad \textit{for } a>0.$$

Observe that if the origin is a contact point then $u\in P^+_\infty(0,M)$ and consequently $\alpha=0$ in Theorem B. Also recall that by the definition, members of $P^+_\infty(M)$ have quadratic growth at infinity.

Before stating the next result we need to define the so called singular set.

Definition 1.3. (*Singular set*) A point x^0 is said to be a singular contact point if $x^0\in\Gamma_0(u)$ and there is a blow up (see Section 3 below) u_0 of u at x^0 which, after translation and rotation, has the representation

$$u_0=\frac{x_1^2}{2}+ax_1x_2, \qquad \text{in } \mathbf{R}^n_+ \qquad (a\neq 0). \tag{1.2}$$

For definiteness we'll use the notation $\mathcal{S}_u(a_0)$ to denote the set of all singular points x^0 such that the blow up of u at the singular point has the representation (1.2), for some $|a|\ge a_0$

The following properties can be listed up for the free boundary $\Gamma(u)$ and the solution u.

Theorem C. *There exists a universal constant $r_0 = r_0(n, M)$, and a modulus of continuity σ ($\sigma(0^+) = 0$) such that if $u \in P_1^+(0, M)$ then the following hold.*

(C1)

$$\partial\Omega \cap B_{r_0} \subset \{x : \ x_1 \leq \sigma(|x|)|x|\}.$$

(C2) *$\partial\Omega \cap B_{r_0}$ is the graph of a C^1 function over $\Pi \cap B_{r_0}$. The C^1 norm is uniform for the class. More exactly the modulus of continuity of the normal vector to the boundary of Ω is $\sigma(r)$.*

(C3) *The singular set $\mathcal{S}_u(a)$ of $\partial\Omega(u)$ lies locally in an $(n-2)$ dimensional C^1-manifold $\Pi' \subset \Pi$. The neighborhood depends on the constant a in (1.2).*

(C4) $\lim_{x\to\partial\Omega(u)} D_{ij}u$ *exists non-tangentially for $x \in \Omega(u) \cap B^+_{1/2}$.*

(C5) *If $\delta_r(u, 0) > \sigma(r)$ for some $r < r_0$, then*

$$\sup_{B_r^+} \frac{|D_e u(x)|}{x_1} \leq \sigma(r), \tag{1.3}$$

for all e orthogonal to e_1.

(C6) *If $n = 2$ then $D_{ij}u \in C(\overline{\Omega(u)} \cap B_{r_0})$. If $n > 2$ then $D_{ij}u$ are continuous in $(\overline{\Omega(u)} \cap B^+_{r_0}) \cup \{0\}$, provided $\overline{\lim}_{r\to 0}\delta_r(u, 0) > 0$.*

For $n = 2$, the statement of part (3) of the theorem means that singular contact points are isolated, while part (5) simply asserts that if the origin is a free boundary point and if there is a free boundary point x^0 that comes too near the origin then the conclusion of the theorem holds.

Observe that Theorem C is point independent, i.e., the same statements hold for all contact points in $B_{1/2}$.

The singular set of the interior points of the free boundary has been studied by L. Caffarelli [Ca3] for the case $u \geq 0$, and by L. Caffarelli and the first author [CSh] for the general case. In [Ca3] using the non-negativity of the solutions one can give a detailed description of the singular set in terms of the blow-up solutions. Notably, Caffarelli's technique for studying the structure of the singular set, seems to fail for the interior singular points when there is no sign restriction on the solution. However, the authors in [CSh] are able to prove that the singular set locally lies on a C^1-manifold of dimension k, where k is the dimension of the linear subspace $\Lambda(u_0)$. A similar analysis works here as well.

Theorem D. *All the results in Theorems A and C can be generalized to hold when Π is replaced by a C^3 surface. More exactly we assume there is a C^3 graph G_0 containing the origin and dividing the unit ball B_1 into two parts, G_0^+ and G_0^-. On one side of this graph, G_0^+ say, we assume that u and Ω ($\subset G_0^+$) solve (1.1) with B^+ replaced by G_0^+. Then there exists a small constant r_0 and a modulus of continuity $\sigma(r)$, depending on n, M, and the C^3-norm of G_0 such that if $0 \in G_0 \cap \Gamma(u)$ with u satisfying the conditions in Definition 1.1 (with the appropriate changes), then all the results above (Theorems A and C) remain valid in this case. The quantities will naturally also depend on the C^3 norm of the surface G_0.*

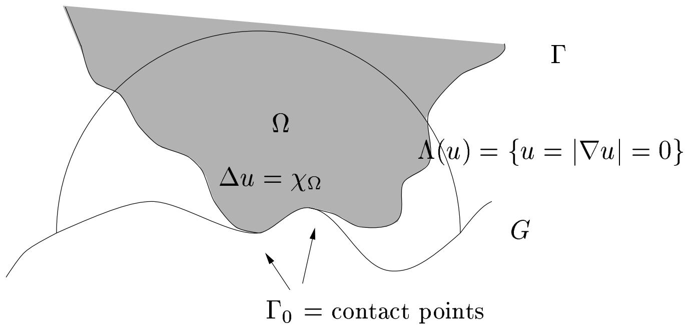

FIGURE 2

Parts (1) and (2) in Theorem C, for the general case, i.e., Theorem D, should be interpreted in terms of the tangent plane of G_0 at the origin.

A fundamental tool in proving our theorems is the following monotonicity lemma.

Lemma 1.4. *[ACF] Let h_1, h_2 be two non-negative continuous sub-solutions of $\Delta u = 0$ in $B(x^0, R)$ $(R > 0)$. Assume further that $h_1 h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$. Then the following function is monotone in r $(0 < r < R)$*

(1.4)

$$\varphi(r) = \varphi(r, h_1, h_2, x^0) = \frac{1}{r^4}\left(\int_{B(x^0,r)} \frac{|\nabla h_1|^2\, dx}{|x - x^0|^{n-2}}\right)\left(\int_{B(x^0,r)} \frac{|\nabla h_2|^2\, dx}{|x - x^0|^{n-2}}\right).$$

More exactly, if any of the sets supp $(h_i) \cap \partial B_r(x^0)$ digresses from a spherical cap by a positive area, then either $\varphi'(r) > 0$ or $\varphi(r) = 0$.

In proving Theorem D we need to use a different type of monotonicity formula, which is similar to the above but relaxes the subharmonicity condition and allows the solutions to have bounded Laplacian from below only.

Lemma 1.5. *[CJK; Theorem 1.6] Recall the assumptions in Lemma 1.4, and replace the subharmonicity assumption by the boundedness (from below) of the Laplacian of h_i, i.e., assume $\Delta h_i \geq -1$. Suppose moreover $|h_i(x)| \leq C|x - x^0|^\beta$ for some $\beta > 0$. Then*

$$\varphi(s_1) \leq (1 + s_2^\beta)\varphi(s_2) + C s_2^\beta, \tag{1.5}$$

where $0 < s_1 \leq s_2 \leq R$.

In Lemma 1.5 if we have $\Delta h_i \geq -C_i$ then we can replace h_i by h_i/C_i and change the constant C in (1.5).

The reader may easily verify that any function verifying (1.5) must have a limit as $r \to 0^+$, i.e.,

$$\lim_{r \to 0^+} \varphi(r) \quad \text{exists} . \tag{1.6}$$

In the sequel we'll use the notation $\varphi(r, D_e u) = \varphi(r, D_e u, x^0)$ in (1.4)-(1.5) with $u \in P_1^+(M)$ and h_1, h_2 replaced by $(D_e u)^\pm$. Here $e \perp e_1$ and $x^0 \in \Pi$, and $D_e u$ is continued into $\{x_1 < 0\}$ in a proper way.

Lemma 1.5 can be used to generalize Theorems A and C to the case of

$$\Delta u = f\chi_\Omega$$

with $f > 0$ and Lipschitz. Since $\Delta(D_e u)^\pm$ will now be bounded from below on Ω.

Remark 1.6. We would like to stress one crucial point in the use of the monotonicity formulas with respect to the functions $u \in P^+(M)$. As discussed above the functions $(D_e u)^\pm$ are admissible for using the monotonicity formula, when $e \perp e_1$, i.e., e is parallel to Π, and $x^0 \in \Pi$.

In the applications of this we use three different continuation of $D_e u$ across the fixed boundary Π.

(1) We consider odd reflection in Π, by setting $u(x) = -u(-x_1, x_2, \cdots, x_n)$, for $x \in \mathbf{R}^n_-$. This is used in connection with the proof of Theorem A, (partially) Theorem B and Theorem C.
(2) We may continue u as zero across Π to the lower half space $\mathbf{R}^n_-$, this is used (partially) in connection with the proof of Theorem B.
(3) In connection with the proof of Theorem D. We consider odd inversion with respect to a sphere; proof of Theorem D.

2. Proof of Theorem A

We will extend $u \in P_1^+(M)$ to the lower half part of the unit ball, by considering odd reflection of u in the plane Π, i.e., we extend u across the plane Π by defining it as $-u(-x_1, x_2, \cdots)$ for $x_1 < 0$ and x in the unit ball.

In this way our functions will have a bounded Laplacian and $D_e u$, for e orthogonal to e_1, will be continuous in the whole unit ball. Moreover $(D_e u)^\pm$ will be subharmonic. Observe also that in [CKS; Theorem I] one only uses the boundedness of the Laplacian of u and the sub-harmonicity of $(D_e u)^\pm$ in order to deduce the quadratic behavior of the solution near the free boundary.

As in [CKS] we set

$$S_j(z,u) = \sup_{B(z,2^{-j})} |u|,$$

and define $\mathbb{M}(z,u)$ to be the maximal subset of $\mathbb{N}$ (natural numbers) satisfying the doubling condition

$$4S_{(j+1)}(z,u) \geq S_j(z,u) \qquad \forall\, j \in \mathbb{M}(z,u).$$

Following [CKS] we consider the next lemma.

Lemma 2.1. *Let $u \in P_1^+(M)$ and $z \in \Gamma(u) \cap B^+_{1/2}$. Then there exists a constant $C_0 = C_0(n)$ such that*

$$S_j(z,u) \leq C_0 M 2^{-2j} \qquad \forall j \in \mathbb{M}(z,u). \tag{2.1}$$

Proof. Extend u, as in the discussion above, to the whole unit ball. Let e be orthogonal to e_1 (x_1-axis) and then follow the steps in [CKS, Lemma 3.2], to end up with $D_e u_0 = 0$ for all such directions. Here u_0 is the limit function. This in particular implies that u_0 is one dimensional, and it depends only on the x_1-direction. Since it is also harmonic it must be linear, i.e., $u_0(x) = ax_1 + b$. But then $u_0(0) = u_0'(0) = 0$ implies $u_0 \equiv 0$. This contradicts [CKS, (3.5)]. Hence (2.1) above holds. □

Observe that the estimate $|u(x)| \le CMd(x)^2$ for all $x \in \Omega(u)$ is equivalent to $|D^2 u| \le C_0 M$. This can be proven following the same lines as that of [CKS; the end of Section 3] in combination with elliptic estimates near the fixed boundary. We leave the details to the reader.

3. Certain Facts

In this and the next sections we will frequently use the blow-up of a function at a point $x^0 \in \Gamma(u)$, i.e., for a given u we consider

$$u_r(x) = \frac{u(rx + x^0)}{r^2}$$

and let r tend to zero, through some subsequence. It is, however, not clear whether the blow-up (the limit function) will not be the zero function. Indeed, if $u(x) = o(|x - x^0|^2)$, then any blow-up will be identically zero. To prevent this we need a non-degeneracy from below, asserted in (3.1) below. However, we first remark that, in view of the maximum principle, in any neighborhood of a free boundary point there are points where u takes positive values.

Non-degeneracy: The function u in (1.1) satisfies

$$\sup_{B^+(x^0,r)} u \ge u(x^0) + C_n r^2, \tag{3.1}$$

for all $x^0 \in \Omega(u)$, where $u(x^0) \ge 0$, and also for all $x^0 \in \Gamma(u)$. Here $C_n = 1/2n$, Also r is small enough so that $B(x^0, r) \subset B(0,1)$. For $B_r(x^0) \subset B_1^+$ (3.1) is also true if $u(x^0) < 0$ with somewhat smaller C_n.

For a proof see [Ca2, CKS].

ε-Close: We say two functions f and g are ε-close to each other in a domain D if

$$\sup_{x \in D} |f(x) - g(x)| < \varepsilon.$$

It can be proven that ε-closeness of $u \in P_r^+(M)$ and $\psi(x) = \frac{((x_1-a)_+)^2}{2}$ $(a > 0)$ implies

$$\sup_{B^+_{r-\sqrt{\varepsilon}}} |Du - D\psi| \le C\sqrt{\varepsilon},$$

with $C = C(n, M)$. We assume here $r > \sqrt{\varepsilon}$.

Blow-up limit: Let A blow-up limit u_0 is a uniform limit on compact subsets of a certain subset of $\mathbf{R}^n$ (to be specified in each case)

$$u_0(x) = \lim_{j\to\infty} \frac{u(r_j x + x^j)}{r_j^2}, \qquad (x^j \in \Gamma(u))$$

where $u \in P_1^+(M)$ and $r_j \to 0$. The function u may even change for different j.

Fact 1. The boundary $\partial\Omega$ has locally finite $(n-1)$-Hausdorff measure; see [CKS], [Ca4] for details.

Fact 2. Let u_j be a blow-up of u and suppose u_j converges to u_0 in C^1_{loc}. Then it it is known that

$$\{u_0 = |\nabla u_0| = 0\} \supset \overline{\lim}\,\{u_j = |\nabla u_j| = 0\},$$

and

$$\text{interior}(\{u_0 = |\nabla u_0| = 0\}) \subset \overline{\lim}\,\{u_j = |\nabla u_j| = 0\};$$

see e.g. ([Ca2], [KS], [CKS]); here $\overline{\lim}$ denotes the limit set of all sequences $\{x_j\}, x_j \in \{u_j = |\nabla u_j| = 0\}$.

Fact 3. Recall the definition of the class $P_1(M)$. It is known [CKS] that if $u \in P_1(M)$ is ε-close to a half-space solution $((x_1)_+)^2/2$, say, then $u \equiv 0$ in $B(0,1/2) \cap \{x_1 < -C\sqrt{\varepsilon}\}$, for some constant $C > 0$. Hence it is easy to deduce that if $u \in P_R^+(M)$ $(R > 4)$ and if u is ε-close in B_R^+ to $((x-1)_+)^2/2$ then $u \equiv 0$ in $\{x \in B_{R-2}^+ : \ x_1 < 1 - c\sqrt{\varepsilon}\}$, for small enough ε. Indeed, by the above statement $u \equiv 0$ in the stripe $\{x = (x_1, X) : \ x_1 \in [1/2, 1 - C\sqrt{\varepsilon}], |X| < R - 5/4\}$. Since u is subharmonic it follows from the maximum principle that for any $y = (1/2, Y)$, $(|Y| \leq R - 7/4)$ either $u \equiv 0$ in $B_{1/2 - C\sqrt{\varepsilon}}(y)$ or $u(x^0) > 0$ for some point $x^0 \in B_{1/2-C\sqrt{\varepsilon}}(y)$. But the latter is impossible as follows from (3.1) and ε-closeness. Thus we obtain $u \equiv 0$ in a wider strip $\{x : \ C\sqrt{\varepsilon} < x_1 < 1 - C\sqrt{\varepsilon}, \ |X| < R - 7/4\}$. By the same reasoning $u \equiv 0$ in $\{x : \ 0 < x_1 < C\sqrt{\varepsilon}, \ |X| < R - 2\}$.

Fact 4. According to Theorem III in [CKS] there exists an ε_0 (small enough) such that if $u \in P_1(0, M)$ is ε_0-close to a half-space solution $((x_1)_+)^2/2$, say, then $\partial\Omega$ is the graph of a C^1 function over Π in B_{t_0} for a universal $t_0 > 0$.

Fact 5. The free boundary $\partial\Omega$ has zero Lebesgue measure, [CKS; General Remarks].

Fact 6. (Stability) Let $u \in P_1^+(0, M)$. Then from the non-degeneracy (3.1) we conclude that for any scaling u_r of u with $x^0 = 0$, we have $0 \in \Gamma(u_r)$. Moreover the same holds for the limit function u_0, when it exists.

Fact 7. Let $u \in P_1(0, M)$ and u_j be a sequence of scalings of u. Then, for a subsequence, the second derivatives of u_j converge to u_0 in L^p, i.e., for some $\{j_m\}$

$$D_{ik} u_{j_m} \to D_{ik} u_0 \qquad \text{in } L^p_{loc}(\mathbf{R}^n_+) - \text{norm},$$

for $1 < p < \infty$. See [CKS; General Remarks c)].

Fact 8. The continuation of a nonnegative subharmonic function v in Ω across $\partial\Omega$ by zero is subharmonic if $v = 0$ on $\partial\Omega$.

4. **Proof of Theorem B**

In this section we will prove Theorem B, i.e., we will classify all global solutions in the class $P_\infty^+(M)$. Obviously, the quadratic growth assumption in the definition of the class $P_\infty^+(M)$ is not redundant, since $u(x) = x_1^2/2 + x_1 x_2 x_3$ is a solution but not of quadratic growth. However, for our local class $P_1^+(0, M)$, we have that scaling of the members of this class remain (in the limit) in the class of global solutions $P_\infty^+(0, M)$. It follows from Theorem A that for the class $P_\infty^+(M)$ the estimate $|D^2 u| \leq CM$ in $\mathbf{R}^n_+$ is true. We leave it to the reader to verify this simple fact.

The classification of global solutions in $P_\infty(0, M)$, that is when we consider the solutions in the entire space rather than in the upper half space, was considered by the first author, L. Caffarelli, and L. Karp in [CKS]. They showed that global solutions are either homogeneous polynomials of second degree or they are convex, i.e., $D_{ee}u \geq 0$ for all directions e. This in turn means that the set $\Lambda(u)$ is convex. They also proved that if the set $\Lambda(u)$ is large enough at infinity then the solution is a half-space solution i.e. $u(x) = ((x_1)_+)^2/2$ in some rotated coordinates.

In our case the situation is rather different. Since we already have the "forcing" set Π, it seems that this information is enough to enable us to prove the theorem under consideration. However, the proof, requires some detailed analysis of the behavior of the free boundary far away. The proof is lengthy, and we have done all efforts to make it clear by adding some details that may not be so obvious for non-experts, but probably trivial to experts.

Since the function vanishes on the set $\{x_1 = 0\}$, and since this set is large, we can use the monotonicity formula more effectively than for the interior case [CKS]. For e orthogonal to e_1, we first extend $(D_e u)^{\pm}$ as zero functions across Π to $\mathbf{R}^n_-$. Then we notice that the scaled functions $u_{r_j}(x) = u(r_j x)/r_j^2$ with $r_j \to \infty$ have uniformly bounded second derivatives. It follows from standard compactness arguments that a subsequence converges uniformly on compact subsets. Taking further a subsequence, we may also assume convergence of the sequence in $\left(W^{2,p}_{loc} \cap C^{1,\alpha}_{loc}\right)(\mathbf{R}^n_+ \cup \Pi)$ for any $1 < p < \infty$ and $0 < \alpha < 1$; see Fact 7.

Indeed, we redefine r_j to get the convergent sequence, and denote by u_∞ the limit function for this sequence. Consider now the monotonicity formula for $0 < r < r_j$ to conclude

$$\varphi(r, D_e u) \leq \varphi(r_j, D_e u) \leq \lim_{r_j \to \infty} \varphi(r_j, D_e u) =: C_e, \tag{4.1}$$

where e is orthogonal to e_1. Also C_e exists by the monotonicity formula and the quadratic growth of u. Now

$$C_e = \lim_{r_j \to \infty} \varphi(s r_j, D_e u) = \lim_{r_j \to \infty} \varphi(s, D_e u_{r_j}) = \varphi(s, D_e u_\infty),$$

for every $s > 0$, i.e., $\varphi(s, D_e u_\infty)$ is constant for all $s > 0$. Since $\{D_e u_\infty = 0\} \supset \{x_1 < 0\}$, Lemma 1.4 implies that either $\varphi'(r) > 0$ for all $r > 0$ or $\varphi(r) \equiv 0$. It thus follows that $C_e = 0$ and consequently by (4.1)

$$0 \leq \varphi(r, D_e u) \leq C_e = 0.$$

We thus conclude that $D_e u \geq 0$ (or ≤ 0); both cases are treated similarly. We also observe that by the strong maximum principle $D_e u > 0$ in (connected components of) Ω or $D_e u \equiv 0$. If there is no direction e orthogonal to e_1 such that $D_e u \not\equiv 0$ then u depends only on the x_1-direction and consequently it has the representation indicated in the theorem.

So suppose for some e (orthogonal to e_1), $D_e u > 0$ in Ω. By rotation (of Π) we may assume that $e = e_2$ (the unit vector directed in the x_2-direction). Fix a point $x^0 \in \Omega$. Then in the $x_2 x_3$-space (denoted $T(e_2, e_3)$) there must be a vector e for which $D_e u(x^0) = 0$. This is obvious, since $D_{-e_2} u < 0$ and it is continuous in e there must exists real numbers a, b such that $D_e u(x^0) = 0$ with $e = ae_2 + be_3$.

Now from the above we also have $D_e u$ does not change sign and it vanishes at x^0. Therefore by the maximum principle it is identically zero in the connected component Ω' of Ω containing x^0. This implies that u restricted to $\Omega' \cap T(e_1, e_2, e_3)$ is two dimensional; here $T(e_1, e_2, e_3)$ means the $x_1 x_2 x_3$-space. Now repeating the same argument for other directions we conclude that u is two dimensional in Ω'. This can also be done for any connected component of Ω.

From the above reasoning it follows that we only need to consider the two dimensional problem. Now suppose the upper half space is free of boundary points, i.e., $\partial\Omega \subset \Pi$. Then $v = D_2 u$ is harmonic in $\{x_1 > 0\}$ and it vanishes on Π. Therefore we can continue v harmonically into $\mathbf{R}^n$. Since v has linear growth (by the assumption) and $v(0) = 0$, it follows by the Liouville's theorem that $v = ax_1 + bx_2$. Now $v = 0$ on Π implies that $b = 0$. Integration gives now $u = ax_1x_2 + g(x_1)$. Next $\Delta u = 1$ in the upper half space gives $g''(x_1) = 1$ there, i.e. $g(x_1) = x_1^2/2 + \alpha x_1 + \beta$. Since $u(0) = 0$ we must have $\beta = 0$. This shows that $u(x) = x_1^2/2 + ax_1x_2 + \alpha x_1$ in the upper half space. This proves the first part of the theorem.

Next suppose there exists $x^1 \in \partial\Omega$ with $x_1^1 > 0$. Then, according to the above analysis and the fact that the free boundary has Lebesgue measure zero (see Fact 5 in Section 3) we conclude that $\Lambda(u)$ has non-void interior. Let us fix a ball $B(x^0, 2r)$ in the interior of $\Lambda(u)$. Since $D_2 u \geq 0$ we must have $u \leq 0$ on the half-cylinder generated by $B(x^0, 2r)$ and directed towards the negative x_2-axis, i.e., $u \leq 0$ in

$$K(x^0, 2r) = \{(x_1, x_2 - s);\ (x_1, x_2) \in B(x^0, 2r)\ s \geq 0\}.$$

From here one infers that for the smaller half cylinder $K(x^0, r)$ we have

$$\partial\Omega \cap K(x^0, r) = \emptyset. \tag{4.2}$$

Indeed if $y \in K(x^0, r) \cap \partial\Omega$ then the strong maximum principle applied to u (a subharmonic function) in $B(y, r)$ gives that $u(y) < 0$ or it is identically zero. Since $u(y) = 0$ the only possibility is that $u \equiv 0$ in $B(y, r)$. Hence $y \notin \partial\Omega(u)$.

Next using (4.2) and that $B(x^0, r) \subset \Lambda(u)$ we conclude $K(x^0, r) \subset \Lambda(u)$. Hence we can translate u in the x_2-direction, by considering $u_m(x) = u(x_1, x_2 - m)$ in $\mathbf{R}^n_+$. One first checks that

$$|u_m(x)| \leq C(1 + |x|^2), \qquad \text{for all positive integers } m. \tag{4.3}$$

This follows from the fact that for any x we evaluate u at $x^m = x - me_2$ and the distance from x^m to the set $K(x^0, r) \subset \Lambda$ is less than or equal to $|x - x^0|$. In other words

$$\mathrm{dist}(x^m, K(x^0, r)) \leq |x - x^0|.$$

Hence the quadratic growth (Theorem A) of u from the free boundary gives the desired bound.

Since $D_2 u \geq 0$ the sequence $\{u_m\}$ is non-increasing, and therefore by compactness it converges to a limit function u_∞. Since $u(x_1, x_2 - m) \geq -C(1 + x_1^2)$, and $D_2 u \geq 0$ there must be a limit for u at $(x_1, -\infty)$. In particular $u_\infty(x) = \lim_m u(x_1, x_2 - m)$ and it is independent of x_2. In other words u_∞ is one dimensional. It is now easy to conclude that $u_\infty = ((x_1 - a)_+)^2/2$ for some constant $a > 0$. From here we see that u becomes nonnegative as x_2 tends to the negative infinity. Since $D_2 u \geq 0$ it means that $u \geq 0$ in the upper half space.

We claim now that u depends only on x_1. If this is true, then we are done using elementary calculus.

First notice that $u \geq 0$ implies that for any direction ν pointing inwards $\mathbf{R}^n_+$ we have

$$D_\nu u \geq 0 \qquad \text{on } \Pi. \tag{4.4}$$

The objective is to prove that

$$D_\nu u \geq 0 \qquad \text{in } \mathbf{R}^n_+, \qquad \text{for all } \nu:\ \nu \cdot e_1 \geq 0. \tag{4.5}$$

Obviously (4.5) implies u is one dimensional.

By (4.4) and the fact that $D_\nu u = 0$ on $\partial\Omega \cap \mathbf{R}^n_+$ we may use the monotonicity formula for $D_\nu u$. Here is how. We first extend u across Π by odd reflection to the largest set for which $D_\nu u \geq 0$ below the plane Π. We also define $D_\nu u$ to be zero in the rest of $\mathbf{R}^n_-$. By this definition we'll have

$$D_\nu u \geq 0 \text{ in } \mathbf{R}^n_-, \tag{4.6}$$

and $\Delta(D_\nu u)^\pm \geq 0$. That $\Delta(D_\nu u)^- \geq 0$ follows from the fact that the support of $(D_\nu u)^-$ is in $\mathbf{R}^n_+$. To deduce the subharmonicity of $(D_\nu u)^+$ one observes that $D_\nu u \geq 0$ on Π so that the odd reflection of u brings us the following information
(a1) $D_{11}u(x) = 1$ for $x \in \Pi \setminus \Lambda(u)$ and $x_1 = 0^+$,
(a2) $D_{11}u(x) = -1$ for $x \in \Pi \setminus \Lambda(u)$ and $x_1 = 0^-$,
(b) $D_{\nu e}u$ is continuous across $\Pi \setminus \Lambda(u)$ for $e \perp e_1$.
Consequently $(D_\nu u)^+$ will be subharmonic in the open set $\{D_\nu u > 0\}$, where u is now the extended function. Since it is also continuously zero outside this set, it must be subharmonic in the whole space.

Next we consider the monotonicity formula for $D_\nu u$ over the whole space $\mathbf{R}^n$. As in the previous situation (4.1) and its succeeding argument (the reader should be aware of the fact that scalings of $D_\nu u$ are uniformly $\mathrm{Lip}(\mathbf{R}^n)$) we will have $\varphi(s, D_\nu u_\infty) = C_\nu$ for all $s \geq 0$. We will prove that $C_\nu = 0$. This will then imply, as in the previous situation, that $D_\nu u$ does not change sign and hence it is non-negative.

Now suppose $C_\nu \neq 0$. Then according to the monotonicity formula both the sets

$$V_s^+ = \{D_\nu u_\infty > 0\} \cap \partial B(0,s), \qquad V_s^- = \{D_\nu u_\infty < 0\} \cap \partial B(0,s),$$

are half-spherical caps (up to $(n-1)$-dimensional zero measure). This, however, is impossible due to the construction of the sets $V_s^\pm$. Indeed, by (4.6) $D_\nu u \geq 0$ in $\mathbf{R}^n_-$, and hence $D_\nu u_\infty \geq 0$ in $\mathbf{R}^n_-$. Then the only possibility for V_s^- to coincide with a half spherical cap for any $s > 0$ is the condition $D_\nu u_\infty \leq 0$ in $\mathbf{R}^n_+$. This in turn implies $u_\infty \leq 0$ in $\mathbf{R}^n_+$. In this case we have $u_\infty \equiv 0$ and again $C_\nu = 0$. This completes the proof. □

5. **Proof of Theorem C**

The proof of Theorem C will now follow easily by using the characterization of global solutions. We first formulate several lemmas.

Lemma 5.1. *Given $\varepsilon > 0$, there exists $\rho = \rho_\varepsilon$ such that if $u \in P_1^+(0,M)$, then for $x^0 \in \partial\Omega(u) \cap B^+(0,\rho_\varepsilon)$ we have*

$$x^0 \in B_{\rho_\varepsilon} \setminus K_\varepsilon, \tag{5.1}$$

where

$$K_\varepsilon := \{x : \ x_1 > \varepsilon\sqrt{x_2^2 + \cdots + x_n^2}\}.$$

Proof. The proof follows using a contradictory argument. So suppose the conclusion of the lemma fails. Then there exists $u_j \in P_1^+(0,M)$ and $x^j \in \partial\Omega(u_j) \cap B_1^+$ with $|x^j| \searrow 0$, and such that (5.1) fails for x^j. Then define $\tilde{u}_j(x) = u_j(d_j x)/d_j^2$ where $d_j = |x^j|$. Now for each function $\tilde{u}_j$ we have a point $\tilde{x}^j = x^j/d_j \in \partial\Omega(\tilde{u}_j) \cap \partial B_1^+$ with $\tilde{x}_1^j > \varepsilon\sqrt{(\tilde{x}_2^j)^2 + \cdots + (\tilde{x}_n^j)^2}$. Next, by standard compactness and for a subsequence, $\tilde{u}_j$ and $\tilde{x}^j$ converge to u_0 and x^0, respectively, and with u_0 a global solution and $x^0 \in \overline{K_\varepsilon} \cap \partial B_1$. Moreover, by Fact 2), both x^0 and the origin are on the free boundary $\Gamma(u_0)$. Since u_0 is a global solution this contradict Theorem B. $\square$

Lemma 5.2. *Given $\varepsilon > 0$, there exists $\rho = \rho_\varepsilon$ such that if $u \in P_1^+(M)$, and $x^0 \in \Gamma(u) \cap B^+(0,1/2)$, and $x_1^0 < \rho_\varepsilon$ then $u \equiv 0$ in $B^+(x^0, 2x_1^0) \cap \{x_1 < x_1^0(1 - C\sqrt{\varepsilon})\}$ and it is $(\varepsilon(x_1^0)^2)$-close to a half-space solution $((x_1 - x_1^0)_+)^2/2$ in $B^+(x^0, 3x_1^0)$.*

Proof. In view of Fact 3) it suffices to prove $\varepsilon(x_1^0)^2$-closeness. So suppose the second statement in the lemma fails. Then for there exist $u_j \in P_1^+(M)$, $x^j \in \partial\Omega(u_j) \cap B_{1/2}^+$, such that

$$\sup_{B^+(x^j, 3d_j)} |u_j(x) - \frac{((x_1 - x_1^j)_+)^2}{2}| > \varepsilon d_j^2,$$

where $d_j = x_1^j \ \to\ 0$.

Set $v_j(x) = u_j(d_j(x - e_1) + x^j)/(d_j)^2$. Then $v_j \in P_{1/2d_j}^+(CMd_j^{-2})$, $e_1 \in \partial\Omega(v_j)$, and the following hold

$$\sup_{B^+(e_1,r)} |D_{kl} v_j| \le CM, \qquad \sup_{B^+(e_1,r)} |v_j| \le CMr^2 \quad \left(\forall\ r \le \frac{1}{2d_j}\right),$$

and

$$\sup_{B^+(e_1,3)} |v_j - ((x_1 - 1)_+)^2/2| > \varepsilon.$$

Now, for a subsequence and in an appropriate space, v_j converges to a global solution $v_0 \in P_\infty^+(CM)$ with $e_1 \in \partial\Omega(v_0)$. But on the other side, we have

$$\sup_{B^+(e_1,3)} |v_0 - ((x_1 - 1)_+)^2/2| \ge \varepsilon.$$

This contradicts the second representation in Theorem B, and completes the proof of the lemma. $\square$

Lemma 5.3. *Given $\varepsilon > 0$, there exists ρ_ε such that if $u \in P_1^+(0, M)$, and $\delta_\rho(u) > \varepsilon$ for some $\rho < \rho_\varepsilon$, then*

$$u > 0 \qquad \text{in } B_\rho \cap K_\varepsilon, \tag{5.2}$$

where K_ε is defined in Lemma 5.1.

Proof. Suppose, towards a contradiction, that (5.2) fails, i.e., there exists $\rho_j \searrow 0$, $u_j \in P_1^+(0, M)$, and $x^j \in \Omega \cap B_{\rho_j} \cap K_\varepsilon$ such that $u_j(x^j) \le 0$, and

$$\delta_{\rho_j}(u_j) > \varepsilon. \tag{5.3}$$

Now consider two different scalings of u_j,

$$\tilde{u}_{\rho_j} = \frac{u_j(\rho_j x)}{\rho_j^2}, \qquad \tilde{u}_{r_j} = \frac{u_j(r_j x)}{r_j^2},$$

where $r_j = |x^j|$. By (5.3) and the fact that $u_j(x^j) \le 0$, one easily verifies that

$$\tilde{u}_{\rho_j} \to P_1 = \frac{1}{2}x_1^2, \qquad \tilde{u}_{r_j} \to P_2 = \frac{1}{2}x_1^2 + \frac{a}{2}x_1 x_2, \qquad \text{in } \mathbf{R}_+^n$$

for some constant a with $|a| \ge \varepsilon$. Observe that we have used the fact that $0 \in \Gamma(u)$, which in conjunction with Fact 6 gives $0 \in \Gamma(P_i)$, $i = 1, 2$. Moreover we have

$r_j \le \rho_j$. Now using the monotonicity formula we will show that this is not possible. Consider the usual odd reflection of u, so that any $u \in P_1^+(0,M)$ will be defined in the whole unit ball.

Now let $0 < \beta < (C_n\varepsilon^4)/4$, and choose j large enough such that

$$\begin{aligned} C_n\varepsilon^4 &\le C_n a^4 = \varphi(1, D_2P_2) \le \varphi(1, D_2\tilde{u}_{r_j}) + \beta = \varphi(r_j, D_2u_j) + \beta \\ &\le \varphi(\rho_j, D_2u_j) + \beta = \varphi(1, D_2\tilde{u}_{\rho_j}) + \beta \le \varphi(1, D_2P_1) + 2\beta \\ &= 2\beta \le \tfrac{C_n\varepsilon^4}{2}. \end{aligned} \tag{5.4}$$

We thus reach a contradiction. The reader should notify that in (5.4) we have used the monotonicity formula at one place only, and while doing this we have fixed u_j. □

Lemma 5.4. *Given $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that if $u \in P_1^+(0,M)$ and $\delta_\rho(u,0) > \varepsilon$ for some $\rho < \rho_\varepsilon$ then*

$$|D_e u(x)| < \varepsilon x_1 \qquad \text{in } B_\rho^+; \qquad\qquad (e \perp e_1).$$

In the above lemma we assume that $\varepsilon \le (8C_1M(n-1))^{-1}$ where C_1 is as in Theorem A.

Proof. We argue by contradiction, using the structure of global solutions. First we prove that the statement holds outside a narrow region of the set Π. More precisely, we claim that the estimate $|D_e u(x)| < \varepsilon x_1/2$ holds in $B_{2\rho}^+ \cap \{x_1 > \varepsilon\rho\}$, for small enough ρ. Indeed if this fails then there exist sequences $u_j \in P_1^+(0,M)$, $\rho_j \to 0$, $x^j \in B_{2\rho_j}^+ \cap \{x_1 > \varepsilon\rho_j\}$ and $\nu_j \perp e_1$ (vectors) such that

$$\delta_{\rho_j}(u_j, 0) > \varepsilon, \qquad\qquad |D_{\nu_j}u_j|(x^j) \ge \varepsilon x_1^j/2.$$

By Lemma 5.3, $u_j > 0$ in $B_{2\rho_j} \cap K_{\frac{\varepsilon}{2}}$ for $2\rho_j < \rho_{\frac{\varepsilon}{2}}$. Scale u_j by ρ_j at the origin, i.e., define

$$\tilde{u}_j(x) = \frac{u_j(\rho_j x)}{\rho_j^2}.$$

Then

$$\tilde{u}_j > 0 \quad \text{ in } K_{\varepsilon/2} \cap B_2^+, \qquad\qquad |D_{\nu_j}\tilde{u}_j|(\tilde{x}^j) \ge \varepsilon\tilde{x}_1^j/2.$$

where $\tilde{x}^j = x^j/\rho_j \in B_2^+ \cap \{x_1 > \varepsilon\}$. As j tends to infinity we can extract convergent subsequences of $\tilde{u}_j$, $\tilde{x}^j$, and ν_j with the corresponding limit values $\tilde{u}_0 \in P_\infty^+(0,M)$, $\tilde{x}^0 \in B_2^+ \cap \{x_1 \ge \varepsilon\}$, and $\nu_0 \perp e_1$ and such that

$$\tilde{u}_0 \ge 0 \quad \text{ in } K_{\varepsilon/2} \cap B_2^+, \qquad\qquad |D_{\nu_0}\tilde{u}_0|(\tilde{x}^0) \ge \varepsilon\tilde{x}_1^0/2.$$

Moreover, $\tilde{u}_0 \in P_\infty^+(0,M)$. Now the classification of global solutions (Theorem B) implies that in $\mathbf{R}_+^n$

$$\tilde{u}_0(x) = \frac{1}{2}x_1^2 + a x_1 x_2,$$

after suitable rotation. Hence $D_2\tilde{u}_0 = a x_1$. Now the non-negativity of $\tilde{u}_0$ in $K_{\varepsilon/2} \cap B_2$ in conjunction with freshman calculus implies that $2|a| \le \varepsilon/2$. Thus $\tilde{x}^0 \in B_2^+ \cap \{x_1 \ge \varepsilon\}$ implies

$$\varepsilon\tilde{x}_1^0 \le 2|D_2\tilde{u}_0(\tilde{x}^0)| = 2|a|\tilde{x}_1^0 \le \varepsilon\tilde{x}_1^0/2, \qquad (\tilde{x}_1^0 > 0)$$

and we reach a contradiction.

Now to prove the lemma we need to estimate $|D_e u|$ in the narrow region $\{x_1 \leq \varepsilon\rho\} \cap B_\rho$. We use a barrier argument. Let $v^\pm(x) = (D_e u)^\pm$ and

$$w(x) = \varepsilon x_1 - \frac{x_1^2}{2\rho} + 4C_1 M \frac{\varepsilon}{\rho}|X - Y|^2,$$

where $Y \in \Pi \cap B_\rho$, and $X = (0, x_2, \cdots, x_n)$. Now $v^\pm$ are subharmonic, w is superharmonic, and $w \geq v^\pm$ on the boundary of the cylinder $Q(Y) = \{x : 0 < x_1 \leq \varepsilon\rho;\ |X - Y| < \rho/2\}$. Now applying the comparison principle we obtain $v^\pm \leq w$ in $Q(Y)$. In particular for $X = Y$ we'll have the estimate $v^\pm(x_1, Y) \leq \varepsilon x_1$ for $0 < x_1 < \varepsilon\rho$. Now varying Y, we come to the desired estimate in the half ball $B^+(0, \rho)$. □

Recall the definition of singular contact points S_u.

Lemma 5.5. *Let $u \in P_1^+(0, M)$ and suppose $x = 0$ is a singular contact point. Then all blow-ups of u at the origin have the same representation as in (1.2), with a fixed a.*

Proof. First let us extend u across Π by odd reflection. Let $r_j \searrow 0$ be such that $u_{r_j} = u(r_j x)/r_j^2$ converges, for a subsequence, and in appropriate spaces, to a global solution

$$u_1 = \frac{1}{2} x^t A x = \frac{1}{2} \sum a_{ij} x_i x_j,$$

where A is a symmetric matrix with entries a_{ij}. More exactly $a_{ij} = 0$ for $\min(i, j) > 1$, and $a_{11} = 1$.

Now let $t_j \searrow 0$ be another arbitrary sequence, and define accordingly u_{t_j}. Then a similar argument gives a limiting polynomial

$$u_2 = \frac{1}{2} x^t B x = \frac{1}{2} \sum b_{ij} x_i x_j.$$

Here $B = (b_{ij})$ is a symmetric matrix, and as above $b_{ij} = 0$ for $\min(i, j) > 1$, and $b_{11} = 1$.

Now for a directional vector $e \perp e_1$, consider the monotonicity formula for $D_e u$. Then $\varphi(r, D_e u)$ is a monotone non-decreasing function of r. Since φ is monotone (Lemma 1.4) the limit, as r tends to zero, exists and

$$\lim_{r \to 0} \varphi(r, D_e u) = \lim_{r \to 0} \varphi(1, D_e u_r) = C a^4.$$

Observe that either $a \neq 0$ or that the blow-up u_0 satisfies $D_e u_0 \equiv 0$.

Replacing r by r_j and then by t_j we obtain

$$\varphi(1, D_e u_1) = \varphi(1, D_e u_2). \tag{5.5}$$

Inserting the polynomial representations of u_1 and u_2 in (5.5) we obtain for all vectors $e \perp e_1$

$$\|Ae\| = \|Be\|, \tag{5.6}$$

where $\|\cdot\|$ denotes the usual vector norm. From here we show that $A = B$. First notice that (5.6) implies that the Kernel of A and B are the same. In particular we may rotate the plane Π in order to obtain $a_{11} = b_{11} = 1$, $a_{ij} = b_{ij} = 0$ for $i + j \geq 4$. In other words we'll have the representation

$$u_1 = \frac{1}{2} x_1^2 + a x_1 x_2, \qquad u_2 = \frac{1}{2} x_1^2 + b x_1 x_2.$$

Next using (5.6) for different choice of directions e we'll have $|a| = |b|$. If $a = 0$ then we are done. Otherwise to see that $a = b$ it suffices to show that $\{u_1 < 0\} \equiv \{u_2 < 0\}$. We consider a point $x^0 \in \partial B_1^+$ such that $u_1(x^0) < 0$ but $u_2(x^0) > 0$, and show that this results in a contradiction. Define the segment $l_0 = \{tx^0 : \ 0 \leq t \leq 1\}$. Then we claim that there exists r_0 such that u does not change sign on $l_0 \cap B_{r_0}$. If this holds then we are done. So suppose this fails. Then there exists $\rho_j \to 0$ such that $u(\rho_j x^0) = 0$. Scale u by ρ_j, i.e. define $u_{\rho_j}(x) = u(\rho_j x)/\rho_j^2$. Applying the monotonicity formula (after odd reflection of u) as above we end up with a limit function (global solution) which has the representation $u_3 = x_1^2/2 + cx_1x_2$. Moreover, the above analysis gives that $|c| = |a|$. Hence $c = a$ or $c = b$, i.e., $u_3 = u_1$ or $u_3 = u_2$. In particular $u_3(x^0) \neq 0$. Using the information that $u_{\rho_j}(x^0) = 0$, we also conclude that $u_3(x^0) = 0$. This is a contradiction. $\square$

For the next lemma we need a definition of lower dimensional cones. We set

$$K_\varepsilon^0(u) := \{x \in \Pi : \ |x \cdot e_A| > \varepsilon |x|\},$$

where $e_A \in \Pi$, $e_A \perp \mathrm{Ker}(A)$, $\|e_A\| = 1$. Here A is the unique matrix corresponding to the blow-up of u at the origin. Observe that the uniqueness of A, and thus that of e_A, up to the factor -1, is guaranteed by Lemma 5.5.

Lemma 5.6. *Given $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(\alpha)$ such that if $0 \in S_u(\alpha)$ then*

$$\Gamma(u) \cap B_{r_\varepsilon} \cap K_\varepsilon^0(u) = \emptyset.$$

The proof of this lemma uses ideas similar to those in (5.4).

Proof. If the conclusion fails, then there exist $u_j \in P_1^+(0, M)$, with $0 \in S_{u_j}(a_j)$ (where $|a_j| \geq \alpha$) and $x^j \in \Gamma(u_j) \cap K_\varepsilon^0(u_j)$, with $r_j := |x^j| \to 0$.

Let A_j be the unique matrix corresponding to the blow-up of u_j; see Lemma 5.5. To simplify the analysis we may consider a rotated system for each function such that the blow up of u_j at the origin is the polynomial $P_j = x_1^2/2 + a_j x_1 x_2$.

Define as usual the rescaled function $\tilde{u}_j(x) = u_j(r_j x)/r_j^2$. Then $\tilde{x}^j = x^j/r_j \in \Gamma(\tilde{u}_j) \cap K_\varepsilon(\tilde{u}_j)$. Also $|\tilde{x}^j| = 1$. Extracting a subsequence of all the ingredients we'll have a limit function u_0, a limit point $x^0 \in \Gamma(u_0) \cap K_\varepsilon^0(u_0)$, and finally a limit value a_0 with $|a_0| \geq a$. The problem is that the matrix A associated to the blow-up u_0 does not need to have the same fixed kernel as all the A_j, and we have no contradiction.

However, the monotonicity formula will help us to fix this small detail. First notice that $0 \in S_{u_j}(a_j)$ implies that

$$C(a_j \nu_j \cdot e_2)^4 = \lim_{r \to 0} \varphi(r, D_{\nu_j} u_j),$$

where ν_j is the directional vector $\overline{0\tilde{x}^j}$. Now applying the monotonicity formula once again, we'll have

$$C(\alpha\varepsilon)^4 \leq C(a_j \nu_j \cdot e_2)^4 \leq \varphi(r_j, D_{\nu_j} u_j) = \varphi(1, D_{\nu_j} \tilde{u}_j), \tag{5.7}$$

for all $j = 1, 2, \cdots$. Letting $j \to \infty$ in (5.7) we'll obtain

$$0 < C(\alpha\varepsilon)^4 \leq \varphi(1, D_{\nu_0} u_0), \tag{5.8}$$

where $\nu_0 = \overline{0x^0}$. Now by Theorem B, u_0 is a two degree homogeneous polynomial of one or two dimensions. Since $x^0 \in \Gamma(u_0)$ we conclude, by homogeneity, that the whole line through x^0 and the origin is on the free boundary $\Gamma(u_0)$. In particular

u_0 is independent of the direction ν_0, and hence $D_{\nu_0} u_0 \equiv 0$. This contradicts (5.8). Hence we have reached a contradiction and the lemma is proved. $\square$

Let A^{x^0} be the matrix corresponding to the blow-up of u at x^0; see the proof of Lemma 5.5. In the next lemma using similar ideas as that of the proof in Lemma 5.6 we can prove that the kernel of A^{x^0} is continuous in x^0 for $x^0 \in S_u(a)$, where the continuity depends on the constant a. For this purpose we need a definition of distance of the matrices. For two $n \times n$-matrices A_1 and A_2 we define $\operatorname{dist}(A_1, A_2)$ as the angle between the corresponding vectors e_{A_1} and e_{A_2}; see the definitions preceding Lemma 5.6.

Lemma 5.7. *Given $\varepsilon > 0$, there exists $r_\varepsilon = r_\varepsilon(a) > 0$ such that if $x^0, x^1 \in S_u(a)$ and $|x^0 - x^1| < r_\varepsilon$, then $dist(A^{x^0}, A^{x^1}) < \varepsilon$.*

The proof follows by Lemma 5.6. We leave the details to the reader.

Proof of Theorem C. For each case we define a modulus of continuity $\sigma(r)$ and a radius r_0, then we take the minimum of all of these as the ones given in the statement of the theorem.

Proof of (C1): For the first statement in the theorem we consider the modulus of continuity $\sigma(t)$ given by the inverse of the relation $\varepsilon \to \rho_\varepsilon$ in Lemma 5.1. Let now $r_0 = \rho_{\{\varepsilon=1\}}$.

Proof of (C2): For any $x^0 \in \partial\Omega$ with $x_1^0 > 0$ and small enough, we can use Lemma 5.2 to conclude that in $B(x^0, 3x_1^0)$ the function u is $(\varepsilon(x_1^0)^2)$-close to the half-space solution $((x_1 - x_1^0)_+)^2/2$, and there are no contact points there. Hence [CKS; Lemma 6.2] applies and we deduce that $D_1 u - u \geq 0$ and $u \geq 0$ in $B(x^0, 3x_1^0/4)$. In particular, by the maximum principle, $D_1 u > 0$ in $\Omega \cap B(x^0, 3x_1^0/4)$. Thus we can apply the result by [AC] (cf. also [Ca3; Theorem 7]) to conclude that $\partial\Omega \cap B(x^0, x_1^0/4)$ is a $C^{1,\alpha}$ graph over Π. Now let us denote by $\sigma(r)$ the modulus of continuity for the inward normal vector to the free boundary that appears in the result by [AC]. Then the modulus of continuity of the inward normal vector to $\partial\Omega \cap B(x^0, x_1^0/4)$ is given by $\sigma(r/x_1^0)$, which obviously deteriorates as x^0 comes closer to the plane Π. The latter depends on the scaling upon application of [AC].

Finally to complete the proof of this part of the theorem we need to show that the (inward) normal vector to $\partial\Omega$ is uniformly continuous. Therefore, by Lemma 5.2, [CKS, Lemma 6.2], and [Ca3 Theorem 7], for given $\varepsilon > 0$ we can choose r_ε small enough such that for $x \in \partial\Omega \cap B^+_{1/2}$ with $x_1 < r_\varepsilon$ we have

$$\text{Angle}(\mathbf{n}_x, e_1) < \frac{\varepsilon}{2}\,, \tag{5.9}$$

where $\mathbf{n}_x$ is the inward normal vector to $\partial\Omega$ at x.

Then for $y^1, y^2 \in \partial\Omega \cap B^+_{1/2}$ we consider three cases:

(I) $y^1, y^2 \in \{x_1 < r_\varepsilon/2\}$,
(II) $y^1, y^2 \in \{x_1 \geq r_\varepsilon/2\}$,
(III) $y^1 \in \{x_1 < r_\varepsilon/2\}$, $\quad y^2 \in \{x_1 \geq r_\varepsilon/2\}$.

In case (I) we may use (5.9) to deduce $\text{Angle}(\mathbf{n}_{y^1}, \mathbf{n}_{y^2}) < \varepsilon$.

In case (II) we take $|y^1 - y^2| \leq s_\varepsilon$ with s_ε such that $\sigma(s_\varepsilon/r_\varepsilon) < \varepsilon$.

In case (III) by restricting the distance $|y^1 - y^2| \leq r_\varepsilon/2$, and using (5.9), we conclude that $\text{Angle}(\mathbf{n}_{y^1}, \mathbf{n}_{y^2}) < \varepsilon$.

Combining all three cases we may define our modulus of continuity as the inverse relation to

$$\varepsilon \quad \to \quad \min\{s_\varepsilon, r_\varepsilon/2\}.$$

Proof of (C3): The third statement in Theorem C, is now an easy consequence of Lemma 5.6 and 5.7. Since the singular points appear only when the set $\Lambda_r(u) \cap \Pi$ lies in a narrow cusp like region (Lemma 5.6) one may argue in a classical way to deduce the statement in part (C3). Obviously the continuity of the kernel of A^x in Lemma 5.7 plays an essential role. For readers convenience we carry out some details.

Let u_0 be a blow up of u at x. As a simple application of the classification in Theorem B, Lemma 5.5, and (C2) in Theorem C, we obtain that all blow-ups u_0 of u at a fixed point (x here) are the same, and moreover the set $\Lambda(u_0, x)$ is a linear space of dimension $(n-2)$. This is actually the tangent space of the contact points at x. Since according to Lemma 5.6, the singular points of $S_u(a)$ have the property that

$$\text{Angle}(\overline{xx^k}, \Lambda(u_0, x)) \quad \to \quad 0, \tag{5.10}$$

as $x^k \in \Lambda(u_0, x)$, and $x^k \to x$. The notation in (5.10) stands for the angle between the vector $\overline{xx^k}$ and the linear subspace $\Lambda(u_0, x)$.

Let the inverse relation $\varepsilon \to r_\varepsilon$ in Lemma 5.6 be denoted by $\sigma(r)$. Then by Lemma 5.6, the angle in (5.10) tends to zero with a uniform speed of $\sigma(r)$, where $r = |x - x^k|$. This (obviously) proves that the set of singular contact points have the property stated in the Theorem. At this step Lemma 5.7 will be crucial. One just uses classical Withney's type extension to show this. See more details in [CSh].

Proof of (C4): Let $x^j \to 0$ non-tangentially; i.e., $x_1^j \geq C|x^j|$ for some $C > 0$. Define

$$u_j(x) = \frac{u(|x^j|x)}{|x^j|^2}; \qquad \tilde{x}^j = \frac{x^j}{|x^j|} \in \partial B_1.$$

Obviously

$$\tilde{x}_1^j \geq C$$

Since by Lemma 5.1 $B(\tilde{x}^j, C/2) \subset \Omega(\tilde{u}_j)$ we have $u_j \to u_0$ (some limit function) in $C^2(B(\tilde{x}^j, C/4))$. In particular

$$D^2 u_j(\tilde{x}^j) = D^2 u(x^j) \to D^2 u_0(x^0).$$

Now by Lemma 5.5 all blow-ups at the origin have the same representation (they are the same function). Hence $u_0 = (x^t A x)/2$ for some symmetric matrix A with entries a_{ij} as in Lemma 5.5. Moreover A is independent of the blow-up, i.e., independent of the choice of x^j. This gives that

$$D^2 u_j(x^j) \to D^2((x^t A x)/2) = \text{fixed}.$$

This proves the claim.

Proof of (C5): The fifth statement uses Lemma 5.4. Here we define the modulus of continuity $\sigma(\tau)$ to be the one given by the inverse relation of $\varepsilon \to \rho_\varepsilon$.

We remark that the thickness condition of the statement in this case is absolutely necessary, since otherwise we have the counterexamples of the type $D_{21} P_2 \equiv a$.

Proof of (C6): The last statement for $n = 2$ is now obvious. We treat the case $n > 2$. First notice that by the techniques in the proof of Lemma 5.3, one may conclude that the assumption $\overline{\lim}_{r\to 0}\delta_r(u,0) > 0$ gives that any blow-up u_0 at the origin is one dimensional, i.e., $u_0 = x_1^2/2$.

Let us now consider an arbitrary sequence $x^j \to 0$ and prove that $D_{\nu e}u(x^j) \to 0$ for $e \perp e_1$ and ν arbitrary. There are two separate cases to be considered:
(a) $d(x^j, \Gamma(u)) = o((d(x^j, \Gamma_0))$, where $d(\cdot)$ denotes the distance function.
(b) $d(x^j, \Gamma(u)) \geq c_0(d(x^j, \Gamma_0)$, for some $c_0 > 0$.

In case (a) the contact points are far away from the point x^j relative to the distance of x^j to the free boundary points in the upper ball. Therefore, in view of (C2), scaling with respect to the nearest point $y^j \in \Gamma(u)$ will eventually lead to a problem as the interior case, and the result follows by [CKS].

In case (b) let $y^j \in \Gamma_0(u)$ be any nearest point to x^j, and set $r_j = |x^j - y^j|$, $z^j = (x^j - y^j)/r_j$. Now scale u as follows

$$u_j(x) = \frac{u(r_j x + y^j)}{r_j^2}.$$

Then $B^+_{c_0}(z^j) \subset \Omega(u_j)$ and by elliptic theory we have $D_{\nu e}u(z^j) \to D_{\nu e}u_0(z^0)$, where $z^0 = \lim z^j$, u_0 is the limit of u_j (maybe for a subsequence). By definition $|z^j| = 1$ and hence $|z^0| = 1$. On the other hand using (C5) we get

$$\sup_{B_1^+} \frac{|D_e u_j(x)|}{x_1} = \sup_{B^+_{r_j}(y^j)} \frac{|D_e u(x)|}{x_1} \leq \sup_{B^+_{r_j+|y^j|}} \frac{|D_e u(x)|}{x_1} \leq \sigma(r_j + 2|y^j|) \to 0,$$

as $j \to \infty$. Hence $\sup_{B_1^+} |D_e u_0(x)| = 0$, for any $e \perp e_1$. Thus for any $e \perp e_1$ and arbitrary direction ν we'll have $D_{e\nu}u_0 \equiv 0$ in B_1^+. Therefore

$$D_{e\nu}u(x^j) = D_{e\nu}u_j(z^j) \to 0, \qquad \text{as } j \to \infty. \qquad \square$$

6. Proof of Theorem D

The proof of Theorem D is much the same as that of the case of the half-ball. The main three steps should be followed as in the half-ball situation. In other words one first proves the $C^{1,1}$ regularity (Theorem A). Since G is uniformly C^3 (observe that we may even change G, i.e., consider a whole class of C^3 surfaces) any blow-up of any sequence of functions in the class $P_1^+(0, M)$ will result in a global solution in the upper half space $\mathbf{R}^n_+$. We thus have global solutions as before in $\mathbf{R}^n_+$, after suitable rotation and translation. Now the classification of such solution is done by Theorem B. The third part of the proof, i.e., Theorem C type analysis is done in a similar way, since it is a local analysis and blow-up always gives a global solution in the upper half space.

Despite the simplicity of the procedure described above, there is a "small" problem. The odd reflection, used in the case of Π (a plane), is not possible in the general case of non-plane surfaces. Also the monotonicity lemma is not applicable in this case, since the tangential derivative $D_e u$, here e is tangential direction at the origin, is not necessarily zero on the rest of G.

Our main concern, in this section, will be the technical difficulties discussed above. This seem to be a non-trivial task, we found out. So let us formulate the problem more rigorously.

First let us in accordance with Definition 1.1 define the class $P_r^+(0, M, M_1)$, which corresponds to all functions having the properties in Definition (1.1) with B^+ replaced by G^+ in Theorem D, and with Π replaced by G. Here M_1 corresponds to the C^3 norm of G. Suppose also the normal vector to G at the origin is parallel to the x_1-axis and that G^+ lies above G; see Picture 2 just after Theorem D.

Definition 6.1. We say a (non-zero) function u belongs to the class $P_1^+(M, M_1)$ if (for some Ω) u satisfies (in the sense of distributions):

(1) $\Delta u = \chi_\Omega$ in G^+, for some $\Omega \subset G^+$,
(2) $u = |\nabla u| = 0$ in $G^+ \setminus \Omega$,
(3) $\|u\|_{\infty, G^+} \le M$,
(4) $u = 0$ on $G \cap B_1(0)$.

Our strategy is very simple. We make the estimates

$$\varphi(r, D_e u, x^0) \le C \qquad \text{for } x^0 \in \Gamma(u)$$

with $B_r(x^0) \cap G \ne \emptyset$. Then for all other (small) r we can use the monotonicity formula, since now we have a solution in the entire ball $B_r(x^0)$.

For a fixed small value $r_0 = r_0(M_1)$ and a suitable rotation, there is a C^3 representation of the surface G in the following manner

$$G = \{x \in B_{r_0} : \ x_1 = \psi(X)\}, \qquad \psi(X^0) = 0, \quad |\nabla\psi(X^0)| = 0,$$

where $X = (x_2, \cdots, x_n)$ and ψ has C^3-norm M_1, say. Also the set $G^+ \cap B_{r_0}(x^0)$ lies above the graph $x_1 = \psi(X)$. For simplicity we will assume $X^0 = 0$.

Lemma 6.2. *Let e_1 be the normal vector to G at the origin, and $e \perp e_1$. Then we have the following estimate*

$$|D_e u(x) + D_1 u(x) D_e \psi(X)| \le C|x_1 - \psi(X)|, \qquad \textit{for } x \in G^+ \cap B_{r_0/2}.$$

Here $C = C(n, M, M_1)$.

Proof. First, suggested by standard techniques in partial differential equations, one transfers the problem to the half ball, by the following change of coordinates

$$y_1 = x_1 - \psi(X), \qquad y_i = x_i, \quad (i = 2, \cdots, n), \tag{6.1}$$

so that $U(y) = u(x)$. Let y^0 and $Q_r^+(y^0)$ be the images of x^0 and $B_r(x^0) \cap G^+$, respectively, under the mapping (6.1). We also define $Q_r^+ = Q_r^+(0)$. Rewrite the Laplacian in this new coordinates. This gives

$$\sum_{i,j=1}^n a_{ij} U_{y_i y_j} + b_1 U_{y_1} = \chi_{\Omega(U)} \quad \text{in } Q_{r_0}^+, \qquad U(0, Y) = |D_e U(0, Y)| = 0, \tag{6.2}$$

where e is orthogonal to e_1 (the x_1 axis). Also the coefficients a_{ij} are represented by

$$a_{ij} = \begin{pmatrix} (1 + |\nabla\psi|^2) & -\nabla\psi \\ -\nabla\psi & I \end{pmatrix}, \qquad b_1 = -\Delta'\psi, \tag{6.3}$$

where Δ' is the $(n-1)$-dimensional Laplacian. Differentiating (6.2) with respect to y_k ($k > 1$, i.e., tangential directions) we obtain

$$\sum_{i,j=1}^n a_{ij} U_{y_k y_i y_j} + \frac{\partial a_{ij}}{\partial y_k} U_{y_i y_j} + b_1 U_{y_1 y_k} + \frac{\partial b_1}{\partial y_k} U_{y_1} = 0, \qquad \text{in } \Omega(U),$$

here $\Omega(U)$ is the transfered domain $\{y_1 > 0\}$. Thus the function $v = U_{y_k}$, $k = 2, \cdots, n$, satisfy the equation of the form

$$Lv = g \qquad \text{in } \Omega(U) \cap Q^+_{r_0}, \tag{6.4}$$

with boundary condition

$$v = 0 \qquad \text{on } \{y_1 = 0\} \cap Q^+_{r_0}, \tag{6.5}$$

where $Lv = a_{ij} v_{y_i y_j}$ and the L^p norms at the right hand side of (6.4) can be estimated for $p < \infty$ as the following

$$\|g\|_{p, \Omega(U) \cap Q^+_{r_0}} \le C_p M. \tag{6.6}$$

with a constant C_p depending on p and G only.

Now we are ready to apply [LU; Theorem 2.1] which gives the estimate

$$\frac{v(y)}{y_1} \le C \left(1 + \|g\|_{p, Q^+_{r_0} \cap \{v > 0\}} \right) \qquad y \in Q^+_{r_0/2} \tag{6.7}$$

for functions $v \in C(\overline{Q^+_{r_0}}) \cap C^2(Q^+_{r_0} \cap \{v > 0\})$ satisfying (6.5) and the inequality

$$-Lv \le g \qquad \text{in } Q^+_{r_0} \cap \{v > 0\}. \tag{6.8}$$

Strictly speaking [LU; Theorem 2.1] is formulated for functions $v \in C(\overline{Q^+_{r_0}}) \cap C^2(Q^+_{r_0})$ satisfying $-Lv \le g$ in $Q^+_{r_0}$. But in fact the proof in [LU] implies the corresponding information about v not for the whole $Q^+_{r_0}$ but for the part of it where v is positive only. It follows from (6.4)–(6.7) that

$$|U_{y_k}| \le C y_1 \qquad \text{in } Q^+_{r_0/2}, \tag{6.9}$$

for $k = 2, \cdots n$. Coming back to x variables we arrive at the statement of the lemma. □

The next lemma will help us to estimate the second derivatives.

Lemma 6.3. *Let u be in the class $P_1^+(M, M_1)$. Suppose also $x^0 \in \Gamma(u) \cap B^+_{1/4}$ and $B_r(x^0) \cap G \neq \emptyset$. Then*

$$\int_{B_r(x^0) \cap V} \frac{|D^2 u(x)|^2}{|x - x^0|^{n-2}} dx \le C_0 r^2, \tag{6.10}$$

where $V = \{x \in G^+ : \ \psi(X) < x_1 < \psi(X) + r_0, \ |X| \le r_0\}$ and C_0 depends on M, M_1 and the dimension n. Here $|D^2 u| = \sum_{ij} |D_{ij} u|^2$.

Proof. We start with rewriting the equation $\Delta u = \chi_\Omega =: f$ in its weak form

$$-\int_V \nabla u \cdot \nabla \eta dx = \int_V f \eta dx \qquad \forall \eta \in C_0^\infty(V).$$

Now changing the variables (as in (6.1)) in the above integrals and using the same notation for η in the new variables, we arrive at

$$-\int_{Q^+_{r_0}} A_{ij} U_{y_j} \cdot \eta_{y_i} dy = \int_{Q^+_{r_0}} \tilde{f} \eta J dy \qquad \forall \eta \in C_0^\infty(Q^+_{r_0}), \tag{6.11}$$

where $\tilde{f}$ is the transfered f,

$$J = \det \left(\frac{\partial x_i}{\partial y_j} \right), \qquad A_{ij} = a_{ij} J.$$

Now if in (6.11) we replace η by η_{y_k} $(k = 2, \cdots n)$ and integrate by parts we arrive at the following relation for the function $v = U_{y_k}$

$$\int_{Q^+_{r_0}} A_{ij} v_{y_j} \eta_{y_i} dy = \int_{Q^+_{r_0}} \left[\tilde{f} J \delta_{ik} - \frac{\partial (J a_{ij})}{\partial y_k} U_{y_j} \right] \eta_{y_i} dy. \tag{6.12}$$

It is clear that (6.12) holds for any $\eta \in W^{1,2}(Q_{r_0})$. Now let $\xi = \xi(|x - x^0|)$ be a cutting function for a ball $B_{2r}(x^0)$, $\xi = 1$ in $B_r(x^0)$, $r \leq r_0/2$. We keep the same notation for ξ in the y variables. In (6.12) insert $\eta = v\Psi_N\xi^2$, where $\Psi_N = \min(\Psi, N)$ and Ψ is the fundamental solution for the divergence operator $\frac{\partial}{\partial y_i}(a_{ij} J \frac{\partial}{\partial y_j})$ with pole at $y^0 = y^0(x^0)$. Here N is a large constant which will tend to infinity.

As a result of this we obtain

$$\begin{aligned} &\int (a_{ij} v_{y_i})(v_{y_j} \Psi_N \xi^2 + 2\Psi_N v \xi \xi_{y_j}) J dy + I \\ &= \int \left[J\tilde{f}\delta_{ik} - \tfrac{\partial (J a_{ij})}{\partial y_k} U_{y_j} \right] \left[v_{y_i} \Psi \xi^2 + v(\Psi_N)_{y_i} \xi^2 + 2v\Psi_N \xi \xi_{y_i} \right] dy, \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} 2I &= \int_{\{\Psi<N\}} J a_{ij} \left(v^2\right)_{y_i} \Psi_{y_j} \xi^2 dy = \int_{\{\Psi<N\}} \left(J a_{ij} (\Psi_N)_{y_j}\right)_{y_i} v^2 \xi^2 dy \\ &\quad -2\int v^2 J a_{ij} (\Psi_N)_{y_j} \xi \xi_{y_i} dy + \int_{\{\Psi=N\}} v^2 J a_{ij} (\Psi_N)_{y_j} \xi^2 \cos(\gamma, y_j) dy \\ &\geq -2\int v^2 J a_{ij} (\Psi_N)_{y_j} \xi \xi_{y_i} dy, \end{aligned} \tag{6.14}$$

where γ is the unit normal vector to $\{\Psi = N\}$ directed towards $\{\Psi > N\}$. Letting $N \to \infty$ in (6.14) and using (6.13)–(6.14) we'll have

$$\begin{aligned} &\int J a_{ij} v_{y_i} v_{y_j} \Psi \xi^2 dy \\ &\leq -\int \left[2J a_{ij} v_{y_i} v \Psi \xi \xi_{y_j} - v^2 J a_{ij} \Psi_{y_j} \xi \xi_{y_i} + C \textstyle\sum_i |v_{y_i} \Psi \xi^2 + v \Psi_{y_i} \xi^2 + 2v\Psi \xi \xi_{y_i}| \right] dy \end{aligned}$$

Now the assumption that $B_r(x^0)$ intersects G implies that $y_1^0 < Cr$. Thus by (6.9) we have $|v(y)| \leq Cr$. Therefore by standard way we can obtain

$$\int_{Q^+_r(y^0)} \frac{|v_{y_i}|^2}{|y - y^0|^{n-2}} \xi^2 dx \leq C_0 r^2, \qquad (i = 1, \cdots, n)$$

where $v = u_{y_k}$ with $k = 2, \cdots, n$. This estimate in conjunction with (6.2) and (6.3) imply

$$\int_{Q^+_r(y^0)} \frac{|U_{y_i y_j}|^2}{|y - y^0|^{n-2}} \xi^2 dx \leq C_0 r^2, \qquad (i, j = 1, \cdots, n).$$

Finally changing the coordinates to the x variables we'll have the estimate (6.10). The proof is completed $\square$

Now we will start the process of scaling and applying the monotonicity lemma as in [CKS; Lemma 3.2]. Recall the definition of $S_j(u)$ from Section 2. Then we claim for $u \in P_1^+(M, M_1)$ there exists a constant $C_0 = C_0(M, M_1, n)$ such that

$$S_{j+1}(z, u) \leq \max\{\frac{S_j(z, u)}{4}, C_0 M 2^{-2j}\}, \qquad \text{for all } j, \tag{6.15}$$

where $z \in \Gamma(u) \cap V$. Here V is as in Lemma 6.3.

Now if (6.15) fails, then there exist sequences $u_j \in P_1^+(M, M_1)$, $z^j \in \Gamma(u_j)$, and k_j ($\to \infty$) such that

$$S_{k_j+1}(z^j, u_j) \geq \max\{\frac{S_{k_j}(z^j, u_j)}{4}, jM2^{-2k_j}\}. \tag{6.16}$$

Two situations may arise:
(a) $B_{2^{-k_j}}(z^j) \subset G^+$, at least for some subsequence of k_j' of k_j.
(b) $B_{2^{-k_j}}(z^j) \cap G \neq \emptyset$, for all k_j with sufficiently large j.

For case (a) we are just in the situation considered in [CKS; Lemma 3.2]. The only difference is that here we cannot apply the monotonicity formula to $\varphi(r, D_e u_j, z^j)$ for any $r \leq 1/2$ because u_j may not be defined in the whole ball $B_r(z^j)$. Let R_j be the maximum radius of $B_r(z^j)$ for which $B_r(z^j) \subset G^+$. By Lemma 6.3 we have $\varphi(R_j, D_e u_j, z^j) \leq C$ for any direction e in $\mathbf{R}^n$ and by the monotonicity formula $\varphi(2^{-k_j'}, D_e u_j, z^j) \leq C$. The problem has now turned into an interior case and we can use the proof in [CKS; Lemma 3.2].

For case (b) we need a more detailed analysis. Now one defines

$$\tilde{u}_j(x) = \frac{u_j(2^{-k_j}x + z^j)}{S_{k_j+1}(z^j, u^j)},$$

in the corresponding domain. More exactly, if

$$\psi_j(X) := \frac{\psi(2^{-k_j}X + Z^j) - z_1^j}{2^{-k_j}}, \qquad G^{+,j} := \{x \in B_1(0) : \ x_1 > \psi_j(X)\},$$

and $G^j := \{x \in B_1(0) : \ x_1 = \psi_j(X)\}$, then by (6.16) we have

$$\sup_{G^{+,j} \cap B_{1/2}} |\tilde{u}_j| = 1, \qquad \sup_{G^{+,j}} |\tilde{u}_j| \leq 4, \qquad |\Delta \tilde{u}_j| = \left| \frac{2^{-2k_j}}{S_{k_j+1}(z^j, u_j)} \chi_{\Omega(\tilde{u}_j)} \right| \leq \frac{1}{jM},$$

in $G^{+,j}$.

Now coming back to the sequence z^j, we see that $z^j \to z^0 \in G$ (for some z^0); by translation we may assume $z^0 = 0$, $\psi(0) = |\nabla\psi(0)| = 0$. Consequently $\psi_j(0) \leq 0$, and

$$|\nabla \psi_j(X)| = |\nabla \psi(2^{-k_j}X + Z^j)| = O(2^{-k_j} + |Z^j|) \quad \to \quad 0,$$

as $j \to \infty$. The derivatives of ψ_j, of order two and three, vanish in the limit as well. In other words ψ_j converges in C^3 to a constant $c \in [-1, 0]$. Let now $V_c = \{x \in B_1(0) : \ x_1 > c\}$. If $c < 0$ then the origin is an interior point of G^+ and the proof in this case is similar to that of case (a). So suppose $c = 0$, so that the origin is a boundary point of V_c. Now using Lemma 6.2 we have, for $e \perp e_1$ and $x \in B_{2^{-k_j}}(z^j) \cap G^+$, the estimate

$$|D_e u_j(x) + D_1 u_j(x) D_e \psi(X)| \leq C|x_1 - \psi(X)| \leq 2C2^{-k_j},$$

For the rescaled function $\tilde{u}_j$ we can use (6.16) and Lemma 6.2 to arrive at

(6.17)
$$\begin{aligned}
&|D_e \tilde{u}_j(x) + D_1 \tilde{u}_j(x) D_e \psi_j(X)| \\
&= \left(\tfrac{2^{-k_j}}{S_{k_j+1}(z^j, u_j)} \right) |D_e u_j(2^{-k_j}x + z^j) + D_1 u_j(2^{-k_j}x + z^j) D_e \psi(2^{-k_j}X + Z^j)| \\
&\leq \left(\tfrac{2C2^{-2k_j}}{S_{k_j+1}(z^j, u_j)} \right) \leq \tfrac{2C}{jM} \quad \to \quad 0,
\end{aligned}$$

as j tends to infinity, and $x \in B_1(0,) \cap G^{+,j}$.

It is better now to transfer the problem to a fixed domain by changing variables as in (6.1) with ψ_j instead of ψ. Denote by $\tilde{z}^j$ and $Q^+_{r,j}$ the images under this mapping of z^j and $B_r(0) \cap G^{+,j}$, respectively. Let also $\tilde{U}_j(y) = \tilde{u}_j(x) = \tilde{u}_j(y_1 + \psi_j(Y), Y)$. For large j one has $Q^+_{1/2,j} \subset B^+_{3/4} \subset Q^+_{1,j}$ and

$$\sup_{Q^+_{1/2,j}} |\tilde{U}_j| = 1, \qquad \sup_{\tilde{B}^+_{3/4}} |\tilde{U}_j| \leq 4 \tag{6.18}$$

$$\|L^{(j)} \tilde{U}_j\|_{\infty, B^+_{3/4}} \leq \frac{C}{j} \tag{6.19}$$

$$\tilde{U}_j = 0 \qquad \text{on } \Pi \cap B_{3/4}, \tag{6.20}$$

$$\tilde{U}_j(\tilde{z}^j) = |\nabla \tilde{U}_j(\tilde{z}^j)| = 0, \qquad \text{and } \tilde{z}^j \to 0 \text{ as } j \to \infty. \tag{6.21}$$

In (6.19) the coefficients of the operator $L^{(j)} = a^{(j)}_{kl} \frac{\partial^2}{\partial y_k \partial y_l} - b^{(j)} \frac{\partial}{\partial y_1}$ are defined by relations similar to (6.3) with ψ_j instead of ψ. It is easy to deduce from (6.18)–(6.20) that the $L^p(B^+_{5/8})$ norms of the second derivatives of $\tilde{U}_j$ are bounded uniformly with respect to j for any $p < \infty$. In particular for some subsequence $\tilde{U}_j \to U_0$ weakly in $W^{2,p}(B^+_{5/8})$ where the limit function U_0 satisfies

$$\Delta U_0 = 0, \quad \text{in } B^+_{5/8} \qquad U_0 = 0 \quad \text{on } \Pi \cap B^+_{5/8},$$

and

$$\sup_{B^+_{1/2}} |U| = 1, \qquad U(0) = |\nabla U(0)| = 0. \tag{6.22}$$

Observe that the left hand side in (6.17) coincides with $|D_e \tilde{U}_j(y)|$. Hence we get $D_e U^0 \equiv 0$ in $B^+_{1/2}$ for any $e \perp e_1$. Therefore $U_0 = U_0(y_1) = by_1 + c = 0$. This contradicts the first statement in (6.22). Thus we have proved (6.15).

From the above analysis it follows that we have a similar estimate as that in (2.1). Now the $C^{1,1}$ regularity follows as in the proof of Theorem A.

Now to prove Theorem C, we argue as in Section 5, without any essential changes. Indeed, Lemmas 5.1 and 5.2 go through without any changes. One needs only to verify that the limit of scalings of G will be the plane Π. This however, is elementary and the reader may verify it easily.

For the proof in Lemma 5.3, we used the monotonicity formula and the odd reflection of u. This however, is not applicable in our case since the surface G is not a plane. Also the function $D_e u$, where e is orthogonal direction to e_1, is not necessarily zero on the rest of G. This is needed in order to apply the monotonicity formula in (5.4).

Now the above reasoning calls for a new type of argument in the general case of Theorem D. We will do this in detail. We first define a new function

$$v(x) = D_1 u D_k \psi + D_k u, \qquad \text{in } G^+$$

where $k = 2, \cdots, n$. Recall that the graph G has the normal e_1 at the origin $0 \in G$. The function v thus defined is zero on $G \cap B_{r_0}$. Now let $B(x^1, r_1)$ be the ball touching G at the origin, and suppose it is located outside G^+, i.e., $B(x^1, r_1) \cap G^+ = \emptyset$, and $0 \in \partial B(x^1, r_1)$.

Next consider an "odd" inversion of the function v with respect to $\partial B(x^1, r_1)$. we call this function also v. In other words we define

$$v(x) = -\frac{r_1^{n-2} v\left(\frac{r_1^2 (x-x^1)}{|x-x^1|^2}\right)}{|x-x^1|^{n-2}}, \qquad \text{for } x \in B(x^1, r_1) \cap \tilde{G^+},$$

where $\tilde{G^+}$ is the inversion of G^+ in the same ball. Let now $r_2 \le r_1/2$. Then v is defined in $B(0, r_2) \setminus (G^+ \cup \tilde{G}^+)$. Extend v as zero function to the rest of $B(0, r_2)$.

Now, a lengthy but elementary, calculations reveals that $\Delta v^\pm \ge -C$ in $B(0, r_2)$. It is also not hard to verify that as we scale v the set where v is zero will converge to Π, the $(n-1)$-dimensional plane, and that the blow-up of v will converge to a global solution in the sense of Theorem B.

At this point one realizes, probably with some efforts, that all analysis involving the monotonicity formula may be carried out by using the monotonicity formula of [CJK] (Lemma 1.5). We leave the obvious details to the reader.

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